	Fall 2013 **MAT 319: Foundations of Analysis**	Fall 2013 **MAT 320: Introduction to Analysis**
Schedule	TuTh 10:00-11:20 Earth&Space 131 (through 10/3: joint lectures in Math P-131)	TuTh 10:00-11:20 Math P-131
Instructor	Thomas Sharland	Samuel Grushevsky
Office hours	M 4pm-5pm, Tu 4pm-5pm and Th 2.30pm-3.30pm in Math 5D148	Tu 11:30-12:00 and W 2:00-3:30 in Math 3-109, Th 11:30-12:30 in P-143
	During the joint lectures please attend the office hours of the professor lecturing	
Recitation	MW 11:00am-11:53am Harriman 116	MW 11:00am-11:53am SB Union 231
TA	Jun Wen	Yury Sobolev
Office hours	W 1:30-2:30 in Math 3-101, Tu 10:00-11:00 and 1:00-2:00 in MLC	W 12:00-1:00 in Math 3-104, MW 10:00-11:00 in MLC
Description	A careful study of the theory underlying topics in one-variable calculus, with an emphasis on those topics arising in high school calculus. The real number system. Limits of functions and sequences. Differentiations, integration, and the fundamental theorem. Infinite series.	A careful study of the theory underlying calculus. The real number system. Basic properties of functions of one real variable. Differentiation, integration, and the inverse theorem. Infinite sequences of functions and uniform convergence. Infinite series.
	The purpose of this course is to build rigorous mathematical theory for the fundamental calculus concepts, sequences and	An introductory course in analysis, required for math majors. It provides a closer and more rigorous look at material

Overview	limits, continuous functions, and derivatives. We will rely on our intuition from calculus, but (unlike calculus) the emphasis will be not on calculations but on detailed understanding of concepts and on proofs of mathematical statements.	which most students encountered on an informal level during their first two semesters of Calculus. Students learn how to write proofs. Students (especially those thinking of going to graduate school) should take this as early as possible.
Prerequisites	C or higher in MAT 200 or permission of instructor; C or higher in one of the following: MAT 203, 205, 211, 307, AMS 261, or A- or higher in MAT 127, 132, 142, or AMS 161	
Textbook	Kenneth Ross *Elementary Analysis: The Theory of Calculus*, **2nd edition**	
Homework	Weekly problem sets will be assigned, and collected in *Wednesday recitation*. The emphasis of the course is on writing proofs, so please try to write legibly and explain your reasoning clearly and fully. You are encouraged to discuss the homework problems with others, but your write-up must be your own work. Your best **10** homeworks will count towards your grade. *Late homework will never be accepted*, but under documented extenuating circumstances the grade may be dropped.	
Grading	Homework: 20%, Midterm I: 25%, Midterm II: 25%, Final: 30%.	

Syllabus/schedule (subject to change)

All joint lectures through 10/3 meet in **Math P-131**.
First recitation on Wed 8/28, second recitation 9/4.
During joint lectures through 10/3, students with last names starting A-L attend recitation in Harriman 116, students with last names M-Z attend recitation in SB Union 231

Tue 8/27	**1.**	Joint class: Introduction, motivation: what are numbers? Natural numbers and induction; integers; rationals. (Tom)	Read pages 1-19
Thu 8/29	**2.**	Joint class: Rational zeros; properties of numbers; concept of a field. (Tom)	**HW due 9/4:** 1.3, 1.4, 1.10, 1.12, 2.2, 2.5, 3.1, 3.4, 3.6
Tue			

9/3		*No class: Labor Day*	
Thu 9/5	**3.**	Joint class: Bounded sets; Completeness axiom for real numbers; Archimedean property. (Tom)	Read pages 20-27; **HW due 9/11:** Parts eghimr of 4.1,4.2,4.3,4.4; & 4.8, 4.10, 4.11, 4.12, 4.14
Tue 9/10	**4.**	Joint class: Infinity, unboundedness. Intro to sequences. (Sam)	Read pages 28-38
Thu 9/12	**5.**	Joint class: Limit of a sequence.(Sam)	**HW due 9/18:** 5.2, 5.6, 7.3, 7.4, 8.1ac
Tue 9/17	**6.**	Joint class: Limit laws for sequences. (Sam)	Read pages 39-55
Thu 9/19	**7.**	Joint class: Divergence to infinity, more formal proofs. (Tom)	**HW due 9/25:** 8.3, 8.6, 8.8, 8.10, 9.1, 9.3, 9.5, 9.12, 9.14
Tue 9/24	**8.**	Joint class: Monotone and Cauchy sequences. (Tom)	Read pages 56-65
Thu 9/26	**9.**	Joint class: Subsequences. (Sam)	**No HW: prepare for the midterm**
Tue 10/1		**Joint Midterm I** in Math P-131.	Practice midterm (Last year's exam)
Thu 10/3	**10.**	Joint class: Subsequences. (Sam)	**HW due 10/9:** 10.1, 10.2, 10.5, 10.8, 10.9, 11.2, 11.4, 11.5, 11.8, 11.9

<u>The following syllabus below is only for **MAT 319**, in **Earth & Space 131**</u>

This is a tentative schedule; it is subject to change.

Tue 10/8	**11.**	Bolzano-Weierstrass Theorem, subsequential limits.	Read pages 66-81
Thu 10/10	**12.**	limsup and liminf, introduction to series.	**HW due 10/16:** 12.1, 12.2, 12.4, 12.6, 12.11, 12.12

Tue 10/15	**13.**	Convergence tests.	Read pages 95-109
Thu 10/17	**14.**	More convergence tests.	**HW due 10/23:** 14.1 parts bf, 14.4 parts bc, 14.8, 14.13, 14.14, 15.1, 15.4, 15.6
Tue 10/22	**15.**	Functions and continuity.	Read pages 123-138
Thu 10/24	**16.**	Combining continuous functions.	**HW due 10/30:** 17.4, 17.5, 17.6, 17.8, 17.10 part c, 17.12, 17.16
Tue 10/29	**17.**	Properties of Continuous functions: EVT, IVT.	Read pages 153-162
Thu 10/31	**18.**	Continuous functions and limits.	**No HW: Prepare for midterm!**
Tue 11/5		**Midterm II**	Practice Midterm II and some solutions and Midterm II Solutions
Thu 11/7	**19.**	Limits.	**HW due 11/13:** 20.4, 20.8, 20.11, 20.13, 20.14, 20.16, 20.17, 20.20
Tue 11/12	**20.**	Limits and Derivatives.	Read pages 223-239
Thu 11/14	**21**	Differentiable functions.	**HW due 11/20:** 28.4, 28.5, 28.7, 28.8, 28.11, 28.14, 28.15, 28.16
Tue 11/19	**22.**	Mean Value Theorem.	Read pages 232-239
Thu 11/21	**23.**	Applications of MVT.	**HW due 12/4:** 29.3, 29.5, 29.6, 29.7, 29.10 (not part c), 29.14, 29.17
Tue 11/26	**24.**	The Riemann Integral	Read pages 269-287
Thu 11/28		**No Class: Happy Thanksgiving!**	**HW due 12/4:** 32.1, 32.2, 32.5, 32.6, 32.7, 32.8
Tue 12/3	**25.**	Properties of the Integral	Read pages 291-296
Thu		Fundamental Theorem of	Prepare for the Final! Here is a Practice

12/5	**26.**	Calculus	exam and some solutions.

Review Session: Wednesday 11th December, 2pm-4pm in Physics 123

Final Exam: Friday, December 13, 11:15AM-1:45PM in ESS 131

Disability Support Services: If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

Academic Integrity: Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/.

Critical Incident Management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students' ability to learn.

LAST NAME: FIRST NAME:

STONY BROOK ID NUMBER:

Problem	1	2	3	4	5	Total
Score						

MAT 319/MAT 320
Analysis
Midterm 1

October 2, 2012

NO BOOKS OR NOTES MAY BE CONSULTED DURING THIS TEST.
NO CALCULATORS MAY BE USED.
Show all your work on these pages!
Total score = 100

1. (40 points) Here **N** represents the counting numbers $\{1, 2, 3, 4, \ldots\}$, **Z** represents the integers, **Q** the rational numbers and **R** the real numbers.

   a. Explain carefully why the equation $x + 5 = 1$ has no solution in **N**.

   b. Explain carefully why the equation $3x = 2$ has no solution in **Z**.

c. Explain carefully why the equation $x^2 = 7$ has no solution in $\mathbf{Q}$.

d. Explain carefully why the least upper bound property (the Completeness Axiom) guarantees that the equation $x^2 = 7$ has a solution in $\mathbf{R}$.

2. (15 points) Prove by induction that the sum of the first n odd integers is equal to n^2, i.e. that
$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

3. (15 points) For a pair (x, y) of real numbers, define $||(x, y)|| = |x| + |y|$. Prove carefully that
$$||(a + c, b + d)|| \leq ||(a, b)|| + ||(c, d)||.$$

4. (15 points) Here $\sin(x)$ is the usual sine function. Show that the sequence $a_1, a_2, a_3, \ldots$ defined by $a_n = \frac{\sin(n)}{n}$ converges, with limit 0.

5. (15 points) Suppose (s_n) is a sequence of positive numbers converging to the limit s. Prove that the sequence $(\sqrt{s_n})$ converges to $\sqrt{s}$. *Hint:* give separate proofs for $s = 0$ and $s > 0$.

END OF EXAMINATION

MAT 319

Practice Midterm II. 28 October, 2013

This is a closed notes/ closed book/ electronics off exam.

You are allowed and encouraged to motivate your reasoning, but at the end your proofs should be formal logical derivations, whether proving that something holds for all, or proving that your example works.

You can use any theorem or statement proven in the book; please refer to it in an identifiable way, eg. "by the completeness axiom", "by the definition of the limit", etc.

You should attempt Problem 1 and three of the remaining four questions. If you attempt all four questions, your total score will be made up of the score for Problem 1 and your best three scores on the remaining questions.

Please write legibly and cross out anything that you do not want us to read.

Each problem is worth 25 points.

Name:						
Problem	1	2	3	4	5	Total
Grade						

Problem 1.

a) Define what it means to say that the series

$$\sum_{n=1}^{\infty} a_n$$

diverges.

b) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

diverges. (Hint: show $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1}-\sqrt{n}$).

c) Define what it means for a function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ to be *sequentially continuous* at a point $x_0 \in \mathrm{dom}(f)$.

d) Suppose that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \text{ and } x \in \mathrm{dom}(f) \implies |f(x) - f(x_0)| < \epsilon$$

(that is, suppose f is continuous at x_0). Show that f is sequentially continuous at x_0.

Problem 2. In this question, you may use standard results about the convergence of series of the form $\sum \frac{1}{n^k}$ and of geometric series. You may also any other results concerning the convergence of series, as long as you state when you are using them. Prove the convergence or divergence of the following.

a)
$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}.$$

b)
$$\sum_{n=1}^{\infty} \frac{n^{2013}}{2^n}$$

c)
$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

d)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at $x = 1$ not continuous at any x in $\mathbb{R} \setminus \{1\}$.

Problem 4.

a) Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $g(x) = 0$ for all $x \in \mathbb{Q}$. Show that $g(x) = 0$ for all $x \in \mathbb{R}$.

b) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in \mathbb{Q}$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

c) Suppose that g is continuous at 0 with $g(0) = 0$ and that $|f(x)| \leq |g(x)|$ for all x. Show that f is continuous at 0.

Problem 5.

a) State the *Extreme Value Theorem.*

b) Give an example of a function $f\colon [0,1] \to \mathbb{R}$ which is not bounded.

c) Give an example of a continuous function $f\colon (0,1) \to \mathbb{R}$ which is not bounded.

d) Let $f\colon [0,4] \cap \mathbb{Q} \to \mathbb{R}$ be given by $f(x) = |x^2 - 2|$. Is f bounded? Does there exist x_0, $y_0 \in [0,4] \cap \mathbb{Q}$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [0,4] \cap \mathbb{Q}$?

MAT 319

Practice Midterm II. 28 October, 2013

This is a closed notes/ closed book/ electronics off exam.

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You should attempt Problem 1 and three of the remaining four questions. If you attempt all four questions, your total score will be made up of the score for Problem 1 and your best three scores on the remaining questions.

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diverges. (Hint: show $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1}-\sqrt{n}$).

c) Define what it means for a function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ to be *sequentially continuous* at a point $x_0 \in \mathrm{dom}(f)$.

d) Suppose that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \text{ and } x \in \mathrm{dom}(f) \implies |f(x) - f(x_0)| < \varepsilon$$

(that is, suppose f is continuous at x_0). Show that f is sequentially continuous at x_0.

Answer 1.

a) Define the nth partial sum of the series to be

$$s_n = \sum_{k=1}^{n} a_k.$$

Then we say the series diverges if the sequence (s_n) does not converge.

b) Following the hint, we write

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})} = \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n} = \sqrt{n+1}-\sqrt{n}.$$

So this means we can write our series as

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=0}^{\infty} \left(\sqrt{n+1}-\sqrt{n}\right).$$

Now we compute the sequence of partial sums (s_n). We find

$$\begin{aligned}
s_n &= \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} \\
&= \sum_{k=0}^{n} \left(\sqrt{k+1}-\sqrt{k}\right) \\
&= (1-0) + \left(\sqrt{2}-1\right) + \cdots + \left(\sqrt{n}-\sqrt{n-1}\right) + \left(\sqrt{n+1}-\sqrt{n}\right) \\
&= \sqrt{n+1}-1.
\end{aligned}$$

Now we see that $\lim_{n\to\infty} s_n = +\infty$, and so this means that the series diverges.

c) A function $f\colon \operatorname{dom}(f) \to \mathbb{R}$ is sequentially continuous at x_0 if for all sequences (x_n) in $\operatorname{dom}(f)$ with $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.

d) Let $\varepsilon > 0$ and x_n be a sequence in $\operatorname{dom}(f)$ with $x_n \to x_0$. By assumption, there exists $\delta > 0$ such that if $x \in \operatorname{dom}(f)$ and

$$|x - x_0| < \delta \quad \text{then} \quad |f(x) - f(x_0)| < \varepsilon.$$

As $x_n \to x_0$, then for this $\delta > 0$ there exists N such that

$$n > N \quad \Longrightarrow \quad |x_n - x_0| < \delta$$

from which it follows that

$$n > N \quad \Longrightarrow \quad |x_n - x_0| < \delta \quad \Longrightarrow \quad |f(x_n) - f(x_0)| < \varepsilon.$$

Hence $f(x_n) \to f(x_0)$ as required.

Problem 2. In this question, you may use standard results about the convergence of series of the form $\sum \frac{1}{n^k}$ and of geometric series. You may also any other results concerning the convergence of series, as long as you state when you are using them. Prove the convergence or divergence of the following.

a)
$$\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}.$$

b)
$$\sum_{n=1}^{\infty} \frac{n^{2013}}{2^n}$$

c)
$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

d)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

Answer 2.

a) We use the comparison test. We have
$$\frac{n^2}{n^3+1} \geq \frac{n^2}{n^3} = \frac{1}{n}.$$
Then since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the original series must also diverge.

b) We use the ratio test. Setting $a_n = \frac{n^{2013}}{2^n}$, we get
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^{2013}}{2^{n+1}}}{\frac{n^{2013}}{2^n}} = \frac{2^n}{2^{n+1}} \frac{(n+1)^{2013}}{n^{2013}} = \frac{1}{2}\left(\frac{n+1}{n}\right)^{2013} \to \frac{1}{2} < 1.$$
So by the ratio test, the series must converge.

c) Again we use the comparison test, this time noting that since
$$\frac{1}{\log n} \geq \frac{1}{n}$$
for all $n \geq 2$ we have that the series diverges.

d) We use the alternating series test. Since $a_n = \frac{1}{\sqrt{n}}$ is decreasing and converges to 0, the alternating series test guarantees the convergence of this series.

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at $x = 1$ not continuous at any x in $\mathbb{R} \setminus \{1\}$.

Answer 3.

- *Continuity at* $x = 1$. Note that $f(1) = 2$. Let $\varepsilon > 0$ and take $\delta = \frac{\varepsilon}{2}$. Then if $|x - 1| < \delta$ then

  $$|f(x) - f(1)| = |2x - 2| = 2|x - 1| < 2\delta = \varepsilon \qquad \text{if } x \in \mathbb{Q}, \text{ or}$$

  $$|f(x) - f(1)| = |(x+1) - 2| = |x - 1| < \delta = \frac{\varepsilon}{2} < \varepsilon \qquad \text{if } x \in \mathbb{R} \setminus \mathbb{Q}.$$

- *Not continuous at* $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. By density of $\mathbb{Q}$, there exists a sequence (x_n) such that $x_n \in \mathbb{Q}$ for all n and $x_n \to x$. Then we get

  $$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2x_n = 2x \neq x + 1 = f(x)$$

  so f is not (sequentially) continuous at x.

- *Not continuous at* $x \in (\mathbb{Q} \setminus \{1\})$. Let $x \in \mathbb{Q} \setminus \{1\}$ and let $x_n = x + \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$ for each $n \in \mathbb{N}$. Then we have $x_n \to x$, but

  $$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n + 1 = \lim_{n\to\infty} (x + \frac{\sqrt{2}}{n} + 1) = x + 1 \neq 2x = f(x)$$

  and so f is not (sequentially) continuous at x.

Problem 4.

a) Let $g\colon \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $g(x) = 0$ for all $x \in \mathbb{Q}$. Show that $g(x) = 0$ for all $x \in \mathbb{R}$.

b) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that if $f(x) = g(x)$ for all $x \in \mathbb{Q}$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

c) Suppose that g is continuous at 0 with $g(0) = 0$ and that $|f(x)| \le |g(x)|$ for all x. Show that f is continuous at 0.

Answer 4.

a) Clearly we only need to show that $g(x) = 0$ for all $x \in \mathbb{R}\setminus\mathbb{Q}$. So let $x \in \mathbb{R}\setminus\mathbb{Q}$ and let x_n be a sequence of rationals such that $x_n \to x$ (this is possible by density of $\mathbb{Q}$). Then we have, by continuity that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0.$$

b) Define $h(x) = f(x) - g(x)$. Then $h(x)$ is the difference of two continuous functions and so is continuous. Furthermore, $h(x) = 0$ for all $x \in \mathbb{Q}$, and so by part (a) we have $h(x) = 0$ for all x. Hence we have $f(x) = g(x)$.

c) First of all, we note that since $|f(0)| \le |g(0)| = 0$, we must have $f(0) = 0$. So now we show f is continuous at 0. Let $\varepsilon > 0$. Then, by continuity of g at 0 we see that there exists $\delta > 0$ such that

$$|x| < \delta \Longrightarrow |g(x)| < \varepsilon.$$

However, $|f(x)| \le |g(x)|$ and so

$$|x| < \delta \Longrightarrow |f(x)| \le |g(x)| < \varepsilon$$

and so f is continuous at 0.

Problem 5.

a) State the *Extreme Value Theorem.*
b) Give an example of a function $f\colon [0,1] \to \mathbb{R}$ which is not bounded.
c) Give an example of a continuous function $f\colon (0,1) \to \mathbb{R}$ which is not bounded.
d) Let $f\colon [0,4] \cap \mathbb{Q} \to \mathbb{R}$ be given by $f(x) = |x^2 - 2|$. Is f bounded? Does there exist x_0, $y_0 \in [0,4] \cap \mathbb{Q}$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [0,4] \cap \mathbb{Q}$?

Answer 5.

a) Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is bounded. Moreover, there exists x_0 and y_0 in $[a,b]$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in [a,b]$.

b) Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x} & 0 < x \le 1 \\ 2013 & x = 0. \end{cases}$$

Then f is not bounded since $f\left(\frac{1}{n}\right) = n$ for all $n \in \mathbb{N}$.

c) Let $f\colon (0,1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Then for the same reasoning as in part (b), f is not bounded.

d) The function f is bounded. To see this, note that

$$\left\{|x^2 - 2| \,:\, x \in [0,4] \cap \mathbb{Q}\right\} \subset \left\{|x^2 - 2| \,:\, x \in [0,4]\right\}$$

and the right side is bounded by the Extreme Value Theorem, since it is the image of a continuous function on a closed interval. However, we see that if we write $A = \{|x^2 - 2| \,:\, x \in [0,4] \cap \mathbb{Q}\}$ we have

$$\sup A = 14 \qquad \text{and} \inf A = 0.$$

However, there does not exist $x_0 \in [0,4] \cap \mathbb{Q}$ such that $f(x_0) = 0$, because the only solutions to $x^2 - 2 = 0$ are irrational.

MAT 319

Midterm II. November 5, 2013

This is a closed notes/ closed book/ electronics off exam.

You are allowed and encouraged to motivate your reasoning, but at the end your proofs should be formal logical derivations, whether proving that something holds for all, or proving that your example works.

You can use any theorem or statement proven in the book; please refer to it in an identifiable way, eg. "by the completeness axiom", "by the definition of the limit", etc.

You should attempt Problem 1 and three of the remaining four questions. If you attempt all four questions, your total score will be made up of the score for Problem 1 and your best three scores on the remaining questions.

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Each problem is worth 25 points.

Name:

Problem	1	2	3	4	5	Total
Grade						

Problem 1.

a) Define what it means to say that the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

b) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges. (Hint: show $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$).

c) Define what it means for a function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ to be continuous at a point $x_0 \in \mathrm{dom}(f)$.

d) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at x_0 and $f(x_0) > 0$. Show that there exists $\delta > 0$ such that $f(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$.

Answer 1.

a) Define the nth partial sum of the series to be

$$s_n = \sum_{k=1}^{n} a_k.$$

Then we say the series converges if s_n converges to a real number.

b) It is easy to show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So now we get

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{1}{n(n+1)} &= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \left(\frac{1}{2} - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Hence $s_n \to 1$ so the series converges by the definition in part (a). (*Remark: Quite a few people used the p-test and comparison test to prove this, which to be honest I had not considered when I wrote the question. I graded it correct as long as all the relevant tests were stated and used correctly.*)

c) The function $f\colon \mathrm{dom}(f) \to \mathbb{R}$ is continuous at x_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathrm{dom}(f)$ and

$$|x - x_0| < \delta \quad \text{then} \quad |f(x) - f(x_0)| < \varepsilon.$$

d) Let $\varepsilon = \frac{f(x_0)}{2} > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$\begin{aligned} x \in (x_0 - \delta, x_0 + \delta) &\Longrightarrow |x - x_0| < \delta \\ &\Longrightarrow |f(x) - f(x_0)| < \varepsilon = \frac{f(x_0)}{2} \\ &\Longrightarrow 0 < \frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2} \\ &\Longrightarrow f(x) > 0. \end{aligned}$$

Remark: No-one got this completely correct. The important thing to realise is that if we know f is continuous, then we are at liberty to choose any $\varepsilon > 0$ we want, and continuity guarantees us the existence of some $\delta > 0$ satisfying the definition. So here we were able to choose $\varepsilon = \frac{f(x_0)}{2}$, for example. No-one actually stated a particular choice for ε, which is what the question requires.

Problem 2.

a) State the *Alternating Series Test.*

b) Prove that the series
$$\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}n}{n^2}$$
converges.

c) Prove or provide a counterexample to the following statements.

i) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges.

ii) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ converges.

Answer 2.

a) Let a_n be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Then the series
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$
converges.

b) We can rewrite the series as
$$\sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$
Now, we know that $\sum \frac{1}{n^2}$ converges by the p-test, and for the second series, we can apply the Alternating Series Test from part (a), since $\frac{1}{n}$ is a decreasing sequence which converges to 0. Hence our original series is the sum of two convergent series, and hence is convergent.

c) i) This is false. Consider the sequence $a_n = \frac{1}{n}$. Then $a_n \to 0$ but $\sum \frac{1}{n}$ diverges.

ii) This is false. Consider the sequence $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. Then $\sum a_n$ converges by the Alternating Series Test but $\sum a_n^2 = \sum \frac{1}{n}$ diverges.

Remark: Everyone attempted this question. Generally the understanding was good, though c) ii) caused a lot of difficulty, as I thought it might (hence I put the counterexample on the practice midterm so it would be fresh in your minds!). However c) i) was generally well done.

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous at 0 and not continuous at any x in $\mathbb{R} \setminus \{0\}$.

Answer 3.

- *Continuity at* 0. Let $\varepsilon > 0$ and set $\delta = \varepsilon$. Then if $|x| = |x - 0| < \delta$ we have

  $$|f(x) - f(0)| = |x - 0| = |x| < \delta = \varepsilon \quad \text{if } x \in \mathbb{Q}$$
  $$|f(x) - f(0)| = |0 - 0| = 0 < \varepsilon \quad \text{if } x \in \mathbb{R} \setminus \mathbb{Q}.$$

  So we see that $|x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon$.
- *Not continuous on* $\mathbb{R} \setminus \mathbb{Q}$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. By denseness of $\mathbb{Q}$, we can take a sequence (x_n) such that $x_n \in \mathbb{Q}$ for all n and $x_n \to x$. But then we see that since $x_n \in \mathbb{Q}$, we must have $f(x_n) = x_n$ for all n. So we have

  $$\lim_{n\to\infty} x_n = x \quad \text{but} \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x \neq 0 = f(x)$$

  and so f is not (sequentially) continuous at x.
- *Not continuous on* $\mathbb{Q} \setminus \{0\}$. Let $x \in \mathbb{Q}$ with $x \neq 0$. Consider the sequence $x_n = x + \frac{\sqrt{2}}{n}$. Then $x_n \in \mathbb{R} \setminus \mathbb{Q}$ for all n and $x_n \to x$. By the definition of f, we must have $f(x_n) = 0$ for all n, and so

  $$\lim_{n\to\infty} x_n = x \quad \text{but} \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq x = f(x).$$

  Hence f is not (sequentially) continuous at x.

Remark: A lot of people attempted this - probably because this was an example done in class. Indeed, a couple got very high scores on this question. I think the main difficulty here was not so much the understanding of the problem, but more on how to write out the proof - this was particularly apparent when proving continuity at 0*. Remember, when proving continuity, it is almost always good to start with the statement "Let* $\varepsilon > 0$*." and then proceed from there. When proving discontinuity, I was looking for a justification of why a sequence* x_n *existed, either by giving a formula or by using density of* $\mathbb{Q}$ *(or* $\mathbb{R} \setminus \mathbb{Q}$*).*

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

a) Prove $f(0) = 0$.
b) Prove $f(-x) = -f(x)$.
c) Suppose f is continuous at 0. Show that f is continuous at all $x \in \mathbb{R}$.

Answer 4.

a) We write
$$f(0) = f(0+0) = f(0) + f(0) = 2f(0),$$
from which it follows that $f(0) = 0$.

b) Now we write
$$0 = f(0) = f(x + (-x)) = f(x) + f(-x).$$
Rearranging this gives $f(-x) = -f(x)$.

c) Let $x_0 \in \mathbb{R}$ be arbitrary and let $\varepsilon > 0$. By continuity of f at 0, we know that for (this) $\epsilon > 0$ there exists $\delta > 0$ such that
$$|x| < \delta \Longrightarrow |f(x)| < \varepsilon.$$
From this, it follows that
$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| = |f(x) + f(-x_0)| = |f(x - x_0)| < \varepsilon$$
and so f is continuous at x_0.

Remark: Of course, the crux of this problem is in part (c); the first two parts were to set up the proof in (c). I don't think anyone got a full answer to this question, though I must admit this is a question I like, and the proof once you see it is very short. For part (c), it is perhaps helpful to notice that f behaves like a linear function $f(x) = kx$ (indeed, you can prove these are the only such continuous functions that have the additive property in the question, but that's another story), and so given $\varepsilon > 0$ the choice of δ at 0 will be the same as the one you require at an an arbitrary $x_0 \in \mathbb{R}$.

Problem 5.

a) State the *Intermediate Value Theorem*
b) Let $f\colon \mathbb{R} \to \mathbb{Z}$ be continuous. Show that f must be constant.
c) Let $f\colon [0,1] \to [0,1]$ be continuous. Show that there exists $c \in [0,1]$ such that $f(c) = c$. Hint: consider the function $h(x) = f(x) - x$.

Answer 5. a) Let $f\colon I \to \mathbb{R}$ be a continuous function on an interval I, and let $a, b \in I$ with $a < b$. Then for all $y \in (f(a), f(b))$ (or for all $y \in (f(b), f(a))$) there exists $c \in (a,b)$ such that $f(c) = y$.

b) Suppose f is not constant. Then there exists $n < m \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ such that $f(x) = m$ and $f(y) = n$. Let $y = n - \frac{1}{2}$ (other choices of y are available); then $y \in (m,n) = (f(x), f(y))$. By the Intermediate Value Theorem, since f is continuous by assumption, there must exist $c \in (x,y)$ (or in (y,x) if $y < x$) such that $f(c) = y$. But $y \notin \mathbb{Z}$, so this is a contradiction, hence f must be constant.

c) Following the hint, let $h(x) = f(x) - x$. Then h is continuous since it is the difference of two continuous functions. Furthermore, since $f(0) \in [0,1]$, we must have

$$h(0) = f(0) - 0 = f(0) \geq 0 \tag{1}$$

and since $f(1) \in [0,1]$ we have

$$h(1) = f(1) - 1 \leq 0. \tag{2}$$

If we have equality in either (1) or (2) then we have found our point such that $f(c) = c$, so suppose $h(0) > 0$ and $h(1) < 0$. Then by the Intermediate Value Theorem there exists $c \in (0,1)$ such that $h(c) = 0$. Therefore $f(c) - c = 0$ and so $f(c) = c$ as required.

Remark: Most people were able to state the Intermediate Value Theorem, but not many tackled the applications in parts (b) and (c) - perhaps this was down to time constraints. A number of people gave heuristic arguments as to how the proofs would work, but a few details were general lacking - this was especially true in (b) where I don't think anyone invoked the Intermediate Value Theorem directly in their solution (though many perhaps were trying to use it implicitly). Part (c) is a classical result about continuity that again makes use of the Intermediate Value Theorem at a vital moment. Remember that when trying to show two functions have a common point on their graph (as here), it is often easier to consider their difference and use the Intermediate Value Theorem to show that this difference must be equal to 0 *somewhere. Note also that part (c) is fairly similar to the example I did in class where I showed that an odd degree polynomial has an real root.*

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Question 1 is worth 80 points. Each other problem is worth 30 points.

Name:								
Problem	1	2	3	4	5	6	7	Total
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Problem 1.

a) Let S be a subset of $\mathbb{R}$.

  (i) Define the greatest lower bound of S, $\inf S$ and least upper bound of S, $\sup S$.

  (ii) State the Completeness axiom for $\mathbb{R}$.

  (iii) Suppose $\inf S = \sup S$. What can you say about S?

b) Let (a_n) be a sequence of real numbers.

(i) Give the definition that (a_n) is bounded.

(ii) Show that if (a_n) is not bounded, then for all $M \in \mathbb{N}$ there exists n with $|a_n| > N$.

(iii) Show that $(a_n) = ((-1)^n(\sqrt{n} - 3))$ is not bounded.

c) Let $f\colon \operatorname{dom}(f) \to \mathbb{R}$ and let $x_0 \in \operatorname{dom}(f)$.

(i) Define what it means for f to be continuous at x_0.

(ii) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Using your definition in part (i), show f is continuous at 0.

d) Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers.

(i) Define the sequence of partial sums for the series $\sum a_n$.

(ii) Define what it mean for the series $\sum a_n$ to diverge.

(iii) Using the definition, show that if $|x| > 1$ then the geometric series $\sum_{n=0}^{\infty} x^n$ diverges.

e) Let $f\colon (a,b) \to \mathbb{R}$ with $x_0 \in (a,b)$ and let $L \in \mathbb{R}$.

(i) Give the definition that $\lim_{x \to x_0} f(x) = L$.

(ii) Give the definition that $f'(x_0) = L$.

(ii) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show f is differentiable at 0 with $f'(0) = 0$.

f) Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function.

(i) State the Fundamental Theorem of Calculus for continuous functions.

(ii) Using the chain rule for derivatives, prove the change of variables formula for integrals of continuous functions f: If u is differentiable and u' is continuous, then

$$\int_a^b (f \circ u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Problem 2. Completeness.

a) State what it means to say $\mathbb{Q}$ is dense in $\mathbb{R}$.
b) Show that if $a < b$ then there exists an irrational number x such that $a < x < b$.
c) Show that if $a < b$ then there are infinitely many rational numbers between a and b.

Problem 3. Sequences.

a) Define what it means to say $(a_n) \to +\infty$.

b) Suppose that $(a_n) \to +\infty$ and (b_n) is bounded. Show that $(a_n + b_n) \to +\infty$.

c) Define what it means to say $(c_n) \to 0$.

d) Show that if $(a_n) \to +\infty$ then $\left(\frac{1}{a_n}\right) \to 0$.

Problem 4. Series.

a) Define what it means for a series $\sum a_n$ to converge.

b) State the Comparison test for convergent series.

c) Show that the following series converge using the comparison test.

(i) $\sum \frac{n}{n^3+3}$

(ii) $\sum \frac{n!}{n^n}$.

Problem 5. Continuity.

a) State the Intermediate Value Theorem.

b) Let f and g be continuous functions on $[a, b]$ with $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

c) Suppose $f\colon [0, 2] \to \mathbb{R}$ and $f(0) = f(2)$. Show that there exists $x, y \in [0, 2]$ such that $y = x + 1$ and $f(x) = f(y)$ (Hint: consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$).

d) Show that if f is an odd degree polynomial, then it has at least one real root.

Problem 6. Differentiation. Recall that an odd function satisfies $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and an even function satisfies $f(-x) = f(x)$.

a) Show that if for all $x, y \in \mathbb{R}$ we have
$$|f(y) - f(x)| \leq (y - x)^2$$
then f is constant.

b) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and even, then f' is an odd function.

c) Show that every differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as a sum
$$f = f_{\text{even}} + f_{\text{odd}}$$
where f_{even} is a differentiable even function and f_{odd} is a differentiable odd function. (Hint: define $f_{\text{even}}(x) = \frac{f(x)+f(-x)}{2}$.)

Problem 7. Integration. Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded.

a) Define what it means for the function f to be integrable on $[a,b]$.

b) Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0,1]$. (You may use the fact that f is integrable if and only if for all $\varepsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$.)

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   (iii) Show that $(a_n) = ((-1)^n(\sqrt{n} - 3))$ is not bounded.

c) Let $f\colon \mathrm{dom}(f) \to \mathbb{R}$ and let $x_0 \in \mathrm{dom}(f)$.
   (i) Define what it means for f to be continuous at x_0.
   (ii) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   Using your definition in part (i), show f is continuous at 0.

d) Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers.
   (i) Define the sequence of partial sums for the series $\sum a_n$.
   (ii) Define what it mean for the series $\sum a_n$ to diverge.
   (iii) Using the definition, show that if $|x| > 1$ then the geometric series $\sum_{n=0}^{\infty} x^n$ diverges.

e) Let $f\colon (a, b) \to \mathbb{R}$ with $x_0 \in (a, b)$ and let $L \in \mathbb{R}$.
   (i) Give the definition that $\lim_{x \to x_0} f(x) = L$.
   (ii) Give the definition that $f'(x_0) = L$.
   (ii) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   Show f is differentiable at 0 with $f'(0) = 0$.

f) Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function.
   (i) State the Fundamental Theorem of Calculus for continuous functions.
   (ii) Using the chain rule for derivatives, prove the change of variables formula for integrals of continuous functions f: If u is differentiable and u' is continuous, then

$$\int_a^b (f \circ u(x))u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du.$$

Solution 1.

a)

(i) A lower bound for the set S is a number ℓ such that $\ell \leq s$ for all $s \in S$. The greatest lower bound for S is a number ℓ such that ℓ is a lower bound for S and if ℓ' is any other lower bound for S, then $\ell \geq \ell'$. An upper bound for the set S is a number L such that $L \geq s$ for all $s \in S$. The least upper bound for S is a number L such that L is a lower bound for S and if L' is any other lower bound for S, then $L \leq L'$.

(ii) Any non-empty subset of S which is bounded above has a least upper bound.

(iii) The set must contain one element. For suppose $s_1, s_2 \in S$ with $s_1 < s_2$. Then $\inf S \leq s_1 < s_2 \sup S$, and so $\inf S \neq \sup S$.

b)

(i) The sequence (a_n) is bounded if there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in N$.

(ii) Suppose not. Then there exists $M \in \mathbb{N}$ such that $M \geq |a_n|$ for all $n \in \mathbb{N}$. But then (a_n) is bounded (since it satisfies the definition of bounded).

(iii) Let $M \in N$ and set $n = (M+4)^2$. Then $|a_n| = (\sqrt{(M+4)^2} - 3) = M + 1 > M$. Hence (a_n) satisfies the condition in (ii) and so is not bounded.

c)

(i) The function f is continuous at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.

(ii) Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Then if $|x| = |x - 0| < \delta$ we have

$$|f(x) - f(0)| = |f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x| < \delta = \varepsilon.$$

d)

(i) The sequence of partial sums is the sequence (s_n) defined by $s_n = \sum_{k=0}^{n} a_k$.

(ii) The series $\sum a_n$ diverges if the sequence of partial sums (s_n) fails to converge.

(iii) Using standard algebra, we find that

$$s_n = \sum_{k=0}^{n} x^n = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Then, since (x^n) fails to converge for $|x| > 1$, the sequence of partial sums fails to converge.

e)

(i) We say $\lim_{x\to x_0} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in (a,b)$ and $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

(ii) We say $f'(x_0) = L$ if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L.$$

(iii) We compute

$$|f'(0)| = \lim_{x\to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0} \left|\frac{x^2 \sin\left(\frac{1}{x}\right)}{x}\right| = \lim_{x\to 0} \left|x \sin\left(\frac{1}{x}\right)\right| \le \lim_{x\to 0} |x| = 0$$

and so $f'(0) = 0$.

f) (i) Let f be continuous on $[a,b]$ and define

$$F(x) = \int_a^x f(t)\,dt.$$

Then if $x_0 \in (a,b)$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

(ii) (Admittedly, this is quite hard...) Clearly $f \circ u$ is continuous. Fix c and define

$$F(u) = \int_c^u f(t)\,dt,$$

so that $F'(u) = f(u)$ by (i). Now define $g = F \circ u$. By the chain rule, we have

$$g'(x) = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$$

and so by the Fundamental Theorem of Calculus

$$\begin{aligned}\int_a^b f \circ u(x) u'(x)\,dx &= \int_a^b g'(x)\,dx = g(b) - g(a) = F(u(b)) - F(u(a)) \\ &= \int_c^{u(b)} f(t)\,dt - \int_c^{u(a)} f(t)\,dt = \int_{u(a)}^{u(b)} f(t)\,dt.\end{aligned}$$

Problem 2. Completeness.

a) State what it means to say $\mathbb{Q}$ is dense in $\mathbb{R}$.
b) Show that if $a < b$ then there exists an irrational number x such that $a < x < b$.
c) Show that if $a < b$ then there are infinitely many rational numbers between a and b.

Solution 2.

a) We say that $\mathbb{Q}$ is dense in $\mathbb{R}$ since between any two real numbers there exists a rational number. That is, suppose $a < b$. Then there exists $q \in \mathbb{Q}$ with $q \in (a, b)$.
b) By density of $\mathbb{Q}$, there exists a rational number q between $a - \sqrt{2}$ and $b - \sqrt{2}$. Hence $q + \sqrt{2}$ is an irrational number between a and b.
c) By density of $\mathbb{Q}$, there exists a rational number q_1 between a and b. Similarly, there exists a rational number r_2 between r_1 and b, and inductively, there exists a rational r_{n+1} between r_n and b. So there is a distinct rational number r_n for each $n \in \mathbb{N}$ between a and b, and so there are infinitely many rational numbers between a and b.

Problem 3. Sequences.

a) Define what it means to say $(a_n) \to +\infty$.
b) Suppose that $(a_n) \to +\infty$ and (b_n) is bounded. Show that $(a_n + b_n) \to +\infty$.
c) Define what it means to say $(c_n) \to 0$.
d) Show that if $(a_n) \to +\infty$ then $\left(\frac{1}{a_n}\right) \to 0$.

Solution 3.

a) For all $M > 0$ there exists N such that

$$n > N \quad \Longrightarrow \quad a_n > M.$$

b) The sequence b_n is bounded so there exists $B > 0$ such that $|a_n| \leq B$ for all $n \in \mathbb{N}$. So let $M > 0$. Since $(a_n) \to +\infty$, there exists N such that

$$n > N \Longrightarrow a_n > M + B.$$

Hence, if $n > N$ we must have

$$a_n + b_n \geq a_n - B > (M + B) - B = M$$

and so $(a_n + b_n) \to +\infty$.

c) For all $\varepsilon > 0$ there exists N such that

$$n > N \quad \Longrightarrow \quad |c_n| < \varepsilon.$$

d) Let $\varepsilon > 0$. Since $(a_n) \to +\infty$, for $M = \frac{1}{\varepsilon}$ there exists N such that if $n > N$ then $a_n > M = \frac{1}{\varepsilon}$. Hence if $n > N$ we get

$$\left|\frac{1}{a_n}\right| < \frac{1}{M} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

and so $\left(\frac{1}{a_n}\right) \to 0$.

Problem 4. Series.

a) Define what it means for a series $\sum a_n$ to converge.
b) State the Comparison test for convergent series.
c) Show that the following series converge using the comparison test.
   (i) $\sum \frac{n}{n^3+3}$
   (ii) $\sum \frac{n!}{n^n}$.

Solution 4.

a) The series $\sum a_n$ converges if the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ converges.
b) Suppose $\sum a_n$ is a series with $a_n \geq 0$ for all n. Then if $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
c)
   (i) Note that

$$\frac{n}{n^3+3} \leq \frac{n}{n^3+3n^3} = \frac{n}{4n^3} = \frac{1}{4n^2}.$$

   Since $\frac{1}{4n^2}$ converges, the comparison test shows that $\sum \frac{n}{n^3+3}$ converges.
   (ii) Now note that

$$\frac{n!}{n^n} = \frac{1 \times 2 \times \cdots \times n}{n \times n \times \cdots \times n} = \frac{1}{n} \times \frac{2}{n} \times \cdots \times \frac{n}{n} \leq \frac{2}{n^2}.$$

   Since $\sum \frac{2}{n^2}$ converges, so does $\sum n!n^n$.

Problem 5. Continuity.

a) State the Intermediate Value Theorem.
b) Let f and g be continuous functions on $[a, b]$ with $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.
c) Suppose $f\colon [0, 2] \to \mathbb{R}$ and $f(0) = f(2)$. Show that there exists $x, y \in [0, 2]$ such that $y = x + 1$ and $f(x) = f(y)$ (Hint: consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$).
d) Show that if f is an odd degree polynomial, then it has at least one real root.

Solution 5.

a) Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then for all $v \in (f(a), f(b))$ (or for all $v \in (f(b), f(a))$) there exists $x_0 \in (a, b)$ with $f(c) = v$.
b) Consider $h(x) = f(x) - g(x)$ on $[a, b]$. Then $h(a) \geq 0$ and $h(b) \leq 0$. If equality holds in either case then we are done, so assume $h(a) > 0 > h(b)$. By the Intermediate Value Theorem there exists $x_0 \in (a, b)$ such that $h(x_0) = 0$, which means $f(x_0) = g(x_0)$.
c) Following the hint, take $g(x) = f(x + 1) - f(x)$. Then $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = f(0) - f(1) = -g(0)$. If $g(0) = 0 = g(1)$ we are done. Suppose then that $g(0) > 0$ (the other case is similar) so that $g(1) < 0$. Then by the Intermediate Value Theorem there exists x_0 in $(0, 1)$ such that $g(x_0) = 0$. But then we must have $f(x_0) = f(x_0 + 1)$.
d) An odd degree polynomial is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

where n is odd and $a_n \neq 0$. Suppose that $a_n > 0$ (otherwise consider $-f(x)$). Then $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Hence there exists a and b in $\mathbb{R}$ with $f(a) < 0 < f(b)$, and so the Intermeidate Value Theorem shows that there must exist $x_0 \in (a, b)$ with $f(x_0) = 0$.

Problem 6. Differentiation. Recall that an odd function satisfies $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and an even function satisfies $f(-x) = f(x)$.

a) Show that if for all $x, y \in \mathbb{R}$ we have
$$|f(y) - f(x)| \leq (y - x)^2$$
then f is constant.

b) Show that if $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable and even, then f' is an odd function.

c) Show that every differentiable function $f \colon \mathbb{R} \to \mathbb{R}$ can be written as a sum
$$f = f_{\text{even}} + f_{\text{odd}}$$
where f_{even} is a differentiable even function and f_{odd} is a differentiable odd function. (Hint: define $f_{\text{even}}(x) = \frac{f(x)+f(-x)}{2}$.)

Solution 6.

a) Using the definition of deriviative, we get for any $x_0 \in \mathbb{R}$ we have
$$|f'(x_0)| = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \lim_{x \to x_0} \frac{(x - x_0)^2}{|x - x_0|} = \lim_{x \to x_0} (x - x_0) = 0.$$
Hence $f'(x_0) = 0$ for all $x_0 \in \mathbb{R}$, and so f is constant.

b) We compute, using $y = -x$ and the fact that $f(x) = f(-x)$ for all x.:
$$\begin{aligned} f'(-x_0) = \lim_{y \to -x_0} \frac{f(y) - f(-x_0)}{y - (-x_0)} &= \lim_{y \to -x_0} \frac{f(y) - f(-x_0)}{y - (-x_0)} \\ &= \lim_{-x \to -x_0} \frac{f(-x) - f(-x_0)}{-x + x_0} \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{-(x - x_0)} \qquad = -f'(x_0) \end{aligned}$$
and so f' is odd.

c) Clearly if we define
$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$
then f_{even} is even and f_{odd} is odd, with $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$ for all x. Furthermore, both are constructed by compositions, additions and multiplication of differentiable functions, and so must be differentiable. (Extra question: can you show that the choices of f_{even} and f_{odd} are unique - you will need to know that the only function which is both even and odd is the zero function.)

Problem 7. Integration. Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded.

a) Define what it means for the function f to be integrable on $[a,b]$.
b) Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0,1]$. (You may use the fact that f is integrable if and only if for all $\varepsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$.)

Solution 7.

a) Define, for a subset $S \subset [a,b]$:

$$\begin{aligned} M(f,S) &= \sup\{f(x) : x \in S\} \quad \text{and} \\ m(f,S) &= \inf\{f(x) : x \in S\} \end{aligned}$$

Let P be a partition of $[a,b]$, a finite ordered subset $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$, and define

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}), \\ L(f,P) &= \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \end{aligned}$$

and

$$\begin{aligned} U(f) &= \inf\{U(f,P) : P \text{ is a partion of } [a,b]\}, \\ L(f) &= \sup\{L(f,P) : P \text{ is a partion of } [a,b]\}. \end{aligned}$$

Then we say f is integrable on $[a,b]$ if $L(f) = U(f)$.

b) (this is more fiddly than I had imagined...) Let $\varepsilon > 0$ and define $r_k^{\pm} = \frac{1}{k} \pm \frac{1}{2^{k+1}}$. Consider the partition P_n given by

$$P_n = \{0 = t_0 < r_n^- < r_n^+ < r_{n-1}^- < r_{n-1}^+ < \cdots < r_1^- < 1\}.$$

Notice on each interval $I_k = [r_k^+, r_{k-1}^-]$ we have $M(f, I_k) = m(f, I_k) = 0$, and on each interval $J_k = [r_k^-, r_k^+]$ we have $M(f, J_k) = 1$ and $m(f, J_k) = 0$. Hence

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= (M(f, [0, r_n^-]) - m(f, [0, r_n^-])) \cdot (r_n^-) \\
&\quad + \sum_{k=1}^{n} (M(f, [r_k^-, r_k^+]) - m(f, [r_k^-, r_k^+])) \cdot (r_k^+, r_k^-) \\
&= (r_n^-) + \sum_{k=1}^{n} (r_k^+, r_k^-) \\
&= \left(\frac{1}{n} - \frac{\varepsilon}{2^{n+1}} \right) + \sum_{k=1}^{n} \frac{2\varepsilon}{2^{n+1}} \\
&= \left(\frac{1}{n} - \frac{\varepsilon}{2^{n+1}} \right) + \sum_{k=1}^{n} \frac{\varepsilon}{2^{n}} \\
&= \left(\frac{1}{n} - \frac{\varepsilon}{2^{n+1}} \right) + \varepsilon \left(1 - \frac{1}{2^{n+1}} \right).
\end{aligned}$$

Now since we can make n as large as we please and ε as small as we please, we can make $U(f, P) - L(f, P)$ arbitrarily small, and so we are done.