MAT 319/320 "Foundations of Analysis/Introduction to Analysis" Fall 2005

General Information

Description and goals: Both MAT 319 and 320 provide a closer, more rigorous look at the fundamental concepts of one-variable calculus. The main focus will be on the key notions of convergence and continuity; the basic facts about differentiation and integration will be presented as examples of how these notions are used. The course provides a good opportunity for students to learn how to read and write rigorous proofs. MAT 320 prepares them for further studies in analysis. Both courses are writing intensive; all students will have the opportunity to complete the proof-oriented component of the Department of Mathematics upper division writing requirement.

Relation between MAT 319 and MAT 320: These two courses will be taught together to begin with. The first Midterm will be taken by all students in class on September 29. The lecture in the following week will still be joint, but after that the classes will split. Students will divide into sections depending on their aptitude and choice and the lecturers' recommendations. (Special arrangements have been made with the Registrar to permit this late change of registration.)
Since the syllabus of MAT 319 is less crowded than that of MAT 320 this course will go more slowly. It is intended primarily for students in the Teacher Preparation Program, and will therefore discuss some topics relevant to future teachers. Students in MAT 319 will also be expected to complete a project. However MAT 319 will not provide as good a foundation for the more advanced subsequent courses in the major such as MAT 322, MAT 324, MAT 364 and MAT 401/2. Any student who is contemplating going to graduate school in a mathematics related field is strongly advised to take MAT 320.

Schedule for the first few weeks: Both classes will meet for lectures on **Tuesday and Thursday 2:20-3:40 in HH112.** The recitations/workshops meet on **Monday and Wednesday 2:20-3:15**, initially in **Physics P112 and P117.** . (Come first to P112; we will treat P117 as an overflow room.) There will be **workshops on Monday Aug 29 and Wednesday Sept 7**, and a **recitation on Wednesday Aug 31**. For the rest of September, workshops will be on Mondays and recitations on Wednesdays.

Instructor for MAT 319: Professor Dusa McDuff.
Office Hours: Tuesday 1-2:15 in Math Undergraduate Office, Wed 11:50-12:45 in office, and by appointment.
Office: 3-111 Mathematics Department. SUNY at Stony Brook.
e-mail: dusa at math.sunysb.edu.

TA: Tanvir Prince
Office Hours: Mon 1:15-2:15 in office, Tuesd 7--9pm in MLC,
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e-mail: prince at math.sunysb.edu.

Textbook: Introduction to Real Analysis, by R. Bartle and D. Sherbert, Third ed, Wiley.

Grading Policy for MAT 319: Homework 25%. Project 15%. Midterms (two) 15% each. Final Exam 30%.

Exam Schedule: Midterm 1: Thu Sept 29 (in class). Midterm 2: TBA (in class). Final: Tuesday Dec 20, 5:00--7:30pm.

Recitations and Workshops: Students are expected to go to both the Monday and Wednesday class until the first Midterm. The workshops are required: students will be working in groups on a problem set that will be handed in and graded. These workshops are intended for the most part to help students learn and review concepts taught in previous classes. However they will count a little towards the homework part of the grade: they will each be worth approximately 1/3 of a homework. The recitations will be more traditional --- problems similar to the homework problems will be discussed. After the class splits, there will be one recitation per week for each class: the MAT 320 recitation on Monday and the MAT 319 class on Wednesday.

Scheduling conflicts: Any student with a scheduling conflict should make appropriate arrangements with their professor. Please tell us about any conflicts as soon as possible.

Homework This is an essential part of the class and is worth a considerable amount of the grade. The homework sets will be posted on the web in PDF format and will be due at 5pm on the due date. Solutions can be handed in to the TA or to the appropriate professor. (Put it under their office door if they are not there.) Late work will receive reduced credit, and will not be accepted after solutions are posted. You may work on your homework with other people (in fact, this is often a good idea), but the work you hand in must be your own, not copied directly from others. You should also list your working partners on the homework you hand in. The first homework will be due on **Thursday Sept 8.**

MAT 319 Project Each student in MAT 319 will work on a project (typically with one or two other students). The exact form of this project will depend on how many students are in the class and will be announced later. It will involve a 5 minute oral presentation backed up by a written paper.

Schedule for presentations I have written your names in alphabetical order within the groups; you should decide yourselves what is the best order for your presentations. The order is more or less (but not exactly) what you asked for on Thursday.

Dec 8	1. Garritano, Hughes, Panoussis
	2. Guastavino, Miller, Scapellati
	3. Pak, Samaroo, Yeung
	4. Bennett, Gounaris
	5. Xu
Dec 12	1. Estrada, Kosta, Langdon
	2. Chen, Tsang
	3. Cohen, Melchin, Samuel
	4. Cheng, Wu
Dec 13	1. Baker, Willie
	2. Agcaian, Falco, Hachmann, VanAcker
	3. Hsieh
	4. McConnell, Oblein
	5. Lauber
	6. Eaton, Gosselin, Phelan

Homeworks etc These will be posted below in pdf format.

Worksheet 1	Solutions to Worksheet 1
Worksheet 2	Solutions to Worksheet 2
Worksheet 3	Solutions to Worksheet 3
Worksheet 4	Solutions to Worksheet 4
Homework 1	Solutions to Homework 1
Homework 2	Solutions to Homework 2
Homework 3 (revised)	Solutions to Homework 3
Homework 4 (due TUESD 9/27)	Solutions
Review for Midterm 1	
Homework 5 (for MAT 319)	
Midterm I	Comments on solutions
Homework 6	Solutions
Homework 7	Solutions
Homework 8	Solutions
Homework 9	Solutions
Midterm II	
Homework 10	

Announcements

- The final exam will be on Tuesday Dec 20 in the usual lecture room HH 112, at 5pm -7:30pm. There are 6 questions worth 15-20 points each. See you there!.

- Here is a review sheet for the final and here is last year's final . I may revise the review sheet over the weekend (I have had to post these in a hurry), but I am not planning to make any serious changes. Just -- watch this space!

- I will have office hours on MONDAY DEc 19 2:30--3:30 in my office and again on Tuesday Dec 20 1--2 in my office.

- You may email me with any questions, if you want your grade for the project etc (though that won't be ready until the end of next week.)

- When you hand in the final version of your project, please also hand in the draft with its comments so I can see how well you addressed them.

- Tanvir will hand back the drafts on Monday Dec 5 2:20-3:15 in his office, and also on Tuesday Dec 6 in HH112 at 2:20--3:40. NOTE: there is no lecture on this Tuesday: instead Tanvir will be in the usual lecture room ready to answer questions.

- Tanvir's extra office hours: today (Thursday Dec 1) at 6-7pm and Friday Dec 2 at 12--1 and 6--7; both in his office. Deadline for project draft is 7pm on Friday in Tanvir's office. (DO NOT hand it in to me -- I will be away.)

- Review session for final exam: Tanvir has booked P 131 (Math) on THURSDAY Dec 15 7-8:30 pm. I will hold some extra office hours on Monday Dec 19, and will also post a review sheet for the exam. (Exam is Tuesday Dec 20.)

- Here is the quiz.

- Here is the list of suggested projects. Email me as soon as you can what your choice of project. Deadline for submission of first draft of paper is Dec 2. The final draft is due Dec 13. There are many other details on the project list.

- Here is the list of theorems and definitions for Midterm 2. Here is a review sheet for the midterm.

- One of you asked about the syllabus for MAT 319. We will do secs 3.1 - 3.4, 3.6, 4.1-4.2, 5.1-5.3 very carefully in lectures and also 6.1-6.2 if time permits. Many of the other topics in the initial chapters of the textbook will be covered in the projects and presentations. In October we will concentrate on Chapters 3 and 4, with lots of homework and an exam (somewhat like Midterm I) in mid November.

- The exam will be in class on Thursday Nov 10. We will start work on the projects the week after that.

For people with disabilities

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.ehs.stonybrook.edu/fire/disabilities.asp

Last modified: 12/1/2005

Math 319/320 Worksheet 1

Name: **ID:**

The first part of this worksheet is a review of basic logic. Recall the symbols

$$\forall = \text{ for all}, \quad \exists = \text{ there exists}, \quad \wedge = \text{ and}, \quad \vee = \text{ or}, \quad \sim = \text{ not}.$$

Remember also that if P and Q are statements the implication $P \Rightarrow Q$ ("P implies Q") is true if P is false and also if P is true and Q is true. Therefore its negation $\sim(P \Rightarrow Q)$ is logically equivalent to $P \wedge (\sim Q)$, i.e. $\sim(P \Rightarrow Q)$ holds iff P is true and Q is false.

Note: this is a slightly edited verion of the sheet that was handed out.

Problem 1. Negate the following statement: "If your glass is half-empty, you are a pessimist or you are thirsty." (Answer in words.)

Problem 2. The context in this problem is the set of all human beings. Let $E(x)$ be "x is educated," $F(x)$ be "x is female" and $O(x)$ be "x is older than 30." Then the statement "every uneducated male is older than 30" can be expressed as

$$\forall x, \Big((\sim E(x) \wedge \sim F(x)) \Longrightarrow O(x) \Big)$$

Express the following statements in a similar way:

(i) Some educated people are younger than 30.

(ii) Every female who is older than 30 is educated.

(iii) No uneducated person is both female and older than 30.

Problem 3. Consider the statement

"For every natural number n, if n^2 is even, then n is even."

Prove this statement in two different ways: (i) by showing that its contrapositive is true; (ii) by showing that its negation is false.
Note: the contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.

Problem 4. On a bumper sticker, I saw the statement

"For every real number x, there is a real number t such that $t(1-t) > x$."

After some thought I conjectured that this statement must be __________. To prove my conjecture carefully, I found a real number ____ such that for every real number ____ the inequality ____________________ held.

Hint: To understand the behavior of the function $f(t) = t(1-t)$ it might be helpful to draw its graph.

Math 319/320 Worksheet 1

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$$\forall = \text{ for all}, \quad \exists = \text{ there exists}, \quad \wedge = \text{ and}, \quad \vee = \text{ or}, \quad \sim = \text{ not}.$$

Remember also that if P and Q are statements the implication $P \Rightarrow Q$ ("P implies Q") is true if P is false and also if P is true and Q is true. Therefore its negation $\sim(P \Rightarrow Q)$ is logically equivalent to $P \wedge (\sim Q)$, i.e. $\sim(P \Rightarrow Q)$ holds iff P is true and Q is false.

Note: this is a slightly edited version of the sheet that was handed out in class.

Problem 1. *Negate the following statement: "If your glass is half-empty, you are a pessimist or you are thirsty." (Answer in words.)*

Your glass is half-empty but you are an optimist who is not thirsty.

or Although your glass is half-empty, you are neither a pessimist nor thirsty.

Problem 2. *The context in this problem is the set of all human beings. Let $E(x)$ be "x is educated," $F(x)$ be "x is female" and $O(x)$ be "x is older than 30." Then the statement "every uneducated male is older than 30" can be expressed as*

$$\forall x, (\sim E(x) \wedge \sim F(x)) \Longrightarrow O(x)$$

Express the following statements in a similar way.

Since x ranges over all human beings, it is implicit in the question that there are some people who are educated and some who are not, some males and some females and also some people over 30 and some under 30. This comment applies particularly to (i) where the answer given assumes that there are educated people, and that the negative of "over 30" is "under 30". When we do actual mathematics, there won't be this ambiguity.... However, it is still true that the questions have several correct (and equivalent) answers.

(i) *Some educated people are younger than* 30.
$\exists x, E(x) \wedge \sim O(x)$.

(ii) *Every female who is older than* 30 *is educated.*
$\forall x, (F(x) \wedge O(x)) \Longrightarrow E(x)$.
or $\sim \exists x, F(x) \wedge O(x) \wedge (\sim E(x))$.

(iii) *No uneducated person is both female and older than* 30.
logically this is equivalent to (ii). So you could use either of the expressions in (ii) or also
$\forall x, \sim E(x) \Longrightarrow \sim \big(F(x) \wedge O(x)\big)$

Problem 3. *Consider the statement*

"For every natural number n, if n^2 is even, then n is even."

Prove this statement in two different ways: (i) by showing that its contrapositive is true; (ii) by showing that its negation is false.
Note: *the contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.*

(i) Suppose that n is odd. We must show that n^2 is odd. But if n is odd we may write $n = 2k+1$ where $k \in \mathbb{N}, k \geq 0$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ is also odd.

(ii) The negation of the implication $P \Rightarrow Q$ is $P \wedge (\sim Q)$. So we must show that it is impossible for n^2 to be even and n to be odd. But if n is odd we may write $n = 2k+1$ where $k \in \mathbb{N}, k \geq 0$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ is also odd. So the hypotheses n^2 odd and n even are indeed contradictory.

Note In this simple case, the basic argument is the same in each case. It is just the framing that varies. But with more complicated arguments the choice of framing sometimes makes a difference.

Problem 4. On a bumper sticker, I saw the statement

"For every real number x, there is a real number t such that $t(1-t) > x$."

After some thought I conjectured that this statement must be __________. To prove my conjecture carefully, I found a real number ____ such that for every real number ____ the inequality ____________________ held.

Hint: To understand the behavior of the function $f(t) = t(1-t)$ it might be helpful to draw its graph.

After some thought I conjectured that this statement must be FALSE. *To prove my conjecture carefully, I found a real number x such that for every real number t the inequality $t(1-t) \geq x$ held.*

Note: The graph of $t(1-t)$ is a parabola cupped downwards and takes its maximum at $t = 1/2$, halfway between its zeros at $t = 0$ and $t = 1$. Therefore the maximum value is $1/4$. So you can take x to be anything $\geq 1/4$.

Math 319/320 Worksheet 2

Problem 1. Fill in the blanks in the following proof that
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

If $x \in A \cup (B \cap C)$ then either $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in$ __________ and so $x \in$ _______________. On the other hand, if $x \in B \cap C$ then $x \in$ __________ and $x \in$ __________. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Now suppose that $x \in$ __________________________. Then $x \in$ __________ and $x \in$ __________.

If $x \in A$ then

On the other hand if $x \notin A$ then

Therefore

Problem 2. It is possible to take intersections and unions of many sets $A_i, i \in I$, not just two. We define

$$\cup_{i \in I} A_i := \{x : \exists i \in I \text{ such that } x \in A_i\}, \qquad \cap_{i \in I} A_i := \{x : x \in A_i \forall i \in I\}.$$

The set I is called the *indexing set*. Often it is the set of the first n integers $\{1, \ldots, n\}$, but sometimes it is the infinite set $\mathbb{N}$ of all positive integers.

(i) Find three subsets A_1, A_2, A_3 of the plane $\mathbb{R}^2$ such that each double intersection $A_i \cap A_j$ is nonempty but the triple intersection $A_1 \cap A_2 \cap A_3$ is empty.

(ii) Find open intervals $A_i = (a_i, b_i) \subset \mathbb{R}$ such that each finite intersection $\cap_{1 \le i \le n} A_i$ is nonempty but the infinite intersection $\cap_{i \in \mathbb{N}} A_i$ is empty.

Problem 3. Let $f : A \to B$ be a function and $C \subset A, D \subset B$. Show that $C \subset f^{-1}(f(C))$ and $f(f^{-1}D) \subset D$.

If f is injective, do either of these inclusions become equalities?

What if f is surjective?

Problem 4. Let A, B be subsets of a universal set U. Simplify the following expressions. You can draw Venn diagrams to help you.

(i) $(A \cap B) \cup (U \setminus A)$

(ii) $A \cup [B \cap (U \setminus A)]$

Math 319/320 Worksheet 2

Problem 1. *Fill in the blanks in the following proof that* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

If $x \in A \cup (B \cap C)$ then either $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in \underline{A \cup C}$ and so $x \in \underline{(A \cup B) \cap (A \cup C)}$. On the other hand, if $x \in B \cap C$ then $x \in \underline{B}$ and $x \in \underline{C}$. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Now suppose that $x \in \underline{(A \cup B) \cap (A \cup C)}$. Then $x \in \underline{A \cup B}$ and $x \in \underline{A \cup C}$.
If $x \in A$ then: $x \in A \cup (B \cap C)$.
On the other hand if $x \notin A$ then $x \in B$ (because $x \in A \cup B$) and $x \in C$ (because $x \in A \cup C$). So $x \in B \cap C$.
Therefore x is either in A or in $B \cap C$, i.e. $x \in A \cup (B \cap C)$.

Problem 2. *It is possible to take intersections and unions of many sets $A_i, i \in I$, not just two. We define*

$$\cup_{i \in I} A_i := \{x : \exists i \in I \text{ such that } x \in A_i\}, \qquad \cap_{i \in I} A_i := \{x : x \in A_i \forall i \in I\}.$$

The set I is called the indexing set. Often it is the set of the first n integers $\{1, \ldots, n\}$, but sometimes it is the infinite set $\mathbb{N}$ of all positive integers.

(i) *Find three subsets A_1, A_2, A_3 of the plane $\mathbb{R}^2$ such that each double intersection $A_i \cap A_j$ is nonempty but the triple intersection $A_1 \cap A_2 \cap A_3$ is empty.*

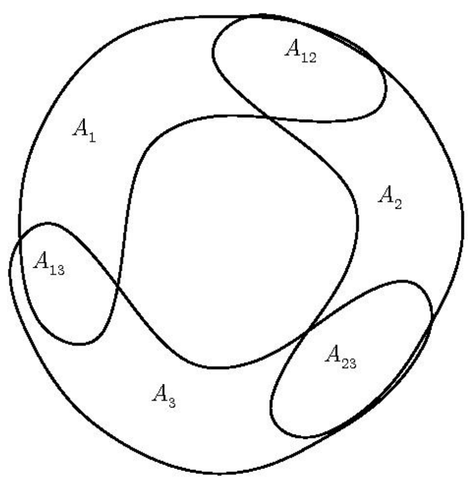


Figure 1. Here I wrote A_{12} to mean $A_1 \cap A_2$, etc.

(ii) *Find open intervals $A_i = (a_i, b_i) \subset \mathbb{R}$ such that each finite intersection $\cap_{1 \le i \le n} A_i$ is nonempty but the infinite intersection $\cap_{i \in \mathbb{N}} A_i$ is empty.*

Take for example, $A_i = (0, 1/i)$.

Problem 3. *Let $f : A \to B$ be a function and $C \subset A, D \subset B$. Show that $C \subseteq f^{-1}(f(C))$ and $f(f^{-1}D) \subseteq D$.*

$f^{-1}(f(C)) = \{a \in A : f(a) \in f(C)\}$. If $x \in C$ then $f(x) \in f(C)$ by defn of $f(C)$, hence x satisfies the condition to be in $f^{-1}(f(C))$.

(In words: $f^{-1}(f(C))$ is the set of all points whose image is contained in the the image $f(C)$ of C. But obviously the points in C have image in $f(V)$.)

If $x \in f^{-1}(D)$ then $f(x) \in D$ by defn of the inverse image. Hence $f(f^{-1}D) \subset D$.

If f is injective, do either of these inclusions become equalities?

f is injective iff $f(x) = f(y)$ implies $x = y$. To say $f(x) \in f(C)$, means that there is $c \in C$ such that $f(x) = f(c)$ (by defn of the set $f(C)$.) Hence if f is injective x must equal c. Since this holds for all x such that $f(x) \in f(C)$, $f^{-1}(f(C)) = C$.

But the second statement about D won't hold unless the image of f contains D, and you can only be sure of this when f is surjective.

(eg take $f : [0, \infty) \to \mathbb{R}, x \mapsto x$ and $D = (-2, -1)$.)

What if f is surjective? Now the first statement need not hold, but the second will. You should find examples here on your own.

Problem 4. *Let A, B be subsets of a universal set U. Simplify the following expressions. You can draw Venn diagrams to help you. (i) $(A \cap B) \cup (U \setminus A)$ and (ii) $A \cup [B \cap (U \setminus A)]$.*

(i) $(A \cap B) \cup (U \setminus A) = B \cup (U \setminus A)$.

Proof: Since $A \cap B \subset B$, $(A \cap B) \cup (U \setminus A) = B \cup (U \setminus A)$.

Now suppose $x \in B \cup (U \setminus A)$. If $x \in U \setminus A$ then $x \in (A \cap B) \cup (U \setminus A)$, as required. So we need to consider the case when $x \notin U \setminus A$. This means that $x \in A$. Since $x \in B \cup (U \setminus A)$, in this case x must be in B. Hence $x \in A \cap B$. So x does lie in $(A \cap B) \cup (U \setminus A)$.

(ii) $A \cup [B \cap (U \setminus A)] = A \cup B$. Here again it is obvious that LHS $\subseteq$ RHS (where LHS = left hand side means the set $A \cup [B \cap (U \setminus A)]$ and RHS = right hand side means $A \cup B$. To show RHS $\subseteq$ LHS we only need to consider the case $x \notin A$ (since if $x \in A$ it is obvious.) But if $x \notin A$ and $x \in B$ then $x \in U \setminus A$ and $x \in B$, i.e. $x \in B \cap (U \setminus A) \subseteq$ LHS.

Math 319/320 Worksheet 3

Name: **ID:**

Problem 1. (i) Construct a function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective.

(ii) Show that if S is an infinite set then there is a function $f : S \to S$ that is injective but not surjective. You may use the fact that S is infinite if and only if there is an injection $\mathbb{N} \to S$.

Problem 2. (i) Consider the set $\mathbb{N} \times \mathbb{N}$ of all ordered pairs (p, q) of integers. Show that it is denumerable, i.e. show how to construct a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

(ii) Show that if A and B are denumerable, disjoint sets, then $A \cup B$ is denumerable.

Problem 3. Let a, b, c, d be real numbers which satisfy $0 < a < b < c < d$.
a) Is it true that $bc < ad$? If it is true prove the inequality. If it is not true give an example of four real numbers which violate the inequality.

b) Is it true that $ca < bd$? If it is true prove the inequality. If it is not true give an example of four real numbers which violate the inequality.

c) Assume that $0 < c^2 < c$ for some real number c. Show that $0 < c < 1$.

Problem 4. a) Show that if x and y are rational numbers then $x+y$ and xy are rational numbers.

b) If $x \neq 0$ is rational and y irrational, show that xy is irrational.

c) If x and y are irrational, is it always true that $x+y$ is irrational? Explain.

Math 319/320 Worksheet 3

Problem 1. (i) *Construct a function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective.*

Define $f(k) = 2k$.

(ii) *Show that if S is an infinite set then there is a function $f : S \to S$ that is injective but not surjective. You may use the fact that S is infinite if and only if there is an injection $\mathbb{N} \to S$.*

Since S is infinite there is an injection $g : \mathbb{N} \to S$. Let $A = g(\mathbb{N})$ be its image. Then $S = A \cup (S \setminus A)$. Then $g : \mathbb{N} \to A$ is bijective and so has an inverse $g^{-1} : A \to \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be as in (i). Then the composite $g \circ f \circ g^{-1} : A \to A$ is well defined, and injective (since f is) but not surjective.

Now define $h : S \to S$ as follows.

If $x \notin A$ define $h(x) = x$.

If $x \in A = f(\mathbb{N})$ define $h(x) = g \circ f \circ g^{-1}(x)$.

Problem 2. (i) *Consider the set $\mathbb{N} \times \mathbb{N}$ of all ordered pairs (p, q) of integers. Show that it is denumerable, i.e. show how to construct a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$.*

We shall obtain a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by constructing an enumeration of the ordered pairs (p, q). To do this, use the diagonal process; ie think of the ordered pairs (p, q) as points in the plane, and then count along the successive diagonals $p + q = k$ for $k = 2, 3, 4, \ldots$. If we count from left to right along these diagonals then the enumeration starts as

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \ldots.$$

(ii) *Show that if A and B are denumerable, disjoint sets, then $A \cup B$ is denumerable.*

By assumption there are bijections $f : \mathbb{N} \to A, g : \mathbb{N} \to B$. Define $h : \mathbb{N} \to A \cup B$ by

$$h(2k - 1) = f(k), k \geq 1, \quad h(2k) = g(k), k \geq 1.$$

Then h is obviously surjective. It is injective because A and B are disjoint. More formally, suppose that $h(x) = h(y)$ but $x \neq y$. Then x, y cannot both be odd since f is injective and they cannot both be even since g is injective. But if one (say x) is odd and the other is even, then $h(x) \in A$ equals $h(y) \in B$. Hence $A \cap B$ is nonempty, contrary to hypothesis.

Problem 3. *Let a, b, c, d be real numbers which satisfy $0 < a < b < c < d$.*

a) Is it true that $bc < ad$? If it is true prove the inequality. If it is not true give an example of four real numbers which violate the inequality.

FALSE: Take $b = 1, c = 2, d = 3$ and $a = 1/4$.

b) *Is it true that $ca < bd$? If it is true prove the inequality. If it is not true give an example of four real numbers which violate the inequality.*

FALSE: take $a = 1, b = 2, c = 3$ and $d = 100$.

c) *Assume that $0 < c^2 < c$ for some real number c. Show that $0 < c < 1$.*

If $c = 1$ then $c = c^2$. So the inequality $c^2 < c$ is not satisfied. If $c > 1$ then $c = 1+a$ where $a = c-1 > 0$. Hence $c^2 = (1+a)^2 = 1+2a+a^2 > 1+a = c$. So the inequality $c^2 < c$ is also not satisfied. Hence, by the trichotomy rule, the only possibility left is that $c < 1$. Since $c > 0$ by assumption and the transitivity rule ($c > c^2$ and $c^2 > 0$ implies $c > 0$), we find $0 < c < 1$.

Problem 4. a) *Show that if x and y are rational numbers then $x + y$ and xy are rational numbers.*

To say x, y are rational means that there are integers p, q, r, s (where $q \neq 0, s \neq 0$) such that $x = p/q, y = r/s$. Then $x + y = (ps + qr)/qs$ and $xy = pr/qs$ and retional.

b) *If $x \neq 0$ is rational and y irrational, show that xy is irrational.*

Argue by contradiction. By assumption $x = p/q$ where $p \neq 0$. Therefore $x^{-1} = 1/x = q/p$ is also a rational number. If xy were rational then $xy(x^{-1}) = y$ would also be rational by (a). But y is irrational by assumption. Therefore xy is irrational.

c) *If x and y are irrational, is it always true that $x + y$ is irrational? Explain.*

NO: Suppose that x is irrational and let $y = 1 - x$. Then y is not rational, since if it were $-y$ would be rational and hence $-y + 1 = x$ would be rational. On the other hand, $x + y = 1$ is rational. (ie the irrationalities of x, y can cancel.)

Math 319/320 Worksheet 4

Name: **ID:**

Problem 1. Find all $x \in \mathbb{R}$ that satisfy the following inequality:
(a) $|x| + |x+1| < 2$

(b) $4 < |x+2| + |x-1| < 5$

Problem 2. Assume that A and B are bounded subsets of $\mathbb{R}$.
(a) Show that $A \cup B$ is a bounded subset of $\mathbb{R}$.

(b) Show that $\sup(A \cup B) = \sup\{\sup\ A, \sup\ B\}$.

Problem 3. Show that nonempty finite subset S always has a supremum and that it contains it. (Hint: Use Mathematical Induction.)

Problem 4. Let S be a subset of $\mathbb{R}$ that is bounded below. Show that $\inf S = -\sup\{-s | s \in S\}$.

Math 319/320 Worksheet 4 solutions

Problem 1. *Find all $x \in \mathbb{R}$ that satisfy the following inequality:*
(a) $|x| + |x+1| < 2$

Method 1: draw a graph.

Method 2: Divide into cases, depending on the signs of x and $x+1$.

Suppose $x \geq 0$. Then we have $2x + 1 < 2$, i.e. $x < 1/2$.

Suppose $-1 \leq x \leq 0$. Then we have $-x + x + 1 < 2$ – always true.

Suppose $x < -1$. Then we have $-x - x - 1 < 2$, i.e. $x > -3/2$.

So answer: $x \in (-3/2, 1/2)$.

(b) $4 < |x+2| + |x-1| < 5$
Case 1: $x > 1$. Then we have $4 < x + 2 + x - 1 = 2x + 1 < 5$. That is, $3 < 2x < 4$ or $x \in (3/2, 2)$.
Case 2: $-2 \leq x \leq 1$. Then we have $4 < x + 2 + 1 - x = 3 < 5$. This is impossible.
Case 3: $x < -2$. Then we have $4 < -x - 2 - x + 1 = -2x - 1 < 5$ or $5 < -2x < 6$ i.e. $-3 < x < -5/2$. So $x \in (-3, -5/2)$.

Thus the answer: $x \in (-3, -5/2) \cup (3/2, 2)$.

Problem 2. *Assume that A and B are bounded subsets of $\mathbb{R}$.*
(a) *Show that $A \cup B$ is a bounded subset of $\mathbb{R}$.*

Since A is bounded there is $K \in \mathbb{R}$ such that $|x| \leq K$ for all $x \in A$. Similarly there is $M \in \mathbb{R}$ such that $|y| \leq M$ for all $y \in B$. Therefore if $z \in A \cup B$ we have

$$|z| \leq K \leq \max(K, M), \text{ if } z \in A, \quad \text{and } |z| \leq M \leq \max(K, M), \text{ if } z \in B.$$

Thus $\max(K, M)$ is a bound for $A \cup B$.

(b) *Show that* $\sup(A \cup B) = \sup\{\sup\ A, \sup\ B\}$.

Let $K := \sup A$ and $M := \sup B$. The argument in (a) shows that $\max(K, M)$ (which is precisely the same thing as $\sup\{\sup\ A, \sup\ B\}$) is an upper bound for $A \cup B$.

To see that $\max(K, M)$ is the supremum of $A \cup B$ it suffices to show that no number p strictly less than $\max(K, M)$ is an upper bound for $A \cup B$.

Without loss of generality we may assume that $K \leq M$. (If not, we can simply rename the sets A, B to make this true.) Then $\max(K, M) = M$. Hence if $p < \max(K, M)$, $p < M$ and so p is NOT an upper bound for B. Thus there is $b \in B$ such that $b > p$. Since $b \in A \cup B$ this means that p is not an upper bound for $A \cup B$. Thus $\max(K, M)$ is the supremum of $A \cup B$.

Problem 3. *Show that nonempty finite subset S always has a supremum and that it contains it. (Hint: Use Mathematical Induction.)*

We will prove this by induction of $n := |S|$, the number of elements in S.

Base case: $n = 1$. Then $S = \{a\}$. Hence $a = \sup S$. Also $a \in S$. Thus the statement holds.

Now suppose that $|S| = k + 1$ and that the statement holds for all sets with k elements. Choose any $x \in S$ and let $T := S \setminus \{x\}$. Then $|T| = k$. Therefore, by the inductive hypothesis, $\sup T$ exists and equals some element $y \in T$.

Case (i): $y \geq x$. Then y is an upper bound for S. And no number smaller than y can be an upper bound for S, since it would also have to be an upper bound for $T \subset S$ and $y = \sup T$. Hence $y = \sup S$. Since $y \in T \subset S$, the statement holds for S.

Case (ii): $y < x$. In this case, x is an upper bound for S since by transitivity of order it is $>$ all the elements in $S \setminus \{x\}$. Any other upper bound u for S must satisfy $u \geq x$ (since $x \in S$.) Hence $x = \sup S$. Thus the statement holds in this case too.

Thus the inductive step is proven. Hence the statement holds by Mathematical Induction.

Problem 4. *Let S be a subset of $\mathbb{R}$ that is bounded below. Show that* $\inf S = -\sup\{-s | s \in S\}$.

Define $T := -S := \{-s : s \in S\}$.

Note that

$$(*) \qquad x \leq s,\ \forall s \in S \iff -x \geq -s := t,\ \forall t \in T.$$

Thus x is a lower bound for S iff $-x$ is an upper bound for T.

Suppose now that $x := \inf S$. Then, $-x$ is an upper bound for T and we just need to see that it is the least upper bound. But if not, there is an upper bound u for T such that $u < -x$. Then $-u > x$ and, by (*), $-u$ is a lower bound for S. But this is impossible; because $x := \inf S$, x must be $\geq$ every lower bound for S, in particular we must have $x \geq -u$.

Math 319/320 Homework 1

Due Thursday, September 8, 2005

Problem 1. Prove the identity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 2. Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ given by $f(x) = x^2/(x-1)$.

(i) Graph it.

(ii) Find $f(A)$ where A is the interval $(1.5, 4)$.

(iii) Find $f^{-1}(B)$ where $B = [1, 4]$.

(iv) Find two subsets C, D of $\mathbb{R} \setminus \{1\}$ such that $f(C) \cap f(D) \neq f(C \cap D)$.

Problem 3. Let $f : A \to B$ be a function and suppose that $C \subseteq A$ and $D \subseteq B$. Are the following statements true or false (for every choice of f, C, D)? Justify your answers by a brief proof or a counterexample.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

Hint: as in question 2, try some examples. You can try functions $f : \mathbb{R} \to \mathbb{R}$ or you can try functions $f : A \to B$ where A and B are finite sets.

Problem 4. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composite $g \circ f$ is surjective. Is g necessarily surjective? What about f? Give brief proofs or counterexamples.

Problem 5. Prove by mathematical induction: $3^{2n} - 1$ is divisible by 8 for all $n \geq 1$.

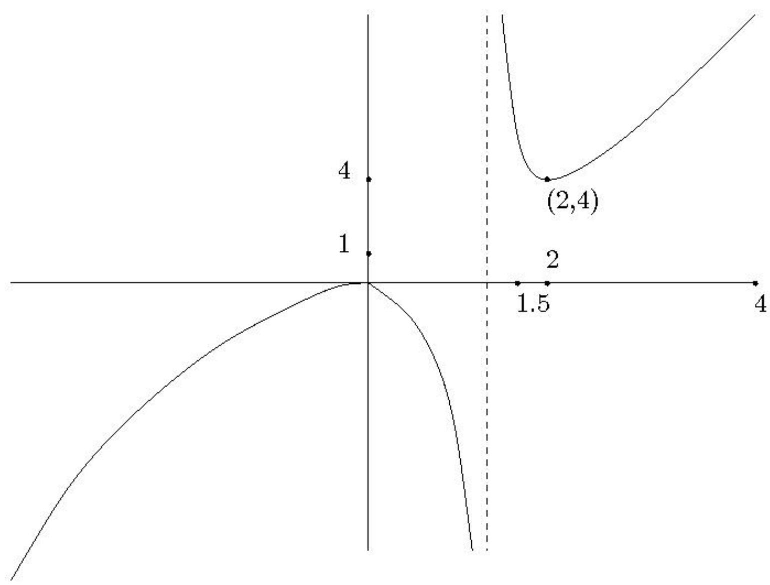
Math 319/320 Solutions to Homework 1

Problem 1. *Prove the identity:* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

1. To show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:
let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $A \cup C$, Hence $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$ then x is in both B and C and hence in both $(A \cup B)$ and $(A \cup C)$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.

2. To show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:
let $x \in (A \cup B) \cap (A \cup C)$. Then x is in $A \cup B$ and in $A \cup C$. In particular, x is in either A or B. If $x \in A$ then $x \in A \cup (B \cap C)$, as required. If $x \notin A$ then $x \in B$. Since $x \in A \cup C$ we must also have $x \in C$. Hence $x \in B \cap C$. Hence again $x \in A \cup (B \cap C)$.

Problem 2. *Consider the function* $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ *given by* $f(x) = x^2/(x-1)$.
(i) Graph it.



(ii) *Find* $f(A)$ *where* A *is the interval* $(1.5, 4)$.
f has a local minimum at 2, and $f(2) = 4$. So answer is $[4, 16/3)$.
(iii) *Find* $f^{-1}(B)$ *where* $B = [1, 4]$. This is the single point 2.
(iv) *Find two subsets* C, D *of* $\mathbb{R} \setminus \{1\}$ *such that* $f(C) \cap f(D) \neq f(C \cap D)$. You could take C, D to be any two different points with the same image. For ex., $f(3) = 9/2$. There is a point $a \in (1, 2)$ that also satisfies the equation $a^2/(a-1) = 9/2$. So take $C = \{a\}, D = \{3\}$.

Problem 3. *Let $f : A \to B$ be a function and suppose that $C \subseteq A$ and $D \subseteq B$. Are the following statements true or false (for every choice of f, C, D)? Justify your answers by a brief proof or a counterexample.*

General comments: As we saw in class the inverse image seems to behave better than the direct image in matters of this sort. So you should be suspicious of (i) – it is more likely that (ii) holds.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

This is FALSE (since f need not be injective). eg take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Let $A = \{-2\}, C = \{2\}$. Then $A \setminus C = \{-2\}$ and so $f(A \setminus C) = \{4\}$. But $f(A) = f(C)$ so $f(A) \setminus f(C) = \emptyset$.

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

This is TRUE. $x \in f^{-1}(B \setminus D)$ iff $f(x) \in B \setminus D$ iff $f(x) \in B$ and $f(x) \notin D$ iff $x \in f^{-1}(B)$ and $x \notin f^{-1}(D)$, that is $x \in f^{-1}(B) \setminus f^{-1}(D)$.

NOTE: here iff = if and only if (a useful shorthand)

Problem 4. *Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composite $g \circ f$ is surjective. Is g necessarily surjective? What about f? Give brief proofs or counterexamples.*

g must be surjective. Proof: since $g \circ f$ is surjective, for every $c \in C$ there is $a \in A$ such that $g \circ f(a) = c$. But $g \circ f(a) = g(f(a))$. Hence there is an element $b \in B$ such that $g(b) = c$, namely $b = f(a)$.

But f need not be surjective because g need not be injective. eg if $f : [0, \infty) \to \mathbb{R}$ is $f(x) = x$, f is not surjective. Define $g : \mathbb{R} \to [0, \infty)$ by $g(y) = y^2$. Then $g \circ f : [0, \infty) \to [0, \infty)$ is surjective.

Problem 5. Prove by mathematical induction: $3^{2n} - 1$ is divisible by 8 for all $n \geq 1$.

The statement $P(n)$ is: 8 divides the integer $3^{2n} - 1$.

Base case: if $n = 1$ then $P(1)$ says that 8 divides $3^2 - 1 = 8$, which is true.

Inductive step: suppose that 8 divides $3^{2k} - 1$. We must show that 8 divides $3^{2k+2} - 1$. But

$$3^{2k+2} - 1 = 3^2 \times 3^{2k} - 1 = 9(3^{2k} - 1) + 9 - 1 = 9(3^{2k} - 1) + 8.$$

Since 8 divides $3^{2k} - 1$ by the inductive hypothesis, it divides $9(3^{2k} - 1)$. It also divides 8. Hence it divides $9(3^{2k} - 1) + 8 = 3^{2k+2} - 1$, as required.

Math 319/320 Homework 2

Due Thursday, September 15, 2005

Problem 1. Show that the set S_{odd} of odd (positive and negative) integers is denumberable by

(a) enumerating them and

(b) giving an explicit formula for the corresponding bijection $f : S_{odd} \to \mathbb{N}$.

Hint: Imitate the following example. An **enumeration** of the set $\mathbb{Z}$ of integers is given by $\{0, 1, -1, 2, -2, 3 \dots\}$. The corresponding explicit bijection $f : \mathbb{Z} \to \mathbb{N}$ is

$$f(0) = 1, \qquad f(k) = 2k, \text{ if } k > 0, \qquad f(k) = -2k + 1, \text{ if } k < 0.$$

Problem 2. Show that for all $n \geq 1$ there is no injection of $\mathbb{N}_n$ onto a *proper* subset of $\mathbb{N}_n$. In other words, any injection $f : \mathbb{N}_n \to \mathbb{N}_n$ is also surjective (and thus bijective.)
(ii) Deduce from (i) that if S is any finite set, there is no injection of S onto a *proper* subset of S.

(iii) Show that (ii) does not hold for the infinite set $S = \mathbb{N}$.

Hint for (i): You can prove this by induction. Or, you can deduce it from Thm B.1 in the textbook (which I did in class) by supposing that there is an injective but nonsurjective map $f : \mathbb{N}_n \to \mathbb{N}_n$ and looking at the composite $g \circ f$ for a suitable map $g : \mathbb{N}_n \to \mathbb{N}_{n-1}$. (Actually, these different approaches boil down to basically the same argument.)

Problem 3. Given a set S we write $\mathcal{P}(S)$ for the set of all its subsets. Note that $\emptyset, S$ are both subsets of S.
(i) List all the subsets of $S = \{1, 2\}$.
(ii) List all the subsets of $S = \{1, 2, 3\}$. (Try to see a relation with (i) – this should give you an idea for the inductive step below.)
(iii) Prove that, for all $n \geq 0$, if a finite set S has n elements then $\mathcal{P}(S)$ has 2^n elements. **Hint:** Use induction. You should have checked this above for $n = 2, 3$, but you need to do the base case $n = 0$ as well as the inductive step.

Problem 4. Prove that $\sqrt{3}$ is irrational.

Problem 5. Let $a, b \in \mathbb{R}$. Show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$.
Hint: Use order properties in the proof.

Bonus Problem 6. Show that if S is a subset of $\mathbb{N}$ that is not contained in any of the sets $\mathbb{N}_n$ then S is denumerable.

Math 319/320 Solutions to Homework 2

Problem 1. *Show that the set S_{odd} of odd (positive and negative) integers is denumberable by (a) enumerating them and (b) giving an explicit formula for the corresponding bijection $f : S_{odd} \to \mathbb{N}$.*

An emumeration: $S_{odd} = \{1, -1, 3, -3, 5, -5, \dots\}$

A bijection $f : S_{odd} \to \mathbb{N}$:

$$f(2k+1) = 2k+1, \ \ k \geq 0, \qquad f(-(2k+1)) = 2k+2, \ \ k \geq 0.$$

Problem 2. (i) *Show that for all $n \geq 1$ there is no injection of $\mathbb{N}_n$ onto a proper subset of $\mathbb{N}_n$.*

Let's prove this by induction. When $n = 1$ $\mathbb{N}_n = \{1\}$ has one element and there is only one map $f : \mathbb{N}_n \to \mathbb{N}_n$, namely the identity (which is bijective.)

Suppose this is true when $n = k$ and consider the case $n = k+1$. Let $f : \mathbb{N}_{k+1} \to \mathbb{N}_{k+1}$ be injective but not surjective. We show below that we can always use f to construct a new map $g : \mathbb{N}_k \to \mathbb{N}_k$ which is injective but not surjective. This contradicts the inductive hypothesis. Hence f cannot exist.

To define g;

case (i): $k+1 \notin f(\mathbb{N}_{k+1})$. Then $f(j) < k+1$ for all $j \leq k+1$. Define $g : \mathbb{N}_k \to \mathbb{N}_k$ by setting $g(j) = f(j), j \in \mathbb{N}_k$. This is injective because f is. I claim that the element $p := f(k+1)$ is not in $g(\mathbb{N}_k)$. To see this, note that because f is injective, there is only one element of $\mathbb{N}_{k+1}$, namely $k+1$ itself, that is mapped to p by f. Therefore $g^{-1}(p)$ is the empty set.

case (ii): suppose that $k+1 = f(j)$ for some $j \leq k+1$. Then because f is not surjective there is $p \leq k$ that is not in the image of f. Define a new map $F : \mathbb{N}_{k+1} \to \mathbb{N}_{k+1}$ by setting:

$$F(i) = f(i), i \in \mathbb{N}_{k+1}, i \neq j, \qquad F(j) = p.$$

Then F is injective but not surjective because $k+1 \notin F(\mathbb{N}_{k+1})$. Now construct g from F as in case (i).

(ii)*Deduce from (i) that if S is any finite set, there is no injection of S onto a proper subset of S.*

Suppose that $f : S \to S$ is an injection. Since S is finite there is a positive integer n and a bijection $g : \mathbb{N}_n \to S$. Then the composite $g^{-1} \circ f \circ g : \mathbb{N}_n \to \mathbb{N}_n$ is injective, since g, f and g^{-1} are. Therefore $g^{-1} \circ f \circ g$ is surjective by part (i). Therefore f is surjective.

(iii)*Show that (ii) does not hold for the infinite set $S = \mathbb{N}$.*

Define $f : \mathbb{N} \to \mathbb{N}$ by $f(k) = k+1)$ for all k.

Problem 3. (i) The subsets of $\{1,2\}$ are

$$\emptyset, \ \ \{1\}, \ \ \{2\}, \ \ \{1,2\}.$$

The subsets of $\{1,2,3\}$ are

$$\begin{array}{cccc} \emptyset, & \{1\}, & \{2\}, & \{1,2\}, \\ \{3\}, & \{1,3\}, & \{2,3\}, & \{1,2,3\}. \end{array}$$

(iii) *Prove that if a set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.*

If S has one element then it has two subsets, the empty set and the set S. Thus $\mathcal{P}(S)$ has 2 elements.

Now suppose the statement is true for $n = k$ and consider a set S with $k + 1$ elements. Suppse $x \in S$ and let $S_0 = S \setminus \{x\}$. Then S_0 has k elements. Every subset of S either lies in S_0 or contains x. By the inductive hypothesis there are 2^k subsets of S_0. Every subset T of S that contains x corresponds to a unique subset T_0 of S_0, namely $T_0 = T \setminus \{x\}$. Moreover, given a subset T_0 of S_0 the subset $T = S_0 \cup \{x\}$is a subset of S containing $\{x\}$. Therefore there is a bijective correspondence between the subsets of S_0 and the subsets of S that contain x. Hence there are 2^k subsets of this kind. So altogether $\mathcal{P}(s)$ has $2^k + 2^k = 2^{k+1}$ elements. Hence the statement holds for $n = k + 1$, and so holds for all n by induction.

Problem 4.*Prove that $\sqrt{3}$ is irrational.*

Suppose that $\sqrt{3}$ is rational are write it as p/q where p, q are positive integers with no common divisor. Then $3 = p^2/q^2$, that is $3q^2 = p^2$. Therefore 3 divides p^2. We claim that 3 must divide p. If not, we may write $p = 3k + r$ for some integer k and where $r = 1$ or 2. Then $p^2 = (3k+r)^2 = 9k^2+6k+r^2$. Since 3 divides p^2, 3 must divide $r^2 = p^2 - 3(3k^2+2k)$. But r^2 is either 1 or 4. So this is impossible. Hence 3 divides p. Therefore $p = 3k$ and the equation $3q^2 = p^2 = 9k^2$ gives $q^2 = 3k^2$. Therefore 3 divides q^2. Arguing as above, it follows that 3 divides q. But this contradicts our asumption that p and q have no common divisor. Hence the original assumption was wrong: $\sqrt{3}$ cannot be rational.

Problem 5. *Show that $a^2 + b^2 = 0$ iff $a = 0$ and $b = 0$.*

If $a = 0 = b$ then $a^2 = 0 = b^2$, so $a^2 + b^2 = 0$.

Therefore we need to show the converse: if $a^2 + b^2 = 0$ then $a = 0 = b$. We will do this by proving the contrapositive. That is, if at least one of a, b is nonzero then $a^2 + b^2 \neq 0$. (in fact in this case we will see that $a^2 + b^2 > 0$.)

To do this, suppose first that $a \neq 0$. Then either $a > 0$ or $-a > 0$ by the trichotomy rule (Def 2.1.5(iii)). If $a > 0$ then $a^2 > 0$ by Def 2.1.5(ii). Similarly, if $-a > 0$ then $(-a)^2 > 0$. Since $(-a)^2 = a^2$, we find that for any $a \neq 0$ $a^2 > 0$. Also, $a^2 \geq 0$ for all $a \in \mathbb{R}$.

Now suppose that at least one of a, b is nonzero. By renaming them if necessary, we may suppose that $a \neq 0$. Then $a^2 > 0$ and $b^2 \geq 0$. Hence $a^2 + b^2 > 0$. (This follows from Def 2.1.5 (i) if $b^2 > 0$ and is obvious if $b^2 = 0$ since in this case $a^2 + b^2 = a^2 > 0$ by hypothesis.)

This completes the proof.

Bonus Problem 6. *Show that if S is a subset of $\mathbb{N}$ that is not contained in any of the sets $\mathbb{N}_n$ then S is denumerable.*

We will define a bijection $\mathbb{N} \to S$ using the principle of induction. The statement $P(n)$ is: there is an injection $f : \mathbb{N}_n \to S$ so that

i) $f(n) \geq n$, and

ii) if $s \in S$ is not in the image $f(\mathbb{N}_n)$ then $s > f(j)$ for all $j \in \mathbb{N}_n$.

By the well ordering principle S has a smallest element, say s_1. Define $f(1) = s_1$. This map satisfies $P(1)$. Suppose by induction that f is defined on $\mathbb{N}_k$ and satisfies (i), (ii) above. Set $f(k+1)$ equal to the smallest element in the set $S \setminus f(\mathbb{N}_k)$. Then $f(k+1)$ is larger that all the elements $f(j), j \leq k$ by (ii). Hence $f(k+1) > f(k) \geq k$ by (i). Hence $f(k+1) \geq k+1$. So (i) holds for f on $\mathbb{N}_{k+1}$. Also because $f(k+1)$ is the smallest element in $S \setminus f(\mathbb{N}_k)$, every element in $S \setminus f(\mathbb{N}_{k+1})$ is larger than every element in $f(\mathbb{N}_{k+1})$. So (ii) holds for f on $\mathbb{N}_{k+1}$.

Therefore we may define $f : \mathbb{N}_n \to S$ satisfying (i) and (ii) for all n.

It follows from (ii) and the inductive construction that

$$f(1) < f(2) < f(3) < \dots$$

ie if $i < j$ then $f(i) < f(j)$. Hence f is injective. Suppose that f is not surjective and let $s \in S \subset \mathbb{N}$ be not in its image. Consider properties (i) and (ii) in the case $k = s$. By hypothesis s is not in the image $f(\mathbb{N}_s)$. Therefore by (ii) s is strictly larger than all elements in the image $f(\mathbb{N}_s)$. Hence $s > f(s)$. But this contradicts (i).

Math 319/320 Homework 3

Due Thursday, September 22, 2005
revised version

Problem 1. Show that
$$\left[\frac{1}{2}(a+b)\right]^2 \leq \frac{1}{2}(a^2+b^2)$$
for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

Problem 2. Assume that $a < x < b$ and $a < y < b$. Show that $|x - y| \leq b - a$. Find a geometric explanation for the obtained inequality.

Problem 3. Let $a, b \in \mathbb{R}$ and $a \neq b$. Show that there exist ϵ-neighborhood $U_\epsilon(a)$ of a and ϵ-neighborhood $V_\epsilon(b)$ of b such that $U_\epsilon(a) \cap V_\epsilon(b) \neq \emptyset$.

Problem 4. Let $S := \{x \in \mathbb{R} | x \geq 0\}$. Show that S has lower bounds, but no upper bounds. Show that $\inf\ S = 0$.

Problem 5. If $S \subset \mathbb{R}$ contains one of its upper bounds, then this upper bound is the supremum of S.

Math 319/320 Homework 3

Due Thursday, September 22, 2005
revised version

Problem 1. *Show that*

$$\left[\frac{1}{2}(a+b)\right]^2 \leq \frac{1}{2}(a^2+b^2)$$

for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.

We must show that $(a^2+2ab+b^2)/4 \leq (a^2+b^2)/2$. Multiplying both sides by 4, we see this is equivalent to $a^2+2ab+b^2 \leq 2a^2+2b^2$ and hence to $0 \leq a^2+b^2-2ab$. But this last inequality holds since $a^2+b^2-2ab = (a-b)^2$ and $x^2 \geq 0$ for all x. (We proved last on the last HW.) This shows that the inequality holds.

Also it shows that we have equality if and only if $0 = a^2+b^2-2ab$, i.e. $(a-b)^2 = 0$. But this holds iff $a = b$. (Again this was proved in last HW.)

Problem 2. *Assume that $a < x < b$ and $a < y < b$. Show that $|x-y| \leq b-a$. Find a geometric explanation for the obtained inequality.*

First proof If $x = y$ the inequality is obvious since $b-a > 0$ by hypothesis. Now assume that $x < y$. We know that $x < b$. Also $a < y$ implies $-y < -a$. Adding we get

$$|x-y| = x-y = x+(-y) < b+(-a) = b-a.$$

Similarly, if $y < x$ we may reverse the roles of x and y to find: $|x-y| = y-x < b-a$. Hence in all cases $|x-y| \leq b-a$. (in fact we have $<$ here.)

Second proof (which I learnt as I was correcting your HW; it's essentially the same but slicker.)

By hypothesis $a < x < b$ and $a < y < b$. Multiply the second inequality by -1 to get $-b < -y < -a$. Then add this to the first inequality to get $a-b < x-y < b-a$. This has the form $-C < Z < C$ where $Z = x-y$ and $C = b-a > 0$. Hence it is equivalent to $|Z| < C$, i.e. $|x-y| < b-a$.

$|x-y|$ is the distance between x and y. So geometrically we are saying that the distance between any two points in the interval (a, b) is at most $b-a$.

Problem 3. *Let $a, b \in \mathbb{R}$ and $a \neq b$. Show that there exist ϵ-neighborhood $U_\epsilon(a)$ of a and ϵ-neighborhood $V_\epsilon(b)$ of b such that $U_\epsilon(a) \cap V_\epsilon(b) \neq \emptyset$.*

The problem here is to show you can choose ϵ large enough that these sets do intersect. So you must specify ϵ.

Proof 1 By renaming a, b we may suppose that $a < b$. Choose $\epsilon = 2(b-a)$, twice the distance between a and b. Then $b \in U_\epsilon(a) = (a-\epsilon, a+\epsilon)$. This is geometrically obvious, since $U_\epsilon(a)$ contains all points whose distance from a is $< \epsilon$ and b has distance $b - a < 2(b-a) = \epsilon$ from a. Since $b \in U_\epsilon(b)$ for any ϵ, b is in the intersection.

But to show it in formulas, note that

$$U_\epsilon(a) = (a - 2b + 2a, a + 2b - 2a) = (3a - 2b, 2b - a).$$

We need to see that $3a - 2b < b < 2b - a$. But $3a < 3b$ implies $3a - 2b < b$; while $b < 2b - a = b + (b - a)$ since $b - a > 0$.

As some of you noticed, any $\epsilon > |b-a|/2$ will do, since then the average $(a+b)/2$ will lie in the intersection. Here is a nice argument to show this:

Proof 2: As above, we may suppose that $a < b$. Note that $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$ and $V_\epsilon(b) = (b - \epsilon, b + \epsilon)$. Since $a < b$, $a - \epsilon < b_\epsilon$ and the only way to have an overlap of these intervals is for $a + \epsilon > b - \epsilon$. (You draw this.) i.e. we need $2\epsilon > b - a$ or $\epsilon > (b-a)/2$. If ϵ satisfies this inequality then the average $(a+b)/2$ is in both intervals.

Problem 4. *Let* $S := \{x \in \mathbb{R} : x \geq 0\}$. *Show that* S *has lower bounds, but no upper bounds. Show that* $\inf\ S = 0$.

Clearly 0 is a lower bound for S. Moreover if $y > 0$ then y is not a lower bound for S since y is *not* $\geq$ the element $0 \in S$. Hence every lower bound for S is ≤ 0. Hence 0 is the greatest lower bound, i.e. $0 = \inf S$.

To show S has no upper bounds: Proof 1

Since every $n \in \mathbb{N}$ is positive and so > 0, then $\mathbb{N} \subset S$. If y were an upper bound for S, we would have $y \geq n$ for all $n \in \mathbb{N}$, in contradiction to Archimedes' Principle. Hence S has no upper bounds.

Proof 2: Suppose that u is an upper bound for S. Then $u \in \mathbb{R}$ and $u \geq 0$, since $0 \in S$. Also $u + 1 > u$ (this is true for all real numbers.) Hence $u + 1 \in \mathbb{R}$ and $u + 1 > u \geq 0$. Hence $u + 1 \in S$. Hence $u \geq u + 1$. But this is impossible, by the trichotomy rule. (we cannot have both $u + 1 > u$ and $u \geq u + 1$.) Hence there is no upper bound.

Some of you combined the two arguments above, but it is simpler (and hence better) to use one OR the other.

Problem 5. *If* $S \subset \mathbb{R}$ *contains one of its upper bounds, then this upper bound is the supremum of* S.

Let y be an upper bound for S and suppose that $y \in S$. We must show that no $z < y$ is an upper bound for S. But if z is an upper bound for S, then $z \geq y$ since $y \in S$ and z is $\geq$ *every* element in S. Therefore z cannot also be $< y$. Hence y is the least upper bound for S, i.e. it is the supremum of S.

Note: this argument is almost the same as the proof in ex. 4 that $0 = \inf S$.

Math 319/320 Homework 4

Due Tuesday, September 27, 2005

Problem 1. Let $I_n := [0, \frac{1}{n}]$ and $J_n := (0, \frac{1}{n}]$, for $n \in \mathbb{N}$. Show that
a) $\cap_{n=1}^{\infty} I_n = \{0\}$
b) $\cap_{n=1}^{\infty} J_n = \emptyset$

Problem 2. Let A and B be bounded nonempty subsets of $\mathbb{R}$. Define $A + B := \{a + b : \ a \in A, b \in B\}$. Show that $\sup(A + B) = \sup A + \sup B$.

Math 319/320 Solutions to Homework 4

Problem 1. *Let* $I_n := [0, \frac{1}{n}]$ *and* $J_n := (0, \frac{1}{n}]$, *for* $n \in \mathbb{N}$. *Show that* $\cap_{n=1}^{\infty} I_n = \{0\}$.

Since $0 \in I_n$ for all n, $0 \in \cap_{n=1}^{\infty} I_n$. If $x \in I_n$ for all n then $0 \le x \le 1/n$ for all n. But if $x > 0$ then $1/x > 0$ and there is by Archimedes' principle an integer k such that $k > 1/x$. This means $x > 1/k$. Hence $x \notin I_k$. Hence $x \notin \cap_{n=1}^{\infty} I_n$. Thus the only point in this intersection is 0.

Show $\cap_{n=1}^{\infty} J_n = \emptyset$. Since $J_n \subset I_n$ for all n, $\cap_{n=1}^{\infty} J_n \subset \cap_{n=1}^{\infty} I_n = \{0\}$. But $0 \notin J_1$. Hence $0 \notin \cap_{n=1}^{\infty} J_n$. Thus this intersection is empty.

Problem 2. *Let* A *and* B *be bounded nonempty subsets of* $\mathbb{R}$. *Define* $A + B := \{a + b : \ a \in A, b \in B\}$. *Show that* $\sup(A + B) = \sup A + \sup B$.

Since A, B are bounded and nonempty these sets have suprema. Let $s = \sup A$ and $t = \sup B$. Then $s \ge a, \forall a \in A$ and $t \ge b, \forall b \in B$. Hence $s + t \ge a + b, \forall a \in A, b \in B$. Hence $s + t$ is an upper bound for $A + B$.

To see it is the least upper bound, it suffices to show that for all $\epsilon > 0$ there is $c \in A + B$ such that $c > s + t - \epsilon$. But because $s = \sup A$ there is $a \in A$ such that $a > s - \epsilon/2$. Similarly, there is $b \in B$ such that $b > t - \epsilon/2$. Then $c = a + b \in A + B$ and $c > s + t - \epsilon$.

Math 319/320 Review for Midterm I

The exam will cover everything in Chapters 1 and 2 up to the bottom of p 47, i.e. not binary representations and decimals. You are expected to know the basic definitions and how to apply them in straightforward examples and arguments. In detail:

• you should know all the definitions in Ch 1 and Ch 2. In the exam we will ask you to state one or two of these.

• we do not expect you to memorize the 9 properties of a field (2.1.1) or the derivation in 2.1 of all the basic properties of the algebraic operations and order on $\mathbb{R}$.

• you should be able to state the important Properties:
(1.2.1) Well-Ordering Property of $\mathbb{N}$
(2.3.6) Completeness property of $\mathbb{R}$
(2.4.4) Archimedean Property
(2.5.2) Nested Intervals Property

• you should know the statements of the most important theorems and have a good idea of their proofs.

We will not ask you to reproduce any of these proofs, but we will expect you to be able to make short arguments using the basic theorems and properties. There will be five short questions on the exam, each worth 10 points.
Note: We will try to phrase the questions in the exam so that it will be clear what you can assume and what you should prove. If you have any questions of this kind during the exam, please ask one of the proctors for clarification.

Here are some sample problems. Some of the questions are too long for the actual exam, and would be shortened.

1: Let $f : A \to B$ be a function and $C \subset A, D \subset B$.
(i) Show that $f\big(C \cap f^{-1}(D)\big) = f(C) \cap D$.

(ii) Give an example to show that $f\big(C \cup f^{-1}(D)\big)$ need not equal $f(C) \cup D$.

2. (i) Define what it means for a function $D : A \to B$ to be injective.
(ii) Let A be the set of all polynomials of the form $p(x) = ax^2 + bx + c$ and B be the set of all polynomials of the form $q(x) = rx + s$, where a, b, c, r, s are arbitrary real numbers. Define a function $D : A \to B$ by $D(p) = p'$, where p' is the derivative dp/dx. Is D injective? Is it surjective?

3. (i) Show that the following statement is false by giving a counterexample.

$$\forall x \in \mathbb{R}, \exists t \in \mathbb{R}, \quad \frac{t^2}{1-x} > 1.$$

(ii) Prove the following statement by contradiction.

$$\sqrt{2} + \sqrt{3} > \sqrt{5}.$$

Note: here we denote by $\sqrt{x}$ the positive square root of a positive real number x. You may use all the standard results on the order properties of the real numbers without proof.

(iii) Prove the following statement by induction: $3^n \geq 2^n + 3n$ for all $n \geq 3$.

4 Are the following statements true or false? Justify your answer with a brief proof or counterexample.

(i) The intersection $\bigcap_{n=1}^{\infty} [n, +\infty)$ is empty.

(ii) The Completeness Property holds in every ordered field.

(iii) If $f, g : [0,1] \to \mathbb{R}$ are bounded then

$$\sup\{f(x) + g(x) : x \in [0,1]\} = \sup\{f(x) : x \in [0,1]\} + \sup\{g(x) : x \in [0,1]\}.$$

5 (i) What does it mean for a set to be countable?

(ii) Show from the definition that the set $\mathbb{Q}$ of rational numbers is countable.

(iii) Show from the definition that if S and T are countable then so is $S \cup T$. First consider the case when S and T are disjoint, and then the general case.

7 State the Nested Intervals Property. Give an example to show that it does not hold for intervals in $\mathbb{Q}$.

8 (i) Let S be a subset of $\mathbb{R}$. What is an upper bound of S? What is the supremum of S?

(ii) Let $S \subset \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound for S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.

Math 319 Homework 5

Due Friday, October 14, 2006

The homework is simply to rewrite the Midterm, making the solutions as perfect as possible. Please write in your own words. I will post the exam questions over the weekend in case any of you need them, but will NOT post answers for now, since you all need practice in expressing yourselves.

Your solutions will be graded both on mathematical correctness and on the precision of your argument. The arguments need NOT be long, just correct and to the point. Sometimes it is better not to use too many words, — but you must use enough for the logical structure of the argument to be clear.

This exercise is a first chance for you to get your solutions to Midterm I in good enough shape for me to accept them towards the writing requirement.

NOTES:

1. You need not rewrite any question that you got essentially correct with a well written solution. So the grade should be at least 16/20, and the argument should be well presented. If you do not want any of your old solutions to count in this way, please submit your exam together with the homework answers.

2. I will grade this homework. Each answer will be graded. You will get the previous grade on any question that you do not rewrite.

3. Since I am grading the HW you can hand them in either to Tanvir at the Wednesday recitation or under my office door by end Friday. (I will pick them up over the weekend.)

Math 319 Midterm I
September 29, 2005

Problem 1. (i) Show that if $f : A \to B$ is injective and $E \subset A$ then $f^{-1}(f(E)) = E$.
(ii) Give an example to show that equality need not hold if f is not injective.

Problem 2. Prove the following statement by induction:

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n, \quad \text{for all } n \in \mathbb{N}.$$

Problem 3. (i) State the Archimedean property of the real numbers.
Use it to prove that if $x > 0$ is any real number then there is a rational number r such that $x > r > 0$.

Problem 4. (i) Let S be a subset of $\mathbb{R}$. Give the definitions of a lower bound of S and of $\inf S$.

(ii) If possible, give examples of sets S with the following properties. If there is no such example, give a brief explanation of why.

(a) a set $S \subset \mathbb{R}$ with no lower bound.

(b) a set $S \subset \mathbb{R}$ with a lower bound but no infimum.

Problem 5. Let us say that a set S has property F if every injection $f : S \to S$ is surjective.
(i) Show that $\mathbb{N}$ does not have property F.
(ii) Show that if S has property F and $T \subset S$ then T has property F.

Math 319 Midterm I – Comments on the solutions
September 29, 2005

Problem 1. *(i) Show that if $f : A \to B$ is injective and $E \subset A$ then $f^{-1}(f(E)) = E$.*

A lot of you made heavy weather with this, though it should be easy. Perhaps one problem was that you didn't enough space on the page for your answer. Anyway, you should simply prove first that $E \subset f^{-1}(f(E))$ and second that $f^{-1}(f(E)) \subset E$. Your argument must be sufficiently clear that one can see where you use the fact that f is injective. You must also be very clear that both E and $f^{-1}(f(E))$ are subsets of A, not B.

Here is a hint for the second step. Suppose that $x \in f^{-1}(f(E))$. Then (by definition) $f(x) \in f(E)$. Since every point in $f(E)$ has the form $f(e)$ for some $e \in E$, this means that $f(x) = f(e)$ for some $e \in E$. If f were an arbitrary function there is nothing more you can say. But since f is injective..... Remember that the conclusion fo this argument is meant to be that $x \in E$!

(ii) *Give an example to show that equality need not hold if f is not injective.* Here you must specify the function $f : A \to B$ and the set E and then verify that $E \neq f^{-1}(f(E))$. Thus you must give A, B, f and E. It is not good enough to say: let $f(x) = x^2$... without mentioning A, B (since this function is injective if the domain is $[0, \infty)$...)

Problem 2. *Prove the following statement by induction:*

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n, \quad \text{for all } n \in \mathbb{N}.$$

Most of you did this okay. But you must explain the logic on the inductive step properly. Some of you just hurriedly wrote down a sequence of equalities without explaining what you were doing. eg say:

assume that $P(k)$ holds, ie that $3 + \cdots + (8k - 5) = 4k^2 - k$. Consider $P(k + 1)$. Then, *by the inductive hypothesis,*

$$3 + 11 + \cdots + (8k - 5) + (8(k + 1) - 5) = 4k^2 - k + (8(k + 1) - 5)....$$

Now do some algebra....

Problem 3. *(i) State the Archimedean property of the real numbers.*

For every $x \in \mathbb{R}$ there is a positive integer n such that $n > x$.

Use it to prove that if $x > 0$ is any real number then there is a rational number r such that $x > r > 0$.

This confused some of you. You are given x and must find r. It is easiest to take r of the form $1/n$, since then the statement is an almost immediate consequence of (a).

Problem 4. *(i) Let S be a subset of $\mathbb{R}$. Give the definitions of a lower bound of S and of* $\inf S$.

x is a lower bound of S if $x \le a$ for all $a \in S$.

$x = \inf S$ if it is a lower bound of S and if for every other lower bound y of S, $x \geq y$.

(There are many acceptable ways to phrase these definitions, but you must get the quantifiers right.)

(ii) *If possible, give examples of sets S with the following properties. If there is no such example, give a brief explanation of why.*

(a) *a set $S \subset \mathbb{R}$ with no lower bound.*

take $S = (-\infty, 0)$. If x were a lower bound for S then $x < -n$ for all $n \in \mathbb{N}$. Hence $-x > n$ for all $n \in \mathbb{N}$, contradicting Archimedes' principle.

(b) *a set $S \subset \mathbb{R}$ with a lower bound but no infimum.*

There is no such example. Any set that is bounded below has an infimum by the completeness axiom. (The completeness axiom axoim says that any set that is bounded above has a supremum, but S is bounded below iff $-S := \{-x : x \in S\}$ is bounded above. And, multiplying by -1 also takes a supremum to an infinum, and conversely.)

Problem 5. *Let us say that a set S has property F if every injection $f : S \to S$ is surjective.*

(i) *Show that $\mathbb{N}$ does not have property F.*

This question confused a lot of you, because property F is defined in terms of the *set of functions* $S \to S$ rather than the *points of* S. Hence it makes no sense to talk about points of S having property F.

If S has property F, EVERY injection $f : S \to S$ is surjective. Hence if a set does NOT have property F THERE IS an injection $f : S \to S$ that is not surjective. Hence to do this bit, you just have to define one function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective.

(ii) *Show that if S has property F and $T \subset S$ then T has property F.*

Here you must show that every injection $g : T \to T$ is surjective (using the fact that every injection $f : S \to S$ is surjective.) The easiest way to do this is by contradiction. Suppose that there is an injection $g : T \to T$ that is NOT surjective. Then use this to construct $f : S \to S$ that is injective, but not surjective. This contradicts the fact that S has property F.

To construct f you just need to extend g. ie define $f(x) = g(x)$ for $x \in T$. Then think of a way to define $f(x)$ for $x \in S \setminus T$ so that it remains injective but is still not surjective. I might help to do a specific example; eg take $T = \mathbb{N}$ and $S = \mathbb{Z}$.

Here I am using a property of functions that might be a little unfamiliar. Suppose that $T \subset S$ and that $g : T \to B$ is any function. Then $f : S \to B$ EXTENDS g if $f(x) = g(x)$ for all $x \in T$.

Math 319 Homework 6

Due Thursday, October 20, 2005

Problem 1. (i) Let $x_n = \dfrac{1}{4n-3}, n \geq 1$. Find an integer K such that

$$x_n < \frac{1}{35}$$

for all $n \geq K$. Explain your reasoning.

(ii) Same as (i) with $x_n = \dfrac{1}{4^n}$.
Hint: use the calculations in 3.1.11 (b).

Problem 2. Use the definition of limit to prove that:

(i) $\lim \dfrac{n}{2n-1} = \dfrac{1}{2}$.

(ii) $\lim \dfrac{\sqrt{n}}{n+1} = 0$.

Problem 3. Prove Bernoulli's inequality:
if $x + 1 > 0$ then $(x+1)^n \geq 1 + nx$ for all $n \geq 1$.

Hint: Prove this by induction on n. Make clear in your argument why you need to assume $x + 1 > 0$.

Problem 4. Show that the sequence $n^2 - \sin n$ has no limit.

Problem 5. (i) Suppose that $\lim z_n = z$ where $z \neq 0$. Show that there is $K \in \mathbb{N}$ such that $z_n \neq 0$ for all $n \geq K$.

(ii) Suppose that $\lim z_n = z$ where $z \neq 0$ and $z_n \neq 0$ for all n. Show there is $\delta > 0$ such that $|z_n| > \delta$ for all n.

Math 319 Solutions to Homework 6

Problem 1. (i) *Let* $x_n = \dfrac{1}{4n-3}, n \geq 1$. *Find an integer* K *such that* $x_n < \frac{1}{35}$ *for all* $n \geq K$. *Explain your reasoning.*

$1/4n - 3 < 1/35$ iff $35 < 4n - 3$ iff $38/4 < n$. This holds for any $n \geq 10$. Therefore we may take $K = 10$ (or any integer ≥ 10.)

(ii) *Same as (i) with* $x_n = 1/4^n$.

Now we want $35 < 4^n$. But $4^3 = 64$ is already large enough. Therefore any $k \geq 3$ will do.

(You might have wanted to use the hint if $1/35$ was replaced with w=something like $1/10000$.)

Problem 2. *Use the definition of limit to prove that:* $\lim \dfrac{n}{2n-1} = \dfrac{1}{2}$.

$$|\frac{1}{2} - \frac{n}{2n-1}| = |\frac{2n-1-2n}{2(2n-1)}| = \frac{1}{4n-2} \leq \frac{1}{n},$$

where the last inequality holds because $n < 4n - 2$ for all $n \geq 1$. Therefore given $\epsilon > 0$ choose $K > 1/\epsilon$. Then

$$|\frac{1}{2} - \frac{n}{2n-1}| \leq \frac{1}{n} \leq \frac{1}{K} < \epsilon, \quad \text{for all } n \geq K.$$

(ii) $\lim \dfrac{\sqrt{n}}{n+1} = 0$.

Note that

$$\frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} < \epsilon$$

if $1/\epsilon < \sqrt{n}$ or $1/\epsilon^2 < n$. Therefore, given $\epsilon > 0$ choose K so that $1/\epsilon^2 < K$. Then $|\frac{\sqrt{n}}{n+1} - 0| < \epsilon$ for all $n \geq K$.

Problem 3. *Prove Bernoulli's inequality: if* $x + 1 > 0$ *then* $(x+1)^n \geq 1 + nx$ *for all* $n \geq 1$.

Base case: if $n = 1$ this says $x + 1 \geq x + 1$, which is true (for all x.)

Now suppose the statement holds for $n = k$ and suppose that $n = k + 1$. We must estimate $(x+1)^{k+1}$. But

$$(x+1)^{k+1} = (x+1)(x+1)^k \geq (x+1)(1+kx)$$

by the inductive hypothesis. (And here we need the factor $x + 1$ to be nonnegative.) Also

$$(x+1)(1+kx) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x.$$

Hence the statement holds for $n = k + 1$ and therefore holds for all n by induction.

Problem 4. *Show that the sequence $n^2 - \sin n$ has no limit.*

This sequence is not bounded above. For if there were an upper bound M then $n^2 - \sin n < M$ which implies $n < n^2 < M + \sin n < M + 1$ for all n. But the integers are not bounded above (By Archimedes).

Since convergent sequences are bounded by Thm 3.2.2, this sequence must diverge.

Problem 5. (i) *Suppose that* $\lim z_n = z$ *where* $z \neq 0$. *Show that there is* $K \in \mathbb{N}$ *such that* $z_n \neq 0$ *for all* $n \geq K$.

Choose $\epsilon = |z|/2$. Then the interval $V_\epsilon(z) = (z - \epsilon, z + \epsilon)$ does NOT contain 0. Since $\epsilon > 0$ there is K so that $z_n \in V_\epsilon(z)$ for all $n \geq K$. Hence $z_n \neq 0$ for all $n \geq K$.

(ii) *Suppose that* $\lim z_n = z$ *where* $z \neq 0$ *and* $z_n \neq 0$ *for all* n. *Show there is* $\delta > 0$ *such that* $|z_n| > \delta$ *for all* n.

Choose K as in (i). Then $|z_n| > |z|/2$ for all $n \geq K$. Hence take

$$\delta = \min\{|z_1|/2, \ldots, |z_{K-1}|/2, |z|/2.$$

Since δ is the minimum of a finite set of positive numbers, $\delta > 0$. To see that $|z_n| > \delta$ for all n we divide in two cases. If $n < K$, then $|z_n| > |z_n|/2 \geq \delta$ by definition of δ. While if $n \geq K$, we use the inequalities $|z_n| > |z|/2 \geq \delta$.

Math 319 Homework 7

Due Thursday, October 27, 2005

Problem 1. Suppose that (x_n) is a sequence with limit 3. Prove carefully from the definition that

$$\lim \frac{1}{x_n} = \frac{1}{3}.$$

Note: I do NOT want you to quote Theorem 3.2.3(b). Instead, adapt the proof to this case.

Problem 2. Suppose that the sequences (x_n) and (z_n) both converge to w and that $x_n \leq y_n \leq z_n$ for all n.

(i) Use Theorem 2.1.10 to show that the sequence $(y_n - x_n)$ converges to 0.

(ii) Use (i) and the sum theorem (3.2.3(a)) to conclude that (y_n) converges to w.
Note This is known as the **Squeeze Theorem.** The book gives a different (more direct) proof in 3.2.7.

Problem 3. Find the limits, justifying your answer carefully. (Use any theorems you like, but say what you are using.)

(i) $x_n = \dfrac{4n^2 - n + 1}{n^2 - 3n}$.

(ii) $(-1)^n \dfrac{n+1}{n^2+2}$.

Problem 4. (i) Show that the sequence $x_n = (n^2 - n)/(n + 1)$ is monotonic.

(ii) Define x_n inductively by the relation $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$. Assume that $x_n \geq x_{n+1} > 0$ for all n. Then (x_n) converges by Theorem 3.3.2. What is its limit?

Problem 5. Consider the sequence defined inductively by: $x_{n+1} = 4 - 3/x_n$.

(i) Show that if $1 < x_1 < 3$ then this is monotonic increasing. What is the limit?

(ii) Show that if $x_1 > 3$ then this is monotonic decreasing. What is the limit?

(iii) Is there a value of x_1 that gives a sequence with limit 1?

Note The points 1 and 3 are special because they are the roots of the quadratic equation $x^2 - 4x + 3 = 0$.

Math 319 Solutions to Homework 7

Problem 1. *Suppose that* (x_n) *is a sequence with limit* 3. *Prove carefully from the definition that* $\lim \frac{1}{x_n} = \frac{1}{3}$.

Since $\lim x_n = 3$ there is K such that $x_n \in (2, 4)$ for $n \geq K$. Then $1/|x_n| \leq 1/2$ for $n \geq K$. Hence when $n \geq K$

$$|\frac{1}{x_n} - \frac{1}{3}| = \frac{|3 - x_n|}{3|x_n|} \leq \frac{2|3 - x_n|}{3} < \epsilon,$$

if $|3 - x_n| < 3\epsilon/2$. Since $\lim x_n = 3$ there is K_1 so that $|3 - x_n| < 3\epsilon/2$ for all $n \geq K_1$. Hence if $K_2 = \max(K, K_1)$, $|\frac{1}{x_n} - \frac{1}{3}| < \epsilon$ for all $n \geq K_2$.

Problem 2. *Suppose that the sequences* (x_n) *and* (z_n) *both converge to* w *and that* $x_n \leq y_n \leq z_n$ *for all* n.
(i) *Use Theorem 3.1.10 to show that the sequence* $(y_n - x_n)$ *converges to* 0.

We know $0 \leq y_n - x_n \leq z_n - x_n$ for all n. Hence $|y_n - x_n| \leq |z_n - x_n|$ for all n. But $\lim z_n - x_n = w - w = 0$ by Theorem 3.2.3(a). Hence, applying Theorem 3.1.10 with $c = 1$, we find that $\lim(y_n - x_n) = 0$.
(ii) *Use (i) and the sum theorem (3.2.3(a)) to conclude that* (y_n) *converges to* w.

Applying Thm 3.2.3(a) to the sum $y_n = (y_n - x_n) + x_n$m, we see that $\lim y_n = 0 + w = w$.

Problem 3. *Find the limits, justifying your answer carefully.*
(i) $x_n = \frac{4n^2 - n + 1}{n^2 - 3n}$.

Assume that $n > 3$ so that $n^2 - 3n \neq 0$. Then

$$\lim \frac{4n^2 - n + 1}{n^2 - 3n} = \lim \frac{4 - 1/n + 1/n^2}{1 - 3/n} = \frac{4 - \lim 1/n + \lim 1/n^2}{\lim(1 - 3/n)} = 4.$$

Here we applied the sum and quotient rules in Thm 3.2.3, using the fact that $\lim(1 - 3/n) = 1 \neq 0$.

(ii) $(-1)^n \frac{n+1}{n^2+2}$.

$|x_n| = (n + 1)/(n^2 + 2) \leq (n + 1)/n^2 \leq 2n/n^2 = 2/n$. Now use the fact that $\lim 1/n = 0$ together with Thm 3.1.10 with $c = 2$ to conclude that $\lim x_n = 0$.

Problem 4. (i) *Show that the sequence* $x_n = (n^2 - n)/(n + 1)$ *is monotonic.*

$x_n \leq x_{n+1}$ iff $(n^2 - n)/(n+1) \leq ((n+1)^2 - n - 1)/(n+2)$ which is true iff $(n+2)(n^2 - n) \leq (n^2 + n)(n + 1)$. This holds iff $n^3 + n^2 - 2n \leq n^3 + 2n^2 + n$, which is TRUE for all n. Hence x_n is monotonic increasing.
(ii) *Define* x_n *inductively by the relation* $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$. *Assume that* $x_n \geq x_{n+1} > 0$ *for all* n. *Then* (x_n) *converges by Theorem 3.3.2. What is its limit?*

If the $\lim x_n = x$ then (because $\lim x_{n+1} = x$ also) x must satisfy the identity $x = \frac{1}{2}(x + 5/x)$. Simplifying we find $x^2 = 5$. Since x_n is monotonic increasaing and always > 0 by hypothesis, the limit is positive and hence $x = \sqrt{5}$.

Problem 5. *Consider the sequence defined inductively by:* $x_{n+1} = 4 - 3/x_n$.
(i) *Show that if* $1 < x_1 < 3$ *then this is monotonic increasing. What is the limit?*

$x_{n+1} \geq x_n$ iff $4 - 3/x_n \geq x_n$. We now want to multiply through by x_n and so need to know whether $x_n \geq 0$ or not.

So, let's do a preliminary calculation, showing by induction that $x_n > 1$ for all n. This is true for $n = 1$ by hypothesis. If it holds for x_k then $3/x_k < 3$; hence $x_{k+1} = 4 - 3/x_k > 1$. Hence it holds for x_{k+1}, and hence for all x_n by induction.

Therefore we find that $x_{n+1} \geq x_n$ iff $4x_n - 3 \geq x_n^2$ iff $0 \geq x_n^2 - 4x_n + 3 = (x_n - 3)(x_n - 1)$. We saw above that $x_n - 1 > 0$ for all n. Hence to finish we need to see that $x_n - 3 \leq 0$ for all n. Again this is true for $n = 1$ by hypothesis. Suppose it holds for $n = k$. Then $0 < x_k \leq 3$ so $3/x_k \geq 1$. Hence $x_{k+1} = 4 - 3/x_k \leq 4 - 1 = 3$. Therefore, by induction, $1 < x_n < 3$ for all n. Hence $(x_n - 3)(x_n - 1) < 0$ and so $x_{n+1} > x_n$ for all n. Therefore the sequence is monotonic increasing and the limit is $\sup\{x_n : n \geq 1\}$ and so is > 1.

But its limit x must satisfy the equation $x^2 - 4x + 3 = 0$ and hence must be 1 or 3. This $x = 3$.

(ii) *Show that if* $x_1 > 3$ *then this is monotonic decreasing. What is the limit?*

Let us first show by induction that $x_n > 3$ for all n. This is true for $n = 1$ by hypothesis. And if it holds for $n = k$ then it holds for $n = k+1$ since $3/x_k < 1$ so that $x_{k+1} = 4 - 3/x_k > 3$.

Since $x_k > 0$ always we can apply the previous reasoning to see that $x_{n+1} \leq x_n$ iff $0 \leq (x_n - 3)(x_n - 1)$. This holds since both factors are positive. The limit must then be 3.

(iii) *Is there a value of* x_1 *that gives a sequence with limit* 1*?*

If we take $x_1 = 1$ then we get $x_n = 1$ for all n and so the sequence does have limit 1. If $x_1 < 1$ the sequence will not have limit 1. I think what happens is thatr it decreases until $x_k \leq 0$ and then $x_{k+1} \geq 4$ and we are in case (ii). (1 is what is called an **unstable** fixed point: such things are discussed in MAT 351.) I think the proof of this fact would be a nice project – I don't want to get into it right now.

Math 319 Homework 8

Due Thursday, November 3, 2005

As always, justify all your answers.

Problem 1. (a) Give an example of a sequence that is bounded above but not bounded below and that has a convergent subsequence.

(b) Explain how to construct a monotonic increasing sequence of rational numbers that converges to $\sqrt{3}$.

Problem 2. Consider the sequences

$$x_n = \sin\left(\frac{n\pi}{4}\right), \quad y_n = \frac{1}{\sqrt{n}}, \quad z_n = x_n y_n.$$

(a) Write down the first 10 terms of (x_n). Find two monotonic subsequences of (x_n) with different limits.

(b) Which terms in the product sequence (z_n) are peaks? (It might help to work out the first 10 terms of z_n.) Find a subsequence of (z_n) that is strictly decreasing (i.e. $z_{n_{k+1}} < z_{n_k}$ for all k), and another that is strictly increasing.

(c) Does (z_n) converge? (Explain which theorems you use in your argument.)

Problem 3. (a) Suppose that $x_n \geq 0$ for all n and that $\lim x_n = 2$. Find a subsequence of $((-1)^n x_n)$ that converges to 2 and another that converges to -2. Does $((-1)^n x_n)$ converge?

(b) Suppose that $x_n \geq 0$ for all n and you are told that $((-1)^n x_n)$ converges. Show that (x_n) converges. What is its limit?

Problem 4. Describe all cluster points of A where
a) $A = \mathbb{Z} \cup (0,1)$,
b) $A = \{1/n : n \in \mathbb{N}\}$.

Problem 5. (a) Consider the intervals $I_k = (1 + \frac{1}{3k+1}, 1 + \frac{1}{3k})$ for $k = 1, 2, 3, 4, \ldots$. Write down I_1, I_2 and I_3 explicitly.

(b) Make an accurate sketch of the set $A = \cup_{k \geq 1} I_k$.

(c) Describe all cluster points of A.

Math 319 Solutions to Homework 8

Problem 1. (a) *Give an example of a sequence that is bounded above but not bounded below and that has a convergent subsequence.*

There are many examples. eg $x_n = 1$ if n is even and $x_n = -n$ if n is odd.

(b) *Explain how to construct a monotonic increasing sequence of rational numbers that converges to $\sqrt{3}$.*

Let $x_1 = 1$. We will construct an increasing sequence of rational numbers x_n so that $\sqrt{3} - 1/n < x_n < \sqrt{3}$ for all n. Then $\lim x_k = \sqrt{3}$. Since

$$\sqrt{3} - 1 < 2 - 1 = 1 = x_1,$$

x_1 satisfies the requirements. Suppose that $x_k \in \mathbb{Q}$ has been chosen so that $x_{k-1} \leq x_k < \sqrt{3}$ and $\sqrt{3} - 1/k < x_k$. Then choose x_{k+1} to be rational number in the interval

$$\left(\max(x_k, \sqrt{3} - 1/(k+1)), \sqrt{3}\right).$$

Problem 2. *Consider the sequences $x_n = \sin\left(\frac{n\pi}{4}\right)$, $y_n = \frac{1}{\sqrt{n}}$, and $z_n = x_n y_n$.*

(a) *Write down the first 10 terms of (x_n). Find two monotonic subsequences of (x_n) with different limits.*

$$x_1 = \frac{1}{\sqrt{2}} = x_9,\ x_2 = 1 = x_{10},\ x_3 = \frac{1}{\sqrt{2}},\ x_4 = 0,\ x_5 = -\frac{1}{\sqrt{2}},\ x_6 = -1, x_7 = -\frac{1}{\sqrt{2}},\ x_8 = 0.$$

This sequence is periodic with period 8 (i.e. $x_{n+8} = x_n$ for all n.) The only monotonic subsequences are eventually constant. eg $x_1, x_4, x_8, x_{3k}, k \geq 3$ is monotonic decreasing with limit 0. $y_k := x_{6+8k}$ is monotonic with limit -1.

(b) *Which terms in the product sequence (z_n) are peaks?*

Note that (y_n) is strictly decreasing, converging to 0. Therefore the terms z_n with $x_n = 1$ are peaks. These are the terms with $n = 2 + 8k$, for $k \geq 0$. Are there any other peaks? The only other possible candidates have $x_n = \frac{1}{\sqrt{2}}$. i.e. $n = 1 + 8k$ or $3 + 8k$ for some $k \geq 0$.

If $n = 1 + 8k$ then $z_n = \frac{1}{\sqrt{2(1+8k)}}$ and $z_{n+1} = \frac{1}{\sqrt{2+8k}}$. Since $8k + 2 < 2(8k + 1)$ for all $k > 0$, we have $z_n < z_{n+1}$ for these k. Hence z_n is not a peak if $k > 0$. But $z_1 = z_2$ is a peak.

If $n = 3 + 8k$ then $z_n = \frac{1}{(\sqrt{2(3+8k)}}$ and the next positive terms are $z_{n+6} = \frac{1}{(\sqrt{2(9+8k)}} < z_n$ and $z_{n+7} = \frac{1}{(\sqrt{(10+8k)}}$. Since $z_{n+8} < z_n$, z_n is a peak iff it is $\geq z_{n+7}$. This happens iff $10 + 8k \geq 2(3 + 8k)$, that is iff $k = 0$. Hence z_3 is the only peak of this form.

Thus, the peaks are z_1, z_2, z_3 and the terms $z_{2+8k}, k > 0$.

Find a subsequence of (z_n) that is strictly decreasing (i.e. $z_{n_{k+1}} < z_{n_k}$ for all k), and another that is strictly increasing.

A strictly decreasing sequence would be a product $x_n y_n$ with x_n constant and positive. eg $z_{2+8k}, k \geq 0$.

A strictly increasing sequence would be a product $x_n y_n$ with x_n constant and negative. eg $z_{6+8k}, k \geq 0$.

(c) *Does* (z_n) *converge?*

Yes it converges to 0 since $|x_n| \leq 1$ for all n. Hence $|z_n| \leq |y_n| \to 0$. (This uses Thm 3.1.10.)

Problem 3. (a) *Suppose that* $x_n \geq 0$ *for all* n *and that* $\lim x_n = 2$. *Find a subsequence of* $((-1)^n x_n)$ *that converges to* 2 *and another that converges to* -2.

The subsequence $x_{2k}, k \geq 1$, converges to 2 and $x_{2k+1}, k \geq 0$ converges to -2.
Does $((-1)^n x_n)$ *converge?* No, because it has two subsequences with different limits. (cf. 3.4.5)

(b) *Suppose that* $x_n \geq 0$ *for all* n *and you are told that* $((-1)^n x_n)$ *converges. Show that* (x_n) *converges. What is its limit?*

Let $(y_n) := ((-1)^n x_n)$ have limit y. Then its subsequence (y_{2k}) also has limit y (by 3.4.2). Since $y_{2k} = x_k \geq 0$ for all k, we have $y \geq 0$ by 3.2.4. Similarly y_{2k+1} converges to y, and because its terms are all ≤ 0 we find that $y \leq 0$. Hence $y = 0$. Then $|x_n| \leq |y_n| \to 0$ also, by Thm 3.1.10.

Problem 4. *Describe all cluster points of* A *where*
a) $A = \mathbb{Z} \cup (0, 1)$,

The cluster points are $[0, 1]$.

Proof: if $c \in [0, 1]$ then $(c - \delta, c + \delta)$ obviously intersects $[0, 1] \setminus \{c\}$ for all $\delta > 0$. Now suppose that $c \in \mathbb{Z}, c \neq 0, 1$. Then $(c - 1/2, c + 1/2)$ meets A only in the point c. Hence c is not a cluster point. The last case is when $c \notin A, c \neq 0, 1$. Then $\delta := \min\{|c-n| : n \in \mathbb{Z}\}$ is a positive number (it is the minimum distance of c from an integer). And $(c-\delta, c+\delta) \cap A = \emptyset$.
b) $A = \{1/n : n \in \mathbb{N}\}$.

In this case the only cluster point is 0. I won't say more now, since it is like (i) and we discussed it in class today.

Problem 5. (a) *Consider the intervals* $I_k = (1 + \frac{1}{3k+1}, 1 + \frac{1}{3k})$ *for* $k = 1, 2, 3, 4, \ldots$. *Write down* I_1, I_2 *and* I_3 *explicitly.*

$I_1 = (5/4, 4/3)$, $I_2 = (8/7, 7/6)$, $I_3 = (11/10, 10/9)$

(b) *Make an accurate sketch of the set* $A = \cup_{k \geq 1} I_k$.

left to you – these intervals are all disjoint and get closer to 1.

(c) *Describe all cluster points of* A.

The cluster points are 1 together with the union of the closed intervals $[1 + \frac{1}{3k+1}, 1 + \frac{1}{3k}]$.

Math 319 Homework 9

Due Tuesday, November 8, 2005

Problem 1. 1: (i) Let $f : (0,5) \to \mathbb{R}$ be the function $f(x) = 2x^2 + 3$. Prove from the definition that $\lim\limits_{x\to 2} f = 11$.

(ii) Find $M, \delta > 0$ so that $|f(x)| \le M$ for all $x \in A \cap (2-\delta, 2+\delta)$.

Problem 2 Let c be a cluster point of A and $f : A \to \mathbb{R}$. Suppose that $\lim\limits_{x\to c} f = L$. Show that there is a $\delta > 0$ such that f is bounded on the set $A \cap (c-\delta, c+\delta)$.

Note: Part (ii) of Problem 1 is an example of this general statement.

Problem 3: Let $A = \{1/n : n \in \mathbb{N}\}$ and let $f : A \to \mathbb{R}$ be the function $f(x) = 1/(1+x)$.

(i) Write down the values of f at the points $x = 1, 1/2, 1/3$.

(ii) Does $\lim\limits_{x\to 0} f$ exist? If so evaluate it.

(iii) Does $\lim\limits_{x\to 1/3} f$ exist? If so evaluate it.

Justify your answers.

Math 319 Homework 9

Problem 1. (i) *Let $f : (0, 5) \to \mathbb{R}$ be the function $f(x) = 2x^2 + 3$. Prove from the definition that* $\lim_{x\to 2} f = 11$.

Suppose given $\epsilon > 0$. We must find $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < |x - 2| < \delta$.

Now $|f(x) - L| = |2x^2 + 3 - 11| = |2x^2 - 8| = 2|x + 2||x - 2|$. If $|x - 2| < 1$ then $1 < x < 3$ and $|x + 2| < 5$. Hence $|f(x) - L| < 10|x - 2| < \epsilon$ if, in addition, $|x - 2| < \epsilon/10$. Therefore, if we take $\delta = \min(1, \epsilon/10)$ the desired inequality will hold.

(ii) *Find $M, \delta > 0$ so that $|f(x)| \le M$ for all $x \in A \cap (2 - \delta, 2 + \delta)$.*

$f(x)$ is an increasing and positive function when $x > 0$. Therefore, if $\delta < 2$, the values of f on $(2 - \delta, 2 + \delta)$ are all $\le M := f(2 + \delta)$. For example, we can take $\delta = 1$ and $M = f(3) = 21$.

Problem 2 *Let c be a cluster point of A and $f : A \to \mathbb{R}$. Suppose that $\lim_{x\to c} f = L$. Show that there is a $\delta > 0$ such that f is bounded on the set $A \cap (c - \delta, c + \delta)$.*

Let $\epsilon = 1$. Then there is $\delta_1 > 0$ so that $|f(x) - L| < 1$ for all $x \in A \cap (c - \delta_1, c + \delta_1), x \neq c$. Hence $|f(x)| \le |L| + |f(x) - L| \le |L| + 1$ for these x.

If $c \notin A$, then we may take $M := |L| + 1$ and $\delta := \delta_1$. If $c \in A$ then we should take $M = \max(|L| + 1, |f(c)|)$ and again $\delta := \delta_1$.

Problem 3: *Let $A = \{1/n : n \in \mathbb{N}\}$ and let $f : A \to \mathbb{R}$ be the function $f(x) = 1/(1 + x)$.*

(i) *Write down the values of f at the points $x = 1, 1/2, 1/3$.*

$f(1) = 1/2, f(1/2) = 2/3, f(1/3) = 3/4$.

(ii) *Does $\lim_{x\to 0} f$ exist? If so evaluate it.*

This limit exists and equals 1. Proof. $|1/(1 + x) - 1| = |x/(1 + x)|$. If $|x| \le 1/2$ then $1 + x \ge 1/2$ so $0 < 1/(1 + x) < 2$. Hence $|1/(1 + x) - 1| < 2|x| < \epsilon$ if we also assume that $|x| < \epsilon/2$. Therefore given $\epsilon > 0$ choose $\delta = \min(1, \epsilon/2)$.

(iii) *Does $\lim_{x\to 1/3} f$ exist? If so evaluate it.*

This limit does not exist because $1/3$ is NOT a cluster point of A. (Hence the limit is not defined,– and something that is not defined does not exist!)

Math 319 Second Midterm

November 10, 2005

Name: **School ID:**

Answer all the following questions, justifying all your statements. If you need more space, please write on the back of the sheets. There are four questions, worth a total of 50 points. Good luck!

Problem 1. (15 points) Prove ONE of the following statements.
EITHER: (i) Prove that a convergent sequence is bounded.

OR: (ii) Prove that if c is a cluster point of A there is a sequence (a_n) with limit c such that $a_n \in A \setminus \{c\}$ for all n.

OR: (iii) If $\lim x_n = x$ and $\lim y_n = y$ then $\lim x_n y_n = xy$.

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Problem 2. (10 points) Define the sequence (x_n) inductively by setting $x_{n+1} = x_n/2 - 1$.
(i) Show that (x_n) is monotone increasing if $x_1 \leq -2$ and is monotone decreasing if $x_1 \geq -2$.

(ii) Suppose that $x_1 = 0$. Does the sequence (x_n) have a limit? If so, what is it?

Problem 3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 2x + x^2$. Prove from the definition that $\lim_{x \to 1} f(x) = 3$.

Problem 4. (15 points) Which of the following sequences (x_n) are convergent? If they are convergent, what are their limits? Prove your claims.

$$\text{(i)} \qquad x_n = \frac{(-1)^n n}{n+3}, \qquad \text{(ii)} \qquad x_n = \frac{(-1)^n n}{n^2+3}, \qquad \text{(iii)} \qquad x_n = \frac{n^2}{n+3}.$$

Math 319 Homework 10

due Thursday December 8

Problem 1. Given $x \in \mathbb{R}$ let $[[x]]$ be the greatest integer $n \in \mathbb{Z}$ that is $\le x$. Thus $[[2]] = 2$ and $[[-1/2]] = -1$. Graph the following functions and determine their points of continuity:

(a) $f(x) := [[x^2]]$ for $x \in \mathbb{R}$,

(b) $g(x) := \dfrac{[[x]]}{x}$ for $x > 0$,

(c) $h(x) := [[2\sin(x)]]$, for $x \in \mathbb{R}$.

Hint: Exercise 4 in Sec 5.1 is similar, and has an answer at the back of the book.

Problem 2. Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$ and that $f(b) = g(b)$. Now define $h : a, c \to \mathbb{R}$ by setting

$$h(x) = \{ \begin{array}{ll} f(x), & \text{if } x \in [a, b] \\ g(x), & \text{if } x \in (b, c]. \end{array}$$

(i) Draw the graphs of functions f, g that satisfy these conditions,

(ii) Prove from the definition that h is continuous at b. (It follows that h is continuous on $[a, c]$, but I am not asking you to prove this.)

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous everywhere.

(i) Suppose that $f(x) = 1$ for all rational numbers x. Show that $f(x) = 1$ for all x.

(ii) What can you say about the values of f if all you know is that $f(1/n) = 0$ for all $n \in \mathbb{N}$?

Problem 4. The function $h(x) := (x-3)(x-4)(x-5)(x-6)(x-8)$ has five roots in the interval $[0, 9]$. Which root is found by the bisection method? Which root is found if you start with the interval $[2, 9]$. Explain your answers.

In the following two questions your answers may quote any theorems from Sections up to 5.2 but not those from Sec 5.3. Problem 5 asks you to prove part of Thm 5.3.4. (Don't do this by repeating the argument given in the book substituting $-f$ for f, but do the argument for f directly.) Problem 6 uses similar ideas.

Problem 5. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and you know that f is bounded below. Prove that there is $c \in [0, 1]$ such that $f(c) = \inf\{f(x); x \in [0, 1]\}$.

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $c \notin f([0, 1])$. Show that there is $\epsilon > 0$ such that $|f(x) - c| > \epsilon$ for all $x \in [0, 1]$.

Hint: Argue by contradiction.

Math 319 Review sheet for Final, Dec 2005

This exam will be out of 90 points: each question will be worth 15 points. The exam will be much like Midterm II, EXCEPT that I will ask you to give ONE definition (one of 3.1.3, 3.4.1, 4.1.1, 4.1.4 and 5.1.1) – so the definitions WILL NOT be on the sheet. There still will be a sheet of statements of theorems. These will be much the same as those on last year's exam, but I may update that (If I do, I will give you some warning.)

As in Midterm II, one question will ask you to prove a part of one of the theorems on the sheet. There will be no (hard) questions on the final about countable/uncountable sets or using the Well-ordering principle for $\mathbb{N}$. (But I might ask about sups and infs.)
NOTE: I have written this sheet in a hurry and may revise it SLIGHTLY at the weekend.

Problem 1. (i) Consider the statement:

$$2 + 5 + \cdots + 3n - 1 = \frac{n(3n+1)}{2}.$$

What does it say when $n = 1, n = 2$, and $n = 3$?
(ii) Use mathematical induction to prove it for all n.

Problem 2. Use mathematical induction to show that $n^3 + 3n^2 - n$ is divisible by 3 for all $n \geq 1$.
Problem 3. Prove ONE of the following results:
EITHER: Let $A = (0, 2)$ and suppose that c is a cluster point of the complement $\mathbb{R} \setminus A$. Prove that $c \notin A$.

OR: **(ii)** Let $f : A \to \mathbb{R}$ be any function and let $s := \sup\{f(x) : x \in A\}$. Show that there is a sequence $x_n \in A$ such that $\lim f(x_n) = s$.

Problem 4. (i) Let $x > 0$. Prove that $\lim_{n\to\infty} \frac{1}{1+nx} = 0$.
(ii) Let $0 < b < 1$. Prove that $\lim b^n = 0$.

(Use Archimedes Principle and Bernoulli's inequality, both of which are now on the sheet)

Problem 5. Suppose that $(x_n), (y_n)$ are sequences such that $\lim x_n = 2$ and $\lim y_n = 3$. Prove from the definitions that there is an integer K such that $x_n < y_n$ for all $n \geq K$.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \frac{3x}{x^2+2}$. Prove from the definition that $\lim_{x\to 1} f(x) = 1$.

Problem 7. Describe examples satisfying the following conditions. Justify your answers.
(i) A bounded set A and a continuous function $f : A \to \mathbb{R}$ that is not bounded.

(ii) A convergent sequence that is not monotonic.

(iii) A countable set that is bounded above but does NOT contain its supremum.

Problem 8. Define the sequence (x_n) recursively by setting $x_1 = 1$ and $x_n = 1 + \frac{x_{n-1}}{2}$.
(i) Show that (x_n) is monotonic increasing.
(ii) Show that it converges and find its limit.

Problem 9. Which of the following sequences are convergent? Justify your answers.

$$\textbf{(i)} \qquad x_n = \frac{(-1)^n \sin n}{n+1}; \qquad \textbf{(ii)} \qquad x_n = \frac{(-2)^n}{n+1}; \qquad \textbf{(iii)} \qquad x_n = \frac{n-3}{2n-1}.$$

Math 319 Final Exam, Dec 14 2004.

Name: **School ID:**

Answer all the following questions, justifying all your statements. Each question is worth 15 points. There are six questions. Good luck!

1: Prove ONE of the following results:

EITHER: Let c be a cluster point of the set $\{x_n : n \geq 1\}$. Show that there is a subsequence of (x_n) that converges to c.

OR: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that x_n are points in $\mathbb{R}$ such that $f(x_n) = n$. Prove that the sequence (x_n) is unbounded.

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2: Prove from the definition of limit and the results on the sheet that $\lim \frac{1}{2^n} = 0$.

3: Consider the function $f : [0, 2] \to \mathbb{R}$ given by $f(x) = \frac{2}{1+x}$. Prove from the definition that $\lim_{x \to 1} f(x) = 1$.

4: Describe examples satisfying the following conditions. Justify your answers.

(i) an infinite subset of $\mathbb{R}$ that has no cluster points.

(ii) a bounded sequence of real numbers that does not converge.

(iii) a function $f : [-1, 1] \to \mathbb{R}$ that is not continuous at $x = 0$.

5: Let (x_n) be a monotonic decreasing sequence and set $B = \{x_n : n \geq 1\}$. Show that the point x_5 is not a cluster point of B.

6: Which of the following sequences are monotonic? Which are convergent?

(i) $x_n = \dfrac{n+1}{2n-1}$; **(ii)** $x_n = (-1)^n \dfrac{n+1}{2n-1}$; **(iii)** $x_n = \dfrac{n^2+1}{2n-1}$.

Def 3.1.3 A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms x_n satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called **divergent**.

Def 3.4.1 Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \ldots)$ is called a **subsequence** of X.

Def 4.1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a **cluster point** of A if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Def 4.1.4. Let $A \subset \mathbb{R}$ and let c be a cluster point of A. A function $f : A \to \mathbb{R}$ is said to have **limit** L **at** c if for all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - c| < \delta,\ x \in A \Longrightarrow |f(x) - L| < \epsilon$.

Def 5.1.1. Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then f is **continuous at** c if for every $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta,\ x \in A \Longrightarrow |f(x) - f(c)| < \epsilon$. If B is a subset of A we say that f is **continous on** B if it is continuous at all points $b \in B$.

Archimedes Principle: For all $x \in \mathbb{R}$ there is an integer $n > x$.

Bernoulli inequality: For all $x \geq 0$ and $n \geq 1$ $(1 + x)^n \geq 1 + nx$.

Thm 3.1.10 Comparison theorem for limits. Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq C a_n$ for all $n \geq m$, then $\lim x_n = x$.

Thm 3.2.2 A convergent sequence of real numbers is bounded.

Thm 3.3.2 Monotone Convergence Theorem. A monotone sequence of real numbers is convergent if and only if it is bounded.

Thm 3.4.2 If $X = (x_n)$ converges to $x \in \mathbb{R}$, every subsequence X' of X converges to x.

3.4.7: Monotone subsequence theorem. Every sequence has a monotone subsequence.

3.4.8: Bolzano–Weierstrass theorem. A bounded sequence of real numbers has a convergent subsequence.

Thm 4.1.8. Sequential criterion Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then $\lim_{x \to c} f = L$ iff for every (x_n) in $A \setminus \{c\}$ with limit c the sequence $(f(x_n))$ converges to L.

Thm 5.1.3. Sequential criterion for continuity: $f : A \to \mathbb{R}$ is continuous at $c \in A$ iff for every (x_n) in A that converges to c the sequence $(f(x_n))$ converges to $f(c)$.

Thm 5.3.2 Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I, i.e. there is M such that $|f(x)| \leq M$ for all $x \in I$.

Thm 5.3.4 Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I, i.e. there are points c, d in I such that $f(c) \leq f(x) \leq f(d)$ for all $x \in I$.

Math 319 Quiz

November 22, 2005

Name: **School ID:**

Question 1 For each of the three statements below give an example.

(i) A bounded monotonic sequence is convergent.

(ii) A sequence that contains two subsequences with different limits is divergent.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be any function and (a_n) be a convergent sequence with limit $c \in \mathbb{R}$. Suppose that $\lim_{x\to c} f(c) = L$. Then $\lim_{n\to\infty} f(a_n) = L$.

Question 2. Prove any ONE of the above statements from the definitions.

Math 319: Suggested projects.

November 15 2005

Here are some possible topics for your project. I would like you to work in pairs (or possibly in threes) on this project. Each group will make a short oral presentation during the last two weeks of the semester highlighting one or two of the main arguments. (Each group should divide the topic into separate parts, so that each member of the group contributes their own bit of the presentation: there should be 5 minutes each.) Each person will also individually write a 6-8 page paper on the topic chosen. (This can be handwritten (especially the formulas etc) but must be legible.) I am most interested in your constructing valid arguments and explaining the mathematics clearly, though you can give a small amount of background. You should outline the whole project but concentrate your calculations on your special part. The essays should describe specific examples, not just be general and vague.

Different groups should work on different projects. So as soon as your group have decided what you want to do, please tell me. Email me at *dusa@math.sunysb.edu.* Give alternatives in case another group has already taken your favorite topic. If you cannot find a suitable group to work with, email me anyway with details of what you would like to do and I will try to put you in touch with people to work with.

Timetable for the rest of the semester. This is quite tight; it will be hard to fit everything in, partly because I am out of town Dec 3-11.

1. I will spend time on Nov 15 and 17 describing the projects, but will also start lecturing on continuous functions Ch 5.1 and Ch 5.3. I will continue lecturing on this subject on Nov 22, Nov 29 and Dec 1. There will be one homework, due Dec 8.
2. The last 20 minutes of the Nov 22 class will be a retest of proof-writing. See the class web page for further details.
3. You must tell me your project title by Nov 22 at the very latest. If you want to choose a different topic from those outlined below, please give me a detailed outline by Nov 22 (earlier if possible.)
4. The first draft of the paper is due **Friday Dec 2.** The final draft is due **Tuesday Dec 13.**
5. I will have extended office hours during the week Nov 30–Dec 2 to discuss the projects with you. I expect you all will have questions: all the problems below require thought.
6. I will be out of town Dec. 3-11, but Tanvir will be available to help you. There will be no class on Tuesday Dec 6; Tanvir will have read the first drafts of the projects and will have comments by then. (If you can get your first draft to me by Thursday Dec 1, I will try to give you feedback before I go away.)
7. Your presentations will be given during class times on **Thursday Dec 8, Monday Dec 12** (at 2:20) and **Tuesday Dec 13.** Attendance will be taken and will form a small part of the grade for the project. Prof Phillips will be in charge of the class on Dec 8.

Projects on Sequences

Project 1: Square roots. Example 3.3.5 gives a sequence that calculates $\sqrt{a}$. This sequence is derived from Newton's method of finding roots of an equation. (You can find this in any first year Calculus text.) Explain this. Use the algorithm to calculate $\sqrt{5}$ correct to 3 decimal places. How many terms do you need? Explain the error estimate. There is another way finding square roots by a long division process, a method that "every school boy used to learn" — perhaps 100 years ago. Find out about this and explain it. Is it based on the same or a different algorithm?

Project 2: The Fibonacci sequence. This is the well known sequence $1, 1, 2, 3, 5, 8, \ldots$ defined inductively by the equation $s_{n+1} = s_n + s_{n-1}$. It turns out that the ratios of successive terms

$r_n = \dfrac{s_{n+1}}{s_n}$ form a sequence that is eventually monotonic and converges to the number known as the Golden Ratio (the positive root of the equation $x^2 = x + 1$.) Use the method of Ex 3.3.5 to explain this. Explore the effect of starting with pairs of different numbers, eg with $1, 5$ or with $-2, 1$. Do these sequences also converge? If so, what are their limits?

Project 3: Contractive sequences Instead of the rule $s_{n+1} = s_n + s_{n-1}$, consider the sequence defined by $s_{n+1} = as_n + (1-a)s_{n-1}$, where $0 < a < 1$. Thus s_{n+1} is a weighted average of s_n and s_{n-1}. This sequence is contractive and converges. The case $a = 1/2$ is explained on p 82/3 in Ex 3.5.6(a). You could discuss a different case (say $a = 1/4$ or $a = 1/3$.) Contractive sequence are also used in Ex 3.5.10 to find roots of certain polynomials. Use this method to find a root of the equation $x^4 + 2x - 2 = 0$ lying between 0 and 1 correct to 5 decimal places. (This is a good project for three people; one could explain contractive sequences and the other two could do the different examples.

Project 4: A divergent sequence. Investigate the sequence (x_n) where $x_n = \sin n$. Show how to construct a subsequence of (x_n) that converges to $+1$ and another that converges to -1. What properties of the sine function and the number π do you use in your argument? Can you show that this sequence has a subsequence that sconverges to any given number between -1 and 1? It may be helpful to look at Example 3.4.6(c) where it is shown that this sequence is divergent.

Projects on series

Project 5: Binary and ternary Decimals. Find the binary and ternary representations of the numbers $3/8, 1/3, 2/7$, and 32. Explain (using the Completeness Axiom) why any infinite sequence $S = .10011100111\dots$ of zeros and ones represents a *unique* real number x_S. Show how to get a geometric series from the ternary representation of 3/8 and sum it to 3/8. Is it true that S is eventually periodic iff x_S is rational? Explain. This project is based on the end of Ch 2.5. See also Ex. 3.7.2 (a) (geometric series). It is somewhat related to Project 8.

Project 6: Variations on the harmonic series. If $\{m_1, m_2, \dots\}$ is the collection of natural numbers that end in 6 then $\sum_{k\geq 1} \frac{1}{m_k}$ diverges. (*Hint:* adapt the proof that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ is divergent.) The next result is more unexpected. Let $\{n_1, n_2, \dots\}$ be the collection of natural numbers that do NOT use the digit 6 in their decimal expansion. (ie 345145 is in this collection while 3456247 is not.) Show that $\sum_{k\geq 1} \frac{1}{n_k}$ converges to a number less than 80. What can you say about $\sum_{k\geq 1} \frac{1}{p_k}$ if $\{p_1, p_2, \dots\}$ be the collection of natural numbers that do not involve 4 in their decimal expansion. *Note:* This is an adaptation of ex 16 on p 263. You will have to read about the convergence of sequences of positive numbers from Section 3.7 and Ch 9, specially Ex. 3.7.2 (a) (geometric series), 3.7.6(b) (harmonic series) and the Comparison test 3.7.7

Project 7: The harmonic series and rearrangements. Show how to rearrange the terms of the alternating harmonic series $\sum_{n\geq 1}(-1)^{n+1}\frac{1}{n} = 1 - 1/2 + 1/3 - 1/4 + 1/5 - ..$ so that the series (a) converges to 2, and (b) diverges. (In a rearrangement you permute the order of the terms but not their signs, eg you might consider $1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 \dots$) Also discuss other patterns of signs in the harmonic series. eg can you decide if $1 - 1/2 - 1/3 + 1/4 - 1/5 - 1/6 + 1/7 - \dots$ converges? (Here signs are $+, -, -, +, -, -, + \dots$) What about $1 - 1/2 - 1/3 + 1/4 + 1/5 + 1/6 - 1/7 - \dots$? (Here there is one $+$, then two $-$, then three $+$ then four $-$ and so on.) What about one $+$, then two $-$, then four $+$ then eight $-$ and so on, ie. where you use powers of 2? *Note:* This is an example of a conditionally convergent sequence, ie. the corresponding sequence of positive terms does not converge. You will find relevant definitions on p 89, and p 255. cf also Ex 3.3.3(b), Ex 3.7.6(b).

Projects on Sets

Project 8: The Cantor set $\mathbb{F}$. This is a paradoxical subset of $[0, 1]$ discovered by Cantor. It is obtained from the unit interval by first removing the open interval $I_1 = (1/3, 2/3)$ (which leaves two intervals each of length 1/3), then removing the middle thirds of these two intervals (leaving four intervals each of length 1/9), then removing the middle thirds of these four intervals which gives you 8 intervals of length 1/27 and so on.... See p 316–318. Give a detailed description of this set $\mathbb{F}$ as an infinite intersection. Show it is closed (cf Def 11.1.2) and that all its points are cluster points. Explain its relation to ternary decimals, ie decimals to base 3. Show that its complement in $[0, 1]$ is the union of infinitely many disjoint open intervals of total length 1. So you might think you'd taken away all the points in $[0, 1]$. But in fact $\mathbb{F}$ has uncountably many points. Explain clearly and in detail why this is so.

Project 9: Countable and Uncountable sets. Give a variety of examples of Countable and Uncountable sets. For example, show that the set of sequences (x_n) where $x_n = 0, 1$, or 2 is uncountably infinite. Show that its subset consisting of sequences with only a finite number of nonzero entries is countably infinite. What about the set of sequences that are eventually periodic i.e. they are periodic if you ignore a finite set of terms at the beginning?

More theoretical projects

Uniform continuity and convergence are the most important concepts covered in MAT 320 and not in MAT 319. Projects 11 and 12 use the Weierstrass M-test (on p 268), which involves uniform convergence.

Project 10: Uniform continuity: Give examples of continuous functions that are not uniformly continuous. Explain why any continuous function defined on the interval $[a, b]$ is uniformly continuous. This concept is important when defining the Riemann integral: explain.

Project 11: Uniform convergence: Explain why the uniform limit of continuous functions is continuous (p 235). Give an example of a sequence of continuous functions that converges pointwise to a discontinous limit. Explain the Weierstrass M-test.

The next two projects concern examples whose discovery astounded the mathematics community of the day. There are many possible references for these examples, and you can use them if you prefer. The main thing is to describe the examples carefully and then explain why they have the properties claimed.

Project 12: A continuous nowhere differentiable function. Give a careful explanation of the example on p 354. Why is your function not differentiable at the point $1/\pi$? (You will need to understand the definition of the derivative from 6.1.1.)

Project 13: A space filling curve. Explain the example on p 355 — or you might be able to find another one online. Explain why your curve goes through an arbitrary point in the square, say $(1/\sqrt{2}, 1/sqrt3)$. Draw some pictures, using a computer graphics system if possible. This is related to fractal curves, snowflakes etc.

Various other projects

Project 14: The fundamental theorem of algebra. This says that any polynomial with complex coefficients has a complex root. Find a proof and explain it. How does your proof apply in the case of the polynomial $x^4 + 4x + 4$? (Why do you know this polynomial has no real roots?)

Project 15: Card Tricks. Martin Gardner has a lovely article in the *College Math Journal* (2000), 173–177, about card tricks. Pick one of the tricks and explain it fully. eg the "nonmessing up theorem" on p 173 or the curiosity on the bottom of p 175. Or you could do one of the challenge problems at the end.

Math 319: Definitions and Theorems for Midterm II

Def 3.1.3 A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms x_n satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called **divergent**.

Thm 3.1.10 Comparison theorem for limits. Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq C a_n$ for all $n \geq m$, then $\lim x_n = x$.

Thm 3.2.2 A convergent sequence of real numbers is bounded.

Thm 3.2.3 (a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively. Then the sequences $X + Y, X - Y, X \cdot Y$ and cX converge to $x + y, x - y, xy$ and cx respectively.
(b) Moreover if $y \neq 0$ and $y_n \neq 0$ for any n then X/Y converges to x/y.

Thm 3.2.6 If $X = (x_n)$ is a convergent sequence and $a \leq x_n \leq b$ for all n then $a \leq \lim x_n \leq b$.

Thm 3.3.2 Monotone Convergence Theorem. A monotone sequence of real numbers is convergent if and only if it is bounded.

Def 3.4.1 Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \ldots)$ is called a **subsequence** of X.

Thm 3.4.2 If $X = (x_n)$ converges to $x \in \mathbb{R}$ then every subsequence X' of X also converges to x.

3.4.5 Divergence Criterion If a sequence X has two convergent subsequences with different limits, then X is not convergent. If X is unbounded, then it diverges.

3.4.7: Monotone subsequence theorem. Every sequence has a monotone subsequence.

3.4.8: Bolzano–Weierstrass theorem. A bounded sequence of real numbers has a convergent subsequence.

Def 4.1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a **cluster point** of A if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Def 4.1.4. Let $A \subset \mathbb{R}$ and let c be a cluster point of A. A function $f : A \to \mathbb{R}$ is said to have **limit** L **at** c if for all $\epsilon > 0$ there is $\delta > 0$ such that

$$0 < |x - c| < \delta,\ x \in A \Longrightarrow |f(x) - L| < \epsilon.$$

Thm 4.1.8. Sequential criterion Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then $\lim_{x \to c} f = L$ iff for every (x_n) in $A \setminus \{c\}$ that converges to c the sequence $(f(x_n))$ converges to L.

Math 319 Review sheet for Second Midterm

This exam will have 5 questions each worth 10 points. The page of the definitions and theorems now posted will be attached to the exam. One question will ask you to prove a part of one of these theorems. You will have a choice here (see Q1 below.) I have tried to make the other questions very straightforward, like the easier homework problems.

Prove ONE of the following results: **(i)** Prove that a monotonic increasing sequence that is bounded above converges.

OR: **(ii)** Suppose (y_n) is a sequence in $\mathbb{R}$ such that $\lim y_n = 0$ and suppose that $|x_n - L| \leq 3|y_n|$ for all $n \geq 10$. Then $\lim x_n = L$.

2: Problem 2. Let $A = \left\{3 + \frac{1}{n} : n \geq 1\right\}$. Which points in $\mathbb{R}$ are cluster points of A? Prove all your claims from the definitions.

3: Suppose that (x_n) is a sequence such that the subsequence (x_{2n}) converges to 1 and the subsequence (x_{2n+1}) converges to 3. Show (from the definition of limit) that (x_n) is not convergent. (Do NOT just quote Thm 3.4.5.)

4: Which of the following sequences are monotonic? Which are convergent? Justify your answers.

$$\textbf{(i)} \qquad x_n = (-1)^n \cos(n\pi); \qquad \textbf{(ii)} \qquad x_n = \frac{n^2}{n+1}; \qquad \textbf{(iii)} \qquad x_n = \frac{\sin n}{n}.$$

5: (i) Give an example of a countably infinite subset of $\mathbb{R}$ that has precisely one cluster point.

(ii) Adapt your example in (i) so that the set has exactly two cluster points.

(iii) Give an example of two nonmonotonic sequences $(x_n), (y_n)$ with positive terms whose product is monotonic.

(iv) Give an example of a sequence that contains a nonconstant monotonic increasing subsequence and a nonconstant monotonic decreasing subsequence. Is there such a sequence that also converges?

Note: I haven't put any questions exactly like Q5 on the test since they are somewhat nonroutine and therefore hard to do in an exam. But I think this question is good practice for review.

6: Suppose that (x_n) is a convergent sequence with limit L and that $x_n \in [0, 2]$ for all n. Prove from the definitions that $L \in [0, 2]$.

7: Let $f : (0, 5) \to \mathbb{R}$ be the function $f(x) = 1/x^2$. Prove from the definition that $\lim_{x \to 2} f = \frac{1}{4}$.