MAT 319 "Foundations of Analysis" Fall 2003

GENERAL INFORMATION

Description and goals: A closer, more rigorous look at the fundamental concepts of one-variable calculus. The main focus will be on the key notions of convergence and continuity; the basic facts about differentiation and integration will be presented as examples of how these notions are used. The course provides a good opportunity for students to learn how to read and write rigorous proofs, and prepares them for further studies in analysis. MAT 319 and 320 will have joint lectures, recitations and workshops for the first 6 weeks of the semester. After that, students will split depending on the lecturers' recommendations and their aptitude and choice. Both courses are writing intensive. Students will have the opportunity to complete the proof-oriented component of the Department of Mathematics upper division writing requirement. In MAT 319, the last 6 Monday slots will be devoted to the presentation of projects. (NOTE: this schedule has been revised; the presentations will be on Mondays Nov 10 and Nov 24 in P131. We will use Dec 1 if necessary.) Each student will work on a project (typically with one other student) and jointly give a 20 minute presentation backed up by a written paper. Students will be expected to attend a total of 3 (revised to : BOTH) of these Monday sessions; either they will be presenting or they will be official "responders", expected to ask questions and give feedback after other people's presentations.

Professor: Professor Dusa McDuff.
Office Hours: Monday and Thursday 1:00-2:00pm. Or by appointment.
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The class will meet on Tuesday and Thursday 2:20-3:40 in ESS 181. There will be a recitation on Friday 12:50-1:45 in Math P131. (NOTE CHANGED CLASSROOM!) For information about what we will see in the class, you can review the syllabus.

Grading Policy: Homework 25%. Project 15%. Midterms (two) 15% each. Final Exam 30%.

Exam Schedule: Midterm 1: Thu Oct 2 (in class). Midterm 2:Thu Nov 20 (in class). Final: Thu Dec 18, 11:00-1:30.

HOMEWORKS & OTHER IMPORTANT DOCUMENTS (AFTER OCTOBER 6TH)

Homework will be posted on the web every Thursday and will be due the next Thursday at 5:00 pm. Doing homework is an absolutely essential part of this course. You are encouraged to work in groups and discuss the assignments with each other. But the work that you hand in must be your own write up. Please note that late homework will never be accepted. All documents posted in this section are in PDF format. The solutions will be available after the due dates.

Homeworkset 1	Solutions
Homeworkset 2	Solutions
Homeworkset 3	Solutions
Homeworkset 4 Due 10/7/03	Solutions
Homeworkset 5. Due 10/10/03	Solutions
Homeworkset 6. Due 10/16/03	Solutions
Homeworkset 7. Due 10/23/03	Solutions
Homeworkset 8. Due 10/30/03	
Homeworkset 9. Due 11/6/03	Solutions

Homeworkset 10. Due 11/13/03 Solutions
Homeworkset 11 Due 12/11/03

These are the worksheets discussed before the split.
Worksheet 1 2 3 4 .

ANNOUNCEMENTS

- The solutions to the Final review sheet is available as of now 7:33 pm on 12/16. Sorry if it is a bit late.
- The class on December 11 will be review. A review sheet is now posted.
- The solutions to Midterm 2 are now available.
- The review sheet for Midterm 2 is now available. It will be discussed in class on Tuesday Nov 18. Note also that there was a typo on Homeworksheet 10 that is now corrected.
- The class on Tuesday Nov 25 will be on the concept of area. It will contain a workshop that will be worth bonus HW points
- The first set of projects went very well. There will be two more on Monday Nov 24 and another two on Monday Dec 1. Both times at 12:50pm in Math P131.
- The class on Nov 11 will contain a short workshop on the current homework problems. There will be a review sheet instead of homework next week to help prepare you for the midterm. The class on Nov 18 is review.
- The class on Nov 4 will be about pancake cutting problems. It will end with a short workshop which will give you bonus HW grade points. The sheet about pancake cutting also contains a sketch of a correct proof of the Intermediate Value theorem (Lemma 22.5 in Lay.) I will hand out in class a version of this sheet with illustrations.
- The class on Nov 6 will be taught by Professor Phillips. It will be on differentiation. The syllabus will be changed: we will do the main aspects of differentiation and integration before the break to leave us time to discuss infinite processes (series) afterwards.
- It is time to start thinking about the project. We will discuss this more in class on Oct 21 and 23; you should sign up during this week. (You can email Prof. McDuff about this if you have questions.)
- Writing requirement: if you have not already completed a piece of writing with proofs, you can resubmit your revised Midterm I for consideration. Some of you have already written this down nicely, but others will have to recopy their work so as to provide complete coherent solutions. For your work to be acceptable, the answers must use your own words, and not be directly taken from the posted solutions. Please submit this before the Thanksgiving break. For those of you interested in completing another piece of writing: you will be able to use the project as a second piece.

FOR PEOPLE WITH DISABILITIES

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.ehs.stonybrook.edu/fire/disabilities.asp

MAT 319 "Foundations of Analysis" Fall 2003

SYLLABUS

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Date	Book Section	Topic
10/9	Section 14	Compact sets; Heine-Borel theorem
10/14	Section 14	Compact sets continued
10/16	Secs 14 and 16	Bolzano--Weierstrass theorem, sequences
10/21	parts of Sec 17,19	Sequences ctd
10/23 - 11/4	Sec 20, 21, 22	Limits and continuity
11/6,11,13	Chapter 6	The Derivative
11/18	Review	
11/20	Midterm 2	
11/25	Sec 29	The Riemann Integral
12/2, 12/4, 12/9	Secs 30 and 31	Integration
12/11	review	

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Last modified: 10/16/2003

Math 319/320 Homework 1

Problem 1. Write (in words) the negation of each of the following statements:

(i) Jack and Jill are good drivers.
(ii) All roses are red.
(iii) Some real numbers do not have a square root.
(iv) If you are rich and famous, you are happy.

Problem 2. Provide a counterexample for each of the following statements:

(i) For every real number x, if $x^2 > 4$, then $x > 2$.
(ii) For every positive integer n, $n^2 + n + 41$ is a prime number.
(iii) No real number x satisfies $x + \frac{1}{x} = -2$.

Problem 3. Recall from calculus that a function f is *increasing* when the following condition holds:

"For all real numbers x and y, if $x \leq y$, then $f(x) \leq f(y)$."

(i) Explain precisely what it means for a function *not* to be increasing.
(ii) Using (i), show that the function $f(x) = x^3 - 3x$ is not increasing.

Problem 4. Show that if $\frac{x}{x-1} \leq 2$, then $x < 1$ or $x \geq 2$. (Hint: Assume $\frac{x}{x-1} \leq 2$ and $x \geq 1$, and conclude that $x \geq 2$.)

Problem 5. Recall that $n! = 1 \cdot 2 \cdots (n-1)n$. Use mathematical induction to show that

$$n! > 2^n$$

for all integers $n \geq 4$.

Problem 6. Use mathematical induction to show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all integers $n \geq 1$. Can you find a direct, induction-free proof of this? (See if you can come up with a "trick" to simplify the sum on the left.)

Math 319/320 Homework 2

Due Thursday September 18, 2003

Problem 1. Prove the identity: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Problem 2. Let $f : A \to B$ be a function. Suppose that C and $\{C_j, j \in \mathbb{N}\}$ are subsets of A and D is a subset of B. Are the following statements true or false? Justify your answers by a brief proof or a counterexample.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

(iii) If f is injective, then $f(\bigcap_{j\in\mathbb{N}} C_j) = \bigcap_{j\in\mathbb{N}} f(C_j)$.

(iv) If f is surjective, then $f(\bigcap_{j\in\mathbb{N}} C_j) = \bigcap_{j\in\mathbb{N}} f(C_j)$.

Problem 3. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composition $g \circ f$ is injective. Is f necessarily injective? What about g? Give brief proofs or counterexamples.

Problem 4. We showed in class that the set of *positive* rational numbers is denumerable. Using this, deduce that the set of *all* rational numbers is denumerable. Give a complete argument that is based on Definition 8.6; do not quote other results from the book without proof.

Bonus Problem. Enumerate the set $\mathbb{Q} \cap [0, 1]$ of rational numbers between 0 and 1 as $q_1, q_2, \ldots, q_j, \ldots$. Find numbers $r_j > 0$ such that

$$[0, 1] \not\subset \bigcup_{j=1}^{\infty} (q_j - r_j, q_j + r_j).$$

This illustrates a surprising fact: The rationals are everywhere "dense" in the real numbers. Nevertheless, when we thicken them up by including each into a little interval, we still may not capture all the reals.

Math 319/320 Homework 3

Due Thursday September 25, 2003

Problem 1. Show that for all real numbers x and y,

$$||x| - |y|| \leq |x - y|.$$

This says that the distance between $|x|$ and $|y|$ is no more than the distance between x and y. (Hint: You should prove the inequalities $-|x-y| \leq |x| - |y| \leq |x-y|$. Both follow from the triangle inequality $|a+b| \leq |a| + |b|$ for appropriate choices of a and b.)

Problem 2. Let $(F, +, \cdot)$ be an ordered field. Carefully prove that

$$x^2 + 1 > 0 \qquad \text{for all } x \in F.$$

(Those of you who are familiar with the field $\mathbb{C}$ of complex numbers will realize that this shows $\mathbb{C}$ cannot be made into an ordered field, since $i^2 + 1 = 0$.)

Problem 3. In each case, find all the upper bounds (if any) and the least upper bound of $S \subset \mathbb{R}$:

- $S = \{a, b, c\}$, where $a > b > c$
- $S = \{n + (-1)^n : n \in \mathbb{N}\}$
- $S = \{x \in [0, \sqrt{10}] : x \text{ is rational}\}$
- $S = \{-\frac{3}{n} : n \in \mathbb{N}\}$

Problem 4. Are the following statements true or false? Justify your answers by a brief proof or a counterexample.

- If $\inf(A) = \sup(A)$, then A consists of a single point.
- If $\sup(A) = \sup(B)$ and $\inf(A) = \inf(B)$, then $A = B$.
- If $\inf(A)$ exists, then for every $\varepsilon > 0$ there is a point $x \in A$ such that $\inf(A) < x < \inf(A) + \varepsilon$.
- If b is an upper bound for S and $b \in S$, then $b = \sup(S)$.

Problem 5. Let A and B be non-empty subsets of the real line which are bounded above (so their supremums exist by the Completeness Axiom). Suppose that for every $x \in A$ there exists a $y \in B$ such that $x \leq y$. Show that

$$\sup(A) \leq \sup(B).$$

(Hint: Assume $\sup(A) > \sup(B)$ and get a contradiction.)

Math 319/320 Homework 4

Due Thursday October 2, 2003

Recap: For a given set $S \subset \mathbb{R}$, we define

- the *interior* of S as
$$\text{int}(S) = \{x : \text{ some neighborhood of } x \text{ is contained in } S\}$$
- the *accumulation set* of S as
$$S' = \{x : \text{every neighborhood of } x \text{ contains a point of } S \text{ other than } x\}$$
- the *closure* of S as
$$\text{cl}(S) = S \cup S'$$
- the *boundary* of S as
$$\{x : \text{every neighborhood of } x \text{ intersects both } S \text{ and } \mathbb{R} \setminus S\}$$

We have the following facts:

- S is open $\iff \mathbb{R} \setminus S$ is closed $\iff S = \text{int}(S)$
- S is closed $\iff \mathbb{R} \setminus S$ is open $\iff S = \text{cl}(S)$.
- Arbitrary unions and finite intersections of open sets are open. Arbitrary intersections and finite unions of closed sets are closed.

Problem 1. Classify the following subsets of $\mathbb{R}$ as open, closed, neither open nor closed, or both open and closed:

- $[-1, 1] \cup \{2\}$
- $\{x \in \mathbb{R} : \sin x \geq 0\}$
- $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right)$
- $\{x \in (0, 1) : x \text{ is irrational}\}$

Problem 2. Find $\text{int}(S)$, S', $\text{cl}(S)$ and $\text{bd}(S)$ for each set S in problem 1.

Problem 3. In each case, give an example of a non-empty set $S \subset \mathbb{R}$ with the corresponding property:

- $S = \text{bd}(S)$
- $S' = \text{bd}(S)$
- $\text{cl}(S) = \text{int}(S)$

Problem 4. Suppose S is a non-empty subset of $\mathbb{R}$ and x is an accumulation point of S. Show that every neighborhood of x contains infinitely many points of S. (Hint: What happens if some neighborhood of x contains only finitely many points of S?)

Bonus Problem. Does there exist a set $S \subset \mathbb{R}$, other than $\emptyset$ and $\mathbb{R}$, such that $\text{bd}(S) = \emptyset$?

Math 319/320 Review Sheet, Sept 26 2003

Here is a list of the topics that may be on the exam. Learn the important definitions; we will ask you to state some.

1. **Logic and techniques of proof**

   Quantifiers and negating quantified statements; proofs by contradiction and counterexample; mathematical induction

2. **Set theory**

   Sets and subsets, equality of sets; union and intersection, difference and complement, de Morgan's Laws; countable and uncountable sets

3. **Functions**

   Injectivity and surjectivity; image and preimage; composition and inverse

4. **The real number system $\mathbb{R}$**

   $\mathbb{R}$ as an ordered field; supremum and infimum, completeness axiom; Archimedean property of $\mathbb{R}$; density of rationals and irrationals in $\mathbb{R}$

5. **Topology of $\mathbb{R}$**

   Neighborhoods; definition of $\mathrm{int}(S)$; S is open iff $S = \mathrm{int}(S)$; $\mathrm{int}(S)$ is always open; arbitrary unions and finite intersections of open sets are open; S is closed iff $\mathbb{R} \setminus S$ is open; arbitrary intersections and finite unions of closed sets are closed; definition of the set S' of accumulation points of S; $\mathrm{cl}(S) = S \cup S'$; $\mathrm{cl}(S)$ is always closed; definition of $\mathrm{bd}(S)$; $\mathrm{cl}(S) = S \cup \mathrm{bd}(S)$

Sample Problems

1. Let A be the set of all continuous functions $f : [0,1] \to [-1,1] \subset \mathbb{R}$. Define the function $E : A \to \mathbb{R}$ by $E(f) = \int_0^1 f(x)\,dx$. Describe the set $E(A)$.

2. Define what is meant by an open subset of $\mathbb{R}$. Show that if A and B are open subsets of $\mathbb{R}$, then $A \cap B$ is also open. Give an example of an infinite collection of open sets $\{A_j\}_{j \in \mathbb{N}}$ such that $\bigcap_{j=1}^{\infty} A_j$ is not open.

3. True or false? Give a brief proof or counterexample.
   (i) Suppose that $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is surjective. Must f be surjective?
   (ii) The set $A = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$ has only one accumulation point.
   (iii) Suppose that b is a boundary point of a subset B of rational numbers. Then b itself is rational.
   (iv) Let A, B be subsets of $\mathbb{R}$ which are bounded above. Set $a = \sup A$, $b = \sup B$ and $c = \sup(A \cup B)$. Then $c = \max\{a, b\}$.

4. Write in words the negation of the following statement:

   "There exists an $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x + y^2 < 3$."

5. Prove the following statement (i) by a direct argument, and (ii) by proving the contrapositive. Is the converse statement true?

   "Let A be a non-empty subset of $\mathbb{R}$. If $b = \sup A$, then b is a boundary point of A."

Math 319 Homework 6

Due Thursday, October 16, 2003

Please base your answers on the definitions and facts mentioned on Homework 4.

Problem 1. Let S and T be subsets of $\mathbb{R}$. Show that
(i) $\mathrm{cl}(\mathrm{cl}(S)) = \mathrm{cl}S$;
(ii) $\mathrm{cl}(S \cup T) = \mathrm{cl}(S) \cup \mathrm{cl}(T)$;
(iii) $\mathrm{cl}(S \cap T) \subseteq \mathrm{cl}(S) \cap cl(T)$;
(iv) Find an example of two disjoint sets that have the same closure. This shows that we need not have equality in (iii).

Problem 2. **(i)** Let S be any subset of $\mathbb{R}$. Show that $\mathrm{cl}(S)$ is the smallest closed subset of $\mathbb{R}$ containing S, i.e. show that if C is any closed subset of $\mathbb{R}$ that contains S then C contains $\mathrm{cl}(S)$.
(ii) Given a subset S let $\overline{S}$ be the intersection of all the closed subsets containing S. Show that $\overline{S}$ is closed. Use (i) to show that $\overline{S} = \mathrm{cl}(S)$.

Problem 3. The Cantor set Let C be the middle third Cantor set as defined in Ex 14.11 in the book. Thus $C = \cap_n A_n$ where

$$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \quad A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1],$$

and in general A_k is obtained from A_{k-1} by removing the open middle third segments from each interval in A_{k-1}. Let $x = 0.a_1a_2a_3\ldots$ be the base 3 expansion of the number $x \in [0,1]$. Thus

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \ldots.$$

Note that $x \in A_1$ iff $a_1 = 0$ or $a_1 = 2$.
(i) Give a similar description of the 4 intervals that make up A_2, and the 8 intervals that make up A_3.
(ii) Show by induction on n that $x \in A_n$ iff $a_k = 0$ or 2 for all $k \in \{1, 2, \ldots, n\}$.
(iii) Deduce that $x \in C$ iff $a_n = 0$ or 2 for all $n \in \mathbb{N}$.
(iv) Find the ternary expansion of $x = 1/4$. Show that $1/4 \in C$.
(v) Describe the endpoints of the intervals in A_k in terms of the ternary expansion. Show that $1/4$ is not the endpoint of any such interval.

Problem 4. For each of the following sets S give an example of an open cover of S that has no finite subcover.
(i) $S = [0, 1)$
(ii) $S = \cup_{n \geq 1}[2n, 2n + 1]$
(iii) $S = \{x \in [0, 1] \ : \ x \text{ is rational}\}$.

Math 319 Homework 7

Due Thursday, October 23, 2003

Problem 1. Let S be a nonempty bounded subset of $\mathbb{R}$. Show that $\sup S$ is a boundary point of S. Show that $\inf S$ is also a boundary point. Deduce that if S is closed it has a maximum and minimum.
Note: this is the idea that was suggested in class; it gives another proof of Lemma 14.4.

Problem 2. (i) Show that if x is an accumulation point of S, every neighborhood of x contains infinitely many distinct points of S.

(ii) Deduce that there is a sequence (s_n) of elements in S that converges to x.

Problem 3. Prove the following statements using Definitions 16.2 and 17.9

(i) for each fixed integer $k \geq 1$

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

(ii)

$$\lim_{n\to\infty} \frac{2n-1}{n+1} = 2.$$

(iii)

$$\lim_{n\to\infty} \frac{2n^2-1}{n+1} = \infty.$$

Problem 4. Consider the sequence $s_n = \sin n$. Write down its first seven terms. Show that it has a subsequence that converges to 0 and another subsequence that converges to 1.
Hint: Try to find n so that n is "close" to $2\pi k$. Therefore consider $\frac{n-2\pi k}{k} = \frac{n}{k} - 2\pi$.

Problem 5. It is time to start work on your project. Pick a topic (from the ones outlined on the separate sheet) and a partner. Write in collaboration with your partner one paragraph on the topic you have selected. State a result that you hope to be able to prove in your presentation, and mention at least one reference on this topic other than Lay. Do this on a separate sheet of paper that you hand in to me Thursday Oct 23. We will have a brief discussion of the projects in class on Tuesday Oct 21. You will have a chance to choose the workshops you will attend, and sign up for them on Thursday Oct 23.

Math 319 Homework 8

Due Thursday, October 30, 2003

Main result this week:

• $\lim_{x\to c} f(x) = L$ if and only if $(f(s_n)) \to L$ for every sequence (s_n) that converges to c and has elements $s_n \neq c$ in the domain of f.

Problem 1. Show using Definition 20.1 that

$$\text{(i)} \qquad \lim_{x\to c} x^2 = c^2;$$

$$\text{(ii)} \qquad \lim_{x\to 2} x^2 - x + 1 = 3;$$

$$\text{(iii)} \qquad \lim_{x\to 1} \frac{x}{x^2 - x + 1} = 1.$$

Problem 2. Let $f : D \to \mathbf{R}$ and let c be an accumulation point of D. Suppose that $\lim_{x\to c} f(x) > 0$. Give two proofs that there is a deleted neighborhood U of c such that $f(x) > 0$ for all $x \in D \cap U$.
(a) Apply Definition 20.1 with suitable $\epsilon > 0$.
(b) Assume this is false, and construct a sequence $s_n \to c$ that does not satisfy the conditions in Theorem 20.13.

Problem 3. Define the function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that

$$\text{(i)} \quad \lim_{x\to 0} f(x) = 0; \qquad \text{(ii)} \quad \lim_{x\to 1} f(x) \text{ does not exist.}$$

Problem 4. True or false? Give a proof or a counterexample. The relevant theorems here are 17.1 and 17.13; be careful how you apply them!

(i) If (s_n) and $(s_n t_n)$ are convergent sequences then (t_n) converges.

(ii) If (s_n) and (s_n/t_n) are convergent sequences that both have nonzero limits then (t_n) converges.

(iii) If $x_n \to c$ then $x_n - c \to 0$.

(iv) Suppose $s_n \neq 0$ for all n. Then $\lim s_n = \infty$ if and only if $\lim s_n = 0$.

Math 319 Homework 9

Due Thursday, November 6, 2003

NOTE: Please read note about classes for week of Nov 4 which is on the schedule page.

Problem 1. (i) Suppose that the sequence s_n converges to L and that t_n is another sequence such that $|s_n - t_n| \to 0$. Prove that $t_n \to L$.
(ii) Give an example of sequences s_n, t_n such that $s_n \to 0$, $|s_n - t_n| \to 1$ but t_n does not converge to 1.
(iii) Suppose that s_n, t_n are as in (ii). Show that t_n has a convergent subsequence.

Problem 2. Define $f : [-\pi, \pi] \to \mathbf{R}$ by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

(i) Show that f has the intermediate value property; i.e. if $-1 \leq a < b \leq 1$ and c is any number between $f(a)$ and $f(b)$ then there is $x \in [a, b]$ such that $f(x) = c$.
(ii) Prove that f is not continuous at $x = 0$.

Problem 3. Let D be a subset of $\mathbf{R}$ and let $f : D \to \mathbf{R}$ be any continuous function. For each of the following, prove or give a counterexample.
(i) If D is an interval then $f(D)$ is an interval.
(ii) If D is bounded then $f(D)$ is bounded.
(iii) If D is bounded and closed then $f(D)$ is bounded and closed.
(iv) If D is open then $f(D)$ is open.
(v) If $D = [a, b)$ (a half open interval) and $f(D)$ is bounded then $f(D)$ is either a closed interval or it is a half open interval.

Problem 4. (A question arising in Thursday's lecture.) Assume the Bolzano–Weierstrass theorem: any infinite subset of a compact set D has an accumulation point. Prove that any sequence (s_n) of elements of a compact set D has a convergent subsequence.

Bonus Problem Explain the gap in the book's proof of Lemma 22.5.

Homework 9. Solutions of selected problems

November 10, 2003

Problem 3

(i) True. We will use a property that characterizes intervals. That is, $S \subset \mathbb{R}$ is an interval $\iff \forall a, b \in S, a < b$, $[a, b] \subset S$.

Now, assume D is an interval. We want to prove that $f(D)$ is also an interval. For that let A, B be any two points in $f(D)$. We need to prove that $[A, B] \subset f(D)$. By definition, there are $a, b \in D$ such that $f(a) = A, f(b) = B$. Now, since D is an interval $[a, b] \subset D$. Then, since tha function is continous and $[a, b]$ is compact, Theorem 22.10 asserts that $f([a, b]) = [m_1, m_2]$ where $m_1 = f(x_1), m_2 = f(x_2)$ are the minimum and maximum of f respectively. Since $f(a), f(b) \in [m_1, m_2]$ and since $[m_1, m_2]$ is an interval, $[f(a), f(b)] \subset [m_1, m_2] = [f(x_1), f(x_2)]$, therefore $[f(a), f(b)] \subset [f(x_1), f(x_2)] = f([a, b]) \subset f(D)$. This proves the statement.

(ii) False. Conterexample: take $f(x) = x^{-1}$ on the interval $(0, 1)$.

(iii) True. "Bounded and closed" is equivalent to "compact".

(iv) False. Counterexample: take $f(x) = \sin(x)$ and $D = (-2\pi, 2\pi)$.

(v) True. Let $D = [a, b)$ and assume $a < b$. $f(D)$ bounded implies that there exist the supremum $S := \sup f(D)$ and $I := \inf f(D)$. Thus $\forall x \in [a, b) I \leq f(x) \leq S$, i.e. $f(x) \in [I, S]$.

Let $\{x_n\}, \{y_n\} \in [a, b)$ sequences such that $\lim_n f(x_n) = S$ and $\lim_n f(y_n) = I$. Since $[a, b) \subset [a, b]$ and $[a, b]$ is compact, there are convergent subsequences (using the same notation!!!) $x_n \to x \in [a, b]$, $y_n \to y \in [a, b]$ in $[a, b]$. Since D is an interval $f(D)$ is also an interval. Therefore $f(D)$ must be one of the intervals $(I, S), (I, S], [I, S), [I, S]$. If $f(D) = (I, S)$, then $x, y \notin [a, b)$. The only possiblity is that $x = y = b$. But this implies that $I = S$ which gives a contradiction since $\phi \neq f(D) \subset (I, S) = \phi$.

Math 319 Homework 10

Due Thursday, November 13, 2003

Problem 1. Show that any continuous map $f : [a, b] \to [a, b]$ has a fixed point.

Hint: Consider the function $h(x) = f(x) - x$ and use the intermediate value theorem.

Problem 2. (i) Let $f : [a, b] \to [a, b]$ be continuous and surjective, and let $g = f \circ f$. Show that g has at least 2 different fixed points.

(ii) Given an example to show that the conclusion fails if f is continuous but not surjective.

Problem 3. (i) Let $I_1 \supset I_2 \supset I_3 \supset \ldots$ be a nested sequence of bounded closed intervals $I_k = [a_k, b_k]$ in $\mathbf{R}$. Show that their intersection $\cap_{k \geq 1} I_k$ is either a single point c or a closed interval $[a, b]$.

(ii) Suppose that $f : I_1 \to \mathbf{R}$ is a continuous function such that $f(a_k) < 0 < f(b_k)$ for all k. Show that f must vanish at some point of the intersection $\cap_k I_k$. Hence if $\cap_k I_k = \{c\}$ then $f(c) = 0$.

This completes the proof of Lemma 22.5 (on the pancake sheet).

Problem 4. (i) Use Definition 25.1 to find the derivative of the function $f(x) = \sqrt{x}, x > 0$.

(ii) For which values of x are the following functions differentiable? Find (by any means, you do not need to go back to the definition) a formula for the derivative wherever it exists.

$$(a)\ \ f(x) = |x - 1|; \qquad (b)\ \ f(x) = |x^2 - 1|, \qquad (c)\ \ f(x) = x|x|.$$

Math 319 Homework 11

Due Thursday, December 11, 2003

Problem 1. (i) Write down an integral that calculates the volume of a hemisphere H of radius r and evaluate it.
(ii) If you put this hemisphere into a cylinder C of base radius r and height r what proportion of the total volume of C does it fill up?

When we took a vote in class, 4 of you said that the cylinder filled up 2/3 of the volume and 8 said it filled up 3/4.
(iii) Now consider a cone A of base radius r and height r inside the cylinder C. Show that the volume of $C - A$ (the outside of the cone) is the same as the volume of H.

Problem 2. Construct a rearrangement of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

that converges to 0. (and justify your answer.)

Problem 3. Show that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \dots$$

diverges. Why does this not contradict the alternating series test?

Problem 4. Find the radius of convergence and interval of convergence for:

$$\sum_{n=1}^{\infty} \frac{3^n}{n} x^n, \qquad \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n.$$

Problem 5 Suppose that the sequence (a_n) is bounded but that the series $\sum a_n$ diverges. Prove that the radius of convergence of $\sum a_n x^n$ is 1.

Problem 6 Remove all terms that have the digit 1 in the denominator from the harmonic series, getting

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{9} + \dots + \frac{1}{20} + \frac{1}{22} + \dots + \frac{1}{99} + \frac{1}{200} + \frac{1}{202} + \dots.$$

Show that the resulting series converges (!) and has sum < 40. **Hint:** Show that there are 8×9 terms with 2 digits in the denominator, $8 \times 9 \times 9$ with 3 digits in the denominator, and so on. Then estimate (very roughly) the size of the terms.
Bonus points will be given for a significantly better estimate of this sum.

Math 319/320 Worksheet 1

Name: **School ID:**

Problem 1. Negate the following statement:

"If your glass is half-empty, you are a pessimist."

Give your answer in words.

Problem 2. The context in this problem is the set of all human beings. Recall the symbols

$$\forall = \text{ for all}, \quad \exists = \text{ there exists}, \quad \wedge = \text{ and}, \quad \vee = \text{ or}, \quad \sim = \text{ not}.$$

Let $P(x)$ be "x is educated," $Q(x)$ be "x is female" and $R(x)$ be "x is older than 30." Then the statement "every uneducated male is older than 30" can be expressed as

$$\forall x : (\sim P(x) \wedge \sim Q(x)) \implies R(x)$$

Express the following statements in a similar way:

(i) Some educated people are younger than 30.

(ii) Every female who is older than 30 is educated.

(iii) No uneducated person is both female and older than 30.

Problem 3. Consider the statement

"For every natural number n, if n^2 is even, then n is even."

Prove this statement in two different ways: (i) by showing that its contrapositive is true; (ii) by showing that its negation is false.

Problem 4. On a bumper sticker, I saw the statement

"For every real number x, there is a real number t such that $t(1-t) > x$."

After some thought, I decided that this statement must be __________. To prove this carefully, I found a real number ____ such that for every real number ____ the inequality ____________________ held.

Math 319/320 Worksheet 2

Problem 1. Fill in the blanks in the following proof of

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

If $x \in A \cup (B \cap C)$, then either $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in$ __________, so $x \in$ ____________________. On the other hand, if $x \in B \cap C$, then $x \in$ __________ and $x \in$ __________, so again $x \in$ ____________________. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Now suppose $x \in$ ____________________. Then $x \in$ __________ and $x \in$ __________. Consider two possibilities: If $x \in A$, then

On the other hand, if $x \notin A$, then

Therefore

Problem 2. Let $\{A_j : j \in \mathbb{N}\}$ and $\{B_j : j \in \mathbb{N}\}$ be two families of sets indexed by the set of positive integers $\mathbb{N}$. Is it true that

$$\bigcup_{j\in\mathbb{N}}(A_j \setminus B_j) = (\bigcup_{j\in\mathbb{N}} A_j) \setminus (\bigcup_{j\in\mathbb{N}} B_j)?$$

Give a proof or counterexample.

Problem 3. Let $f : A \to B$ be a function, $C \subset A$ and $D \subset B$. Show that

$$C \subset f^{-1}(f(C)) \qquad \text{and} \qquad f(f^{-1}D) \subset D.$$

If f is injective, do either of these inclusions become equalities?

What if f is surjective?

Problem 4. Let A and B be subsets of a universal set U. Simplify the following expressions. It might be helpful to draw Venn diagrams.

(i) $(A \cap B) \cup (U \setminus A)$

(ii) $A \cup [B \cap (U \setminus A)]$

Math 319/320 Worksheet 3

Name: **School ID:**

Problem 1. Consider a non-empty set $S \subset \mathbb{R}$. When should we call S "bounded below"? Formulate the definition of a "lower bound" and the "greatest lower bound" for S. The greatest lower bound of S, if exists, is called the *infimum* of S and is denoted by $\inf(S)$.

Problem 2. Find $\sup(S)$ and $\inf(S)$ in each case. Justify your answers.

- $S = \left\{ \dfrac{(-1)^n}{n} : n \in \mathbb{N} \right\}$

- $S = (-1, 0] \cup [1, 2] \cup \{3\}$.

Problem 3. The real number system has the *Archimedean property (AP)*, which can be formulated as follows:

"For every real number $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \dfrac{1}{n} < x$."

Below we show that (AP) is a consequence of the Completeness Axiom for $\mathbb{R}$. Fill in the blanks:

Suppose (AP) fails. Then we can find some __________ such that _______________ for all $n \in \mathbb{N}$. In other words, the real number $1/x$ should be an _______________ for $\mathbb{N}$. By _______________________, $\mathbb{N}$ must have a least upper bound $b \in \mathbb{R}$. Since _______________, $b - 1$ cannot be an upper bound for $\mathbb{N}$, so there must be an $n \in \mathbb{N}$ such that _______________. But this means _______________, which contradicts the definition of $b = \sup(\mathbb{N})$. The contradiction shows that (AP) must hold.

Problem 4. Using the Archimedean property, carefully show that if

$$S = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\},$$

then $\sup(S) = 1$.

Math 319/320 Worksheet 4

Name: **School ID:**

Problem 1. Find $\text{int}(S)$, S', $\text{cl}(S)$ and $\text{bd}(S)$ if

- $S = [0,1] \cup (1,2)$
- $S = \bigcap_{n=1}^{\infty} \left[0, 1 + \frac{2}{n}\right)$
- $S = [0, +\infty) \cap \mathbb{Q}$

Problem 2. True or false? Give a short proof or counterexample.

- If S is open and T is closed, then $S \setminus T$ is open.

- If S is not open, then S is closed.

- If S is closed, then $S' \subset S$.

Problem 3. Let $S \subset \mathbb{R}$ be non-empty. Show that $\mathrm{int}(S)$ is an open set.

Math 319 Review Sheet for Final: Solutions

Question 1. State Rollé's Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and noninjective. Show that there is a point $c \in \mathbb{R}$ such that $f'(c) = 0$.

Solution: Since f is non injective there are points $a < b \in \mathbb{R}$ such that $f(a) = f(b)$. Since f is continuously differentiable we may apply Rollé's theorem to the restriction of f to the interval $[a, b]$. Hence there is $c \in (a, b)$ such that $f'(c) = 0$.

Question 2. State the Mean Value Theorem. Use it to show that $\sin x \leq x$ for $x \geq 0$. (You can use any familiar differentiation formulas you want.)

Solution: Let $f(x) = \sin(x)$. We argue by contradiction. Note that $f(0) = 0$. Therefore the given condidition holds for $x = 0$. Therefore, if the result is false there is $x > 0$ such that $f(x) > x$. Since f is continuously differentiable we may apply the Mean Value Theorem to the interval $[0, x]$ to conclude that there is $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} > 1,$$

by choice of x. But $f'(y) = \cos(y) \leq 1$ for all y. Therefore such a c does not exist. A contradiction. Hence the result must be true.

Question 3. (i) Sum the series $2 - 2/3 + 2/9 - 2/27 + \ldots$.
Solution: This is a geometric series with $a = 2$ and $r = -\frac{1}{3}$. Hence the sum is

$$S = \frac{a}{1 - r} = \frac{2}{4/3} = \frac{3}{2}.$$

CHECK: the answer should be < 2 and $> 2 - 2/3 = 4/3$.

(ii) Write out the first five terms of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n + 1}.$$

Does it converge? Explain why or why not.

Solution: First five terms are

$$-\frac{1}{3} + \frac{2}{5} - \frac{3}{7} + \frac{4}{9} - \frac{5}{11} + \ldots$$

This does not satisfy the requirements of the alternating series test because $|a_n| \not\to 0$. In fact $\frac{n}{2n+1} \to \frac{1}{2}$.

This observation is enough to show that the series cannot converge: there is a theorem stating that if the series $\sum_n b_n$ converges then the sequence b_n must converge to 0.

Question 4. (i) State the ratio test for the convergence of a series $\sum_{n \geq 1} a_n$. Give the formula for the radius of convergence of the power series $\sum_{n \geq 1} a_n x^n$.

Solution: Ratio test: If $L := \lim \frac{|a_{n+1}|}{|a_n|}$ exists and is < 1 then the series $\sum_{n\geq 1} a_n$ converges.

Assuming that L exists as above (but it needn't be < 1) then the radius of convergence R is $1/L$ if $L \neq 0, \infty$, and it is ∞ if $L = 0$. If the sequence $\frac{|a_{n+1}|}{|a_n|}$ diverges to ∞ then $R = 0$.

(ii) Find the radius of convergence:

$$(i)\ \sum_{n\geq 1} \frac{n!}{2^n} x^n, \qquad (ii)\ \sum_{n\geq 1} \frac{n}{3^n} x^n, \qquad (iii);\ \sum_{n\geq 1} \frac{2^n}{n!} x^n.$$

Solution: For (i):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{2^{n+1}} \frac{2^n}{n!} = \frac{n+1}{2} \to \infty, \quad R = 0.$$

For (ii):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n+1}{3^{n+1}} \frac{3^n}{n} = \frac{3(n+1)}{n} \to 3, \quad R = \frac{1}{3}.$$

For (iii):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \frac{2}{n+1} \to 0, \quad R = \infty.$$

Question 5. (i) Define what it means for the sequence (s_n) to converge.
(ii) Suppose that $s_n \to L$. Show from the definition that $s_n - s_{n+1} \to 0$.
Solution: To show that $s_n - s_{n+1} \to 0$ we must find for every $\epsilon > 0$ an N such that

$$n \geq N \Longrightarrow |s_n - s_{n+1}| < \epsilon.$$

Now $|s_n - s_{n+1}| = |s_n - L + L - s_{n+1}| \leq |s_n - L| + |L - s_{n+1}|$. Since $s_n \to L$, for all $\epsilon > 0$ there is N such that $|s_n - L| < \epsilon/2$ when $n \geq N$. With this N we have

$$n \geq N \Longrightarrow |s_n - s_{n+1}| \leq |s_n - L| + |L - s_{n+1}| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as required.

The next parts of this question are good review: I would not put them in this form on the exam!

(iii) Deduce the result in (ii) from Theorem 17.1 (a) and (b).
Solution: Assume that (s_n) and (t_n) are convergent. The Theorem says that (s_n+t_n) and (cs_n) are also convergent (where $c \in \mathbb{R}$) and that $\lim(s_n + t_n) = \lim s_n + \lim t_n$, and $\lim cs_n = c \lim s_n$. Hence taking $c = -1$ we see that $\lim(s_n - t_n) = \lim s_n - \lim t_n$. Apply this with the given sequence (s_n) which converges to L and with $t_n := s_{n+1}$. Then $t_n \to L$.

NOTE: You can either treat this as obvious — it follows immediately from the defn of convergence — or you can notice that (t_n) is a subsequence of (s_n) and so converges to L by the theorem about subsequences. (I don;t have the book with me here to get the precise reference.

Going back to the proof: we now have $s_n - s_{n+1} = s_n - t_n$ and this converges to $\lim s_n - \lim t_n = L - L = 0$ by the theorem.

(iv) Suppose that s_n is the nth partial sum of a convergent series $\sum_{n\geq 1} a_n$. Use (ii) to show that $a_n \to 0$.
Solution: By definition $s_n = \sum_{k=1}^n a_k$. Therefore $s_n - s_{n+1} = -a_{n+1}$. Hence $-a_{n+1} \to 0$. Multiplying by $c = -1$ and renumbering we find $a_n \to 0$.

(v) How does (iv) help with Question 3 (ii)?

Solution: This proves the criterion for convergence that was used in 3(ii).

Question 6 State what it means for $f : [a, b] \to \mathbb{R}$ to be continuous at $c \in (a, b)$. Show that $\lim_{x\to 2} \frac{1+x}{2+x} = 3/4$.

Solution: Let $x = 2 + h$. Consider

$$|f(2+h) - f(2)| = \left|\frac{3+h}{4+h} - \frac{3}{4}\right| = \left|\frac{4(3+h) - 3(4+h)}{4(4+h)}\right| = \left|\frac{h}{4(4+h)}\right| < \left|\frac{h}{4\times 3}\right|,$$

if $|h| < 1$ so that $4 + h > 3$. Given $\epsilon > 0$ choose $|\delta| \leq \min(1, \epsilon)$. Then, because $\delta < 1$,

$$|h| < \delta \Longrightarrow |f(2+h) - f(2)| < \frac{|h|}{12} < \frac{|\delta|}{12}$$

which is $< \epsilon$ by choice of δ. Hence $\lim_{x\to 2} \frac{1+x}{2+x} = 3/4$.

Question 7 Let A, B be subsets of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. Define the inverse image $f^{-1}(A)$. Show that $f\big(f^{-1}(A)\big) \subset A$. If f is injective is $f\big(f^{-1}(A)\big) = A$ always? What about the case when f is surjective?

Solution: $f^{-1}(A) = \{x \in \mathbb{R} : f(x) \in A\}$. Therefore if $x \in f^{-1}(A)$ $f(x) \in A$. This implies that $f\big(f^{-1}(A)\big) \subset A$.

The set $f\big(f^{-1}(A)\big)$ is always contained in the image of f. Therefore if f is not surjective, ie.e if $f(\mathbb{R}) \neq \mathbb{R}$ we can always find A such that $f\big(f^{-1}(A)\big) \neq A$. For example take

$$A = \mathbb{R} \setminus f(\mathbb{R}) = \big\{y \in \mathbb{R} \,|\, \nexists x \in \mathbb{R} : f(x) = y\big\}.$$

Thus it is irrelevant whether f is injective. On the other hand, if f is surjective, then for all $y \in A$ there is $x \in \mathbb{R} : f(x) = y$; then $x \in f^{-1}(A)$ and so $f(f^{-1}(A)) = A$. Hence there is equality when f is surjective.

The next question reviews some important concepts. It is too hard for the exam.

Question 8 Let $A \subset [0, 1]$ be a denumerable set, i.e. it is countable and has infinitely many elements. Define $f : \mathbb{R} \to \mathbb{R}$ by setting $f(x) = 1$ is $x \in A$, $f(x) = 0$ if $x \notin A$.

Show that that there is a point $c \in [0,1]$ at which f is not continuous. **Hint:** First do this with $A = \{1/n : n \geq 1\}$. What theorem tells you something about A? Apply this to do the general case.

Solution: As we discussed in class, this question is incorrectly stated. I should have asked for a point c where $\lim_{x \to c} f(x)$ does not exist. Let is suppose that this is the question. Use the Bolzano–Weierstrass theorem to find an accumulation point $c \in [0,1]$ of the bounded infinite set A. Then I claim that $\lim_{x \to c} f(x)$ does not exist. **Proof:** for each n consider the set $A \cap (c - 1/n, c + 1/n)$. Since c is an accumulation point of A this set is nonempty. Therefore there is $x_n \in A \cap (c - 1/n, c + 1/n)$. But because A is denumberable and the reals are uncountable, there also is $y_n \in \big((c - 1/n, c + 1/n) \setminus A\big)$. Then $x_n \to c$ and $y_n \to c$ by construction. But $f(x_n) = 1$ and $f(y_n) = 0$. Therefore there is no single number L such that $f(s_n) \to L$ for *every* sequence $s_n \to c$. Hence $\lim_{x \to c} f(x)$ does not exist.

NOTE: with the question as given there is no need to use special properties of A. f is not continuous at all the points of the closure of A.

Math 319 Review Sheet for Final

The format of the final will be much like Midterm 2. You will be asked to state definitions and theorems and to do short calculations and make short proofs like those on the homework and review sheets. There will be 9 questions each worth 10 points of which you should do 8. Some will be easier thsan others.

Syllabus: Selected parts of that covered by Midterms 1 and 2 plus Mean Value Theorem and series.

Most important definitions and concepts:
function, injective, surjective;
countable and uncountable sets;
supremum, infimum, completeness axiom;
open and closed sets; boundary and accumulation points;
convergence of sequences; subsequences;
continuity and differentiability for functions
convergence of series; geometric, harmonic, alternating series
power series, and radius of convergence

Most important theorems Note: Unless you are told you must work form the definition you are allowed to quote these theorems in your answers (provided you state them clearly.)
Bolzano–Weierstrass theorem for sets (14.6) and sequences (19.7);
convergent sequences have unique limits and are bounded;
criteria for nonconvergence: cf Thm (19.4) and ex(19.6) for sequences and Thm (20.10) for limit of function;
continuous functions are bounded and have max and min values on closed intervals;
Intermediate Value theorem;
differentiable functions are continuous;
Rolle's Theorem and Mean Value Thorem; applications of MVT to get estimates (as in exercise 26.5);
comparison test for convergence (Thm 33.1 (a)); ratio test (Thm 33.8);
radius of convergence of power series (Thm 34.3 with $R = \lim \frac{a_{n+1}}{a_n}$ (assuming this exists).

Sample questions All the sample questions on the other review sheets are relevant. Some of the questions below are longer than those on the actual exam.

Question 1. State Rollé's Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and noninjective. Show that there is a point $c \in \mathbb{R}$ such that $f'(c) = 0$.

Question 2. State the Mean Value Theorem. Use it to show that $\sin x \leq x$ for $x \geq 0$. (You can use any familiar differentiation formulas you want.)

Question 3. (i) Sum the series $2 - 2/3 + 2/9 - 2/27 + \ldots$.

(ii) Write out the first five terms of the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{2n+1}.$$

Does it converge? Explain why or why not.

Question 4. (i) State the ratio test for the convergence of a series $\sum_{n\geq 1} a_n$. Give the formula for the radius of convergence of the power series $\sum_{n\geq 1} a_n x^n$.

(ii) Find the radius of convergence:

$$\sum_{n\geq 1} \frac{n!}{2^n} x^n, \qquad \sum_{n\geq 1} \frac{n}{3^n} x^n, \qquad \sum_{n\geq 1} \frac{2^n}{n!} x^n.$$

Question 5. (i) Define what it means for the sequence (s_n) to converge.
(ii) Suppose that $s_n \to L$. Show from the definition that $s_n - s_{n+1} \to 0$.

The next parts of this question are good review: I would not put them in this form on the exam!
(iii) Deduce the result in (ii) from Theorem 17.1 (a) and (b).
(iv) Suppose that s_n is the nth partial sum of a convergent series $\sum_{n\geq 1} a_n$. Use (ii) to show that $a_n \to 0$.
(v) How does (iv) help with Question 3 (ii)?

Question 6 (i) State what it means for $f : [a, b] \to \mathbb{R}$ to be have a limit at $c \in (a, b)$.

(ii) Show that $\lim_{x\to 2} \frac{1+x}{2+x} = 3/4$.

Question 7 Let A, B be subsets of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. Define the inverse image $f^{-1}(A)$. Show that $f\big(f^{-1}(A)\big) \subset A$. If f is injective is $f\big(f^{-1}(A)\big) = A$ always? What about the case when f is surjective?

The next question reviews some important concepts. It is too hard for the exam.

Question 8 (i) State what it means for $f : [a, b] \to \mathbb{R}$ to be continuous at $c \in (a, b)$.
(ii) Let $A \subset [0, 1]$ be a denumerable set, i.e. it is countable and has infinitely many elements. Define $f : \mathbb{R} \to \mathbb{R}$ by setting $f(x) = 1$ is $x \in A$, $f(x) = 0$ if $x \notin A$. Show that that there is a point $c \in [0, 1]$ at which f is not continuous. **Hint:** First do this with $A = \{1/n : n \geq 1\}$. In the general case, what theorem tells you something relevant about A that will help you find the point c?

Math 319 Second Midterm. Solutions

November 20, 2003

We will only solve the second part of each problem.

Problem 1. (ii) Show that every polynomial $a_0 + a_1x + \cdots + a_kx^k$ of odd degree and with real coefficients $a_0, \ldots, a_k$ has at least one real root.
Solution: The first thing to note is that polynomials are continuous functions. Let $f(x) = a_0 + a_1x + \cdots + a_kx^k$. Now, since k is a n odd number, if $x < 0$ then $x^k < 0$ and if $x > 0$, $x^k > 0$. Without lost of generality, we can assume $a_k > 0$, otherwise we can multiply by -1. Let $x_1 << 0$ (this means x_1 is really big in the negative direction), so that $a_0 + a_1x_1 + \cdots + a_kx_1^k \simeq a_kx_1^k$. Since $x_1^k > 0$, we have $f(x_1) = a_0 + a_1x_1 + \cdots + a_kx_1^k < 0$. Analogously consider $x_2 >> 0$, so that $a_0 + a_1x_2 + \cdots + a_kx_2^k \simeq a_kx_2^k$. Then $f(x_2) > 0$. Then $f(x_1) < 0 < f(x_2)$. By the intermediate value theorem, there is a number c such that $x_1 < c < x_2$ and $f(c) = 0$.
Problem 2.
(ii) Guess the limit

$$\lim_{x\to 1}\left((x-1)\sin\frac{1}{x-1}\right).$$

Then prove your guess is correct. **Solution:** We claim $\lim_{x\to 1}\left((x-1)\sin\frac{1}{x-1}\right) = 0$. For that, let $\epsilon > 0$. Now, consider $\delta = \epsilon$. Note that $\forall x \in \mathbb{R}, |\sin(x)| \leq 1$. Therefore, if x is such that $|x-1| < \delta$, we have

$$|(x-1)\sin\frac{1}{x-1}| < |x-1| < \delta = \epsilon.$$

This proves the claim.
Problem 3. (ii) Give a careful proof that any convergent sequence is bounded.
Solution: Let x_n a convergent sequence, say $x_n \to x$. Consider $\epsilon = 1$. Then, there is a $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n > N$. Hence $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$ for $n > N$. Take $M = \max\{x_1, x_2, \ldots, x_N, 1 + |x|\}$. It is clear that $\forall n \in \mathbb{N}\ \ |x_n| < M$.
Problem 4. (ii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x) = 0$ for all *rational* Numbers x. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.
Solution: Let x be any real number. Since the rational numbers are *dense* in the reals we have that there is a sequence $\{x_n\}$ of rational numbers that converges to x. Thus, since all the x_n are rationals $f(x_n) = 0$. Then, $\lim_{n\to\infty} f(x_n) = 0$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x)$. It follows that $f(x) = 0$.
Problem 5. (ii) Define $f : (-1, 1) \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ x & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is not differentiable at $x = 0$.
Solution: To prove that f is not differentiable at $x = 0$, it is enough to prove that the limit

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

does not exist. To see that, let $x_n \to 0$, $y_n \to 0$ be two convergent sequences where $\forall n, x_n \neq 0$ is rational and $y_n \neq 0$ is irrational. Then

$$\lim_{n\to\infty}\frac{f(x_n)}{x_n} = \lim_{n\to\infty}\frac{0}{x_n} = 0, \qquad \text{and} \quad \lim_{n\to\infty}\frac{f(y_n)}{y_n} = \lim_{n\to\infty}\frac{y_n}{y_n} = 1.$$

This proves that the limit does not exist.

Math 319 Review Sheet for Midterm 2

This exam will cover sections 14, 16, 17, 20, 21, 22, 25. This is a lot of material and I do not expect you to know it all. Section 13 is basic; I will not examine it specifically, but your should know all its definitions (especially accumulation point and the fact that a set S is closed iff its contains all its accumulation points.) Here are the main points. The exam will ask you to state some definitions and theorems and also to prove some theorems, but only from the list below. Other questions will be calculations. Concentrate on understanding the questions from Homeworks 7 through 10, specially the easier ones. Solutions will be posted very soon.

Section 14 Know the statements (not proofs) of Theorems 14.5 (Heine–Borel) and 14.6 (Bolzano–Weierstrass) and Lemma 14.4. There will be no questions about open covers. **Section 16** Def 16.2 and its use in examples; statements and proofs of Thms 16.8, 16.13, 16.14. Examples 16.5, 16.6, 16.12.
Section 17 Thm 17.1 (statement and proof of a,b,c) and its use in examples Ex 17.2
Section 19 Subsequence; statement and proof of Thms 19.4, 19.7
Section 20 Defn 20.1; Thms 20.8 and 20.10 (statement and proof); Ex 20.5, 20.6
Section 21 Defn 21.1 for $D =$ interval and Thm 21.2 (d) (statement), examples of continuous and discontinuous functions.
Section 22 Thm 22.2 and Cor 22.3 (statement; take compact to mean "closed and bounded"); Thm 22.6 (statement)
Section 25 Defn 25.1 and use in examples; Thm 25.6 (statement and proof)
Question 1. Let $f : [0, 2] \to \mathbb{R}$ be a function and $c \in (0, 1)$. Define what it means for $\lim_{x\to c} f(x)$ to exist. Use this definition to show that $\lim_{x\to 1} 2x^2 + x + 1 = 4$. There could be a very similar question about convergence of sequences or the derivative of a function.
Question 2. Define what it means for the sequence (s_n) to converge. Suppose that $s_{2n} = \frac{1}{n}$ for all $n \geq 1$ while $s_{2n+1} = 1 + \frac{1}{n}$ for all $n \geq 1$. Use the definition to show that (s_n) does not converge.
Question 3. State the intermediate value theorem for continuous functions. Use it to prove that there are at least two values of x such that $\sin x = x^4 - 4$. You may assume that $\sin x$ is a continuous function.
Question 4. Suppose that $s_n \geq 0$ for all $n \geq 1$ and that $s_n \to L$. Show that $L \geq 0$.
Question 5 Let $f : [a, b] \to \mathbb{R}$ be continuous at $c \in (a, b)$. Show that f is bounded on some neighborhood of c.
Question 6 Suppose that $f : \mathbb{R} \to \mathbb{R}$ does not have a limit at point c. Show that there is a sequence $(s_n) \to c$ such that $f(s_n)$ does not converge. (This is half of Thm 20.8.)
Question 6 Define the derivative of f at the point c. Prove that the function $f(x) = |x|$ is not differentiable at the point $x = 0$.

MAT 319: Pancake cutting and other matters
November 2, 2003

First a result about functions on the circle S^1. We may identify the circle with the real numbers modulo 2π. Thus the point on the circle with coordinate t has (x, y) coordinates $(\cos t, \sin t)$. Therefore its antipodal point $(-x, -y)$ has coordinate $t+\pi$. A function on the circle $t \mapsto f(t)$ may be thought of as a **periodic** function on $\mathbf{R}$, i.e. a function such that $f(t+2\pi) = f(t)$. Therefore we know what continuity means. This is equivalent to saying that a neighborhood of a point t on the circle is a little open arc containing t, ad then using the neighborhood definition of continuity.

Informally, the next lemma says that any continuous real valued function defined on the circle that takes opposite values on antipodal points must be zero somewhere.

Lemma 1: *Let $f : S^1 \to \mathbf{R}$ be a continuous function on the circle such that $f(t) = -f(t+\pi)$. Then there is $z \in S^1$ such that $f(z) = 0$.*
Proof: Take any point t_0. If $f(t_0) = 0$ we have found the point we want. If not, one of $f(t_0), f(t_0+\pi) = -f(t_0)$ is positive and the other is negative. Therefore the result follows by applying the intermediate value theorem to the restriction of f to one of the arcs joining t_0 to $t_0 + \pi$.

Borsuk–Ulam theorem *If $f : S^1 \to \mathbf{R}$ is continuous, there is some point t_0 such that $f(t_0) = f(t_0 + \pi)$.*

This says that if you imagine measuring the temperature at all points on the earth on the great circle through the poles and the place where we stand, there is a pair of antipodal points with precisely the same temperature.

Proof: Consider the function $g(t) = f(t) - f(t+\pi)$. This is continuous and has the property that $g(t) = -g(t+\pi)$. By Lemma 1, g vanishes somehwere. At this point t_0, $f(t_0) = f(t_0 + \pi)$.

Pancake cutting theorem *Suppose given two pancakes (bounded regions of the plane). Then it is possible to divide them both into parts of equal area by one cuut of the knife.*

For example, if the pancakes are circular, cut along the line joining their centers. We will assume that the pancakes are regions A, B of the plane that do have a well defined area. (I am not attempting to define **area**, which one would have to do using the notion of integration.)
Proof: Enclose both regions (pancakes) A, B by a large circle C centered at O. Define a function $f : C \to \mathbf{R}$ as follows. For each point $t \in C$ consider the diameter Ot and make cuts perpendicular to this diameter. Exactly one of these cuts will divide the pancake A into two equal halves. This cut will divide B into two, usually unequal

pieces which we label B_1 and B_2, naming B_1 so that it is on the same side of the cut as is the point t. Then define

$$f(t) = \text{area}(B_1) - \text{area}(B_2).$$

"Clearly" $f(t)$ is a continuous function of t: as t changes the areas only change a little. (I am not going to prove this.)

Now let us compare $f(t)$ with $f(t+\pi)$. We get the same diameter, and hence the same cut divides A in half, but now the labels B_1 and B_2 are interchanged. Therefore

$$f(t+\pi) = \text{area}(B_2) - \text{area}(B_1) = -f(t).$$

Hence by the Lemma there is a point where $f(t) = 0$. ie the cut divides both A and B in half!.

Questions

Problem 1. Suppose you just have one (very irregular) pancake – imagine the cook is a klutz. Show that it is possible to divide it into 4 equal parts by two perpendicular cuts.

You will get bonus HW points for doing Problem 1 and handing it in today.

There is a similar argument in Lay showing that any region (with smooth boundary) in the plane can be enclosed by a square.

Problem 2. Imagine now that you have three muffins in space. Show that you can divide all of them in half by one knife cut, if you just choose the right direction. **Hint:** You need to generalize Lemma 1 to deal with the case of two functions f, g on the sphere such that $f(-p) = -f(p)$ and $g(-p) = -g(p)$ where $-p$ is the point antipodal to p. (We can discuss this in class.)

Corrected proof of Lay: Lemma 22.5 (Intermediate Value theorem)

If $f : [a,b] \to \mathbf{R}$ is continuous and such that $f(a) < 0 < f(b)$, then there is $c \in (a,b)$ such that $f(c) = 0$.

First, let's describe a subdivision process. We are given an interval $I_0 = [a_0, b_0]$ such that $f(a_0) < 0$ and $f(b_0) > 0$. Let m be the midpoint of $[a_0, b_0]$. If $f(m) = 0$, stop: we have found the point we want. If not, $f(m)$ is either positive or negative. If $f(m) > 0$ set $I_1 := [a_0, m]$, if $f(m) < 0$ set $I_1 := [m, b_0]$. Relabel the endpoints of I_1 as a_1, b_1. Then $f(a_1) < 0$ and $f(b_1) > 0$. Now repeat the process, obtaining intervals $I_2 = [a_2, b_2], \ldots, I_k = [a_k, b_k], \ldots$.

If the process stops we have a zero. Otherwise we get an infinite sequence of nested intervals $I_0 \supset I_1 \supset I_2 \ldots$ such that f is negative on the left endpoint and positive on the right endpoint. By compactness (see Cor 14.8) there is a point x in the intersection of all these intervals. The continuity of f implies that $f(x) = 0$. (This last point is on your homework.)

Math 319 Projects

General guidelines: Student will work in pairs. You will have 15 minutes to give a focussed short talk that presents a proof or argument. Both of you should speak. You are also expected to write a 5 page paper on the topic. There should be a section containing definitions and examples, one with a brief history of the question, and a third section containing the proofs. This paper is due Tuesday December 2.

We expect you to talk either with Eduardo or myself to discuss the details of what you will say. The outlines below are very brief, and you may modify them if one of us agrees. Many of the projects stem form sections in Lay that we will not cover, and so that gives you a natural place to start, but you are expected to find some other references.

1. **Countable and uncountable subsets.** Prove (for example) that the number of finite subsets of $\mathbb{N}$ is countable but the number of arbitrary subsets of $\mathbb{N}$ is uncountable. (Lay: Sec 8)

2. **Axioms for set theory** Why is there no universal set? Explain two paradoxes; how does the axiom of separation resolve Russell's paradox? (Lay: sec 9.)

3. **What is a real number?** Explain their construction from the rationals using Cauchy sequences, and the relation to decimals. Prove the axiom of completenss using your definition. (Lay: sec 18 is a place to start.)

4. **Metric spaces** Describe some different metrics on the plane $\mathbf{R}^2$ and explore their differences; eg. show that there may be several different shortest paths between two points; show that closed bounded sets need not be compact. (Lay: sec 15)

5. **Differentiability for functions of a complex variable** Define differentiability and show that a differentiable function preserves angles. (starting point: these functions are called holomorphic or complex analytic; look at a book on complex analysis.)

6. **The fundamental theorem of algebra** Sketch a proof: Every polynomial has at least one complex root. (Starting point: Gauss gave the first proof)

7. **A continuous but nowhere differentiable function** constructed by Weierstrass, need to look at a more advanced analysis book. uses Fourier series.

8. **A continuous bounded curve in the plane with infinite length** or find a space filling curve, if you prefer. This is in the realm of ideas of fractal sets.

Dates:

Projects 1 and 2 will be presented Monday Nov 3;
projects 3 and 4 on Monday Nov 10;
projects 5 and 6 on Monday Nov 24; and
projects 7 and 8 on Monday Dec 1.

We will use the time slot 12:50– 1:45 pm. Each of you is expected to appear at three of these meetings. You will present at one, and at the other two you should ask questions and provide comments. (This will not need preparation.)