MAT 319: Foundations of Analysis

Fall 2019

Course Description: A careful study of the theory underlying topics in one-variable calculus, with an emphasis on those topics arising in high school calculus. The real number system. Limits of functions and sequences. Differentiation, integration, and the fundamental theorem. Infinite series.

Textbook: ***Robert G. Bartle and Donald R. Sherbert, Introduction to Real Analysis, 4th edition***

Instructor: Yu Li, Math Tower 4-101B. Office Hours: TuTh 12:00-1:00, Email: yu.li.4@stonybrook.edu.

TAs: James Seiner, Ruijie Yang

Class schedule: TuTh 10:00-11: 20 Javits 103

Homework: Weekly problem sets will be assigned, and collected on Wednesday or Thursday recitation. The emphasis of the course is on writing proofs, so please try to write legibly and explain your reasoning clearly and fully. You are encouraged to discuss the homework problems with others, but your write-up must be your own work. Late homework will never be accepted, but under-documented extenuating circumstances the grade may be dropped.

Week	Lectures	Homework
9/30	3.4, 3.5	3.3:4,7,10,11,13; 3.4:3,9,12,16
10/7	3.6, 4.1, 4.2	3.5:5,9,13; 3.6: 6,9; 4.1: 4,6,12
10/14	5.1,5.2	4.2:3,5,10; 5.1:4,11,15
10/21	5.2,5.3,5.4	5.2:2,5,8,11; 5.3:3,13,18
10/28	5.6, Second midterm	5.4:2,3,5,14
11/4	5.6,6.1	5.6:1,5,9,12; 6.1:4,7,9,12
11/11	6.2,6.3	6.1:13,14,15; 6.2:1,4,6,11
11/18	6.4	6.3:3,8,11; 6.4:1,10,14,15
11/25	7.1, Thanksgiving	7.1:1,2,6,8,15
12/2	7.2, 7.3	

Exams: The second midterm exam is in-class on **Oct. 31.** The final exam is on **Dec. 19, 8:00 am-10:45 am** and the room is **Javits 103**.

If you register for this course you must make sure that you are available at these times, as there will be **no make-ups** for missed exams.

The course grade is computed by the following scheme:

Homework: 20%

Midterm Test I: 20%

Midterm Test II: 20%

Final Exam: 40%

Help: The **Math Learning Center** (MLC) is located in Math Tower S-235, and offers free help to any student requesting it. It also provides a locale for students wishing to form study groups. The MLC is open 9 am-7 pm Monday through Friday. A list of graduate students available for hire as private tutors is maintained by the Undergraduate Mathematics Office, Math Tower P-143.

Disability Support Services (DSS)

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.stonybrook.edu/ehs/fire/disabilities

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Critical Incident Management Statement

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures.

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Ċ	Final Exam Practice Pro...	Yu Li, Dec 5, 2019, 9:20	v.1	ď
Ċ	Final practice exam ans...	Yu Li, Dec 8, 2019, 5:08	v.1	ď
Ċ	Midterm 2 Practice Pro...	Yu Li, Oct 19, 2019, 7:42	v.1	ď
Ċ	Practice exam answers....	Yu Li, Oct 26, 2019, 12:1	v.1	ď

Comments

Midterm 2 Practice Problems

Problem 1. Let the sequence (x_n) be defined as

$$x_n = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is odd;} \\ \frac{1}{n^2} & \text{if } n \text{ is even.} \end{cases}$$

Is (x_n) convergent?

Problem 2. Suppose $\lim_{n\to\infty} x_n = a > 0$. Prove that there exists a $K \in \mathbb{N}$ such that

$$\frac{a}{2} < x_n < 2a$$

for any $n \geq K$.

Problem 3.

1. Let the function f be defined as

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Prove that f is continuous at 0.

2. Let the function f be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Prove that f is discontinuous everywhere.

Problem 4. Give examples of functions f and g such that f and g do not have limits at c, but fg has the limit at c.

Problem 5. Suppose for any $x \in [-1, 1]$, $|f(x)| \leq 2|x|$. Prove that f is continuous at 0.

Problem 6. Let f be a continuous function on $[0, 1]$ such that $f(x) \in [0, 1]$ for any $x \in [0, 1]$. Prove that there exists a $c \in [0, 1]$ such that $f(c) = c$.

Problem 7.

1. Let (x_n) be a sequence such that $|x_{n+1} - x_n| < 2^{-n}$ for any $n \in \mathbb{N}$. Prove that (x_n) is convergent.

2. Is the result still true if we only assume $|x_{n+1} - x_n| < \frac{1}{n}$ for any $n \in \mathbb{N}$?

Problem 8. Let f and g be continuous functions on (a, b) such that $f(r) = g(r)$ for each rational number $r \in (a, b)$. Prove $f(x) = g(x)$ for all $x \in (a, b)$.

Problem 9.

1. Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some $k > 0$, then f is uniformly continuous on $[0, \infty)$.
2. Prove $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Problem 10. Let f be a continuous function on $[0, 1]$ such that $f(x) \in \mathbb{Q}$ for any $x \in [0, 1]$. Prove that f is constant.

1) Let (x_n) be defined by

$$x_n = \begin{cases} 1 + \frac{1}{n} & n \text{ odd} \\ \frac{1}{n^2} & n \text{ even} \end{cases}$$

is (x_n) convergent?

No. By Divergence Criterion for sequences, we need only find two subsequences of (x_n) with different limits. Consider (x_{2k}), (x_{2k-1})

$x_{2k} = \frac{1}{(2k)^2} \leq \frac{1}{k}$. Since $\frac{1}{(2k)^2} \geq 0$ and $\lim_{k\to\infty} \frac{1}{k} = 0$,

we have $\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} \frac{1}{(2k)^2} = 0$ by the squeeze thm.

However, $x_{2k-1} = 1 + \frac{1}{2k-1}$. Note that for any $\varepsilon > 0$,

$\exists\, M \in \mathbb{N}$ s.t. $M > \frac{\frac{1}{\varepsilon}+1}{2}$ (Archimedian property). So, $\left|\frac{1}{2M-1}\right| < \varepsilon$.

So $\forall\, k \geq M$, we have $\left|\frac{1}{2k-1}\right| < \varepsilon$. So $\lim_{k\to\infty} \frac{1}{2k-1} = 0$.

Thus, $\lim_{k\to\infty} x_{2k-1} = 1 + \lim_{k\to\infty} \frac{1}{2k-1} = 1$.

So $\lim_{k\to\infty} x_{2k} \neq \lim_{k\to\infty} x_{2k-1}$, and thus (x_n) does not converge.

2) Suppose $\lim_{n\to\infty} x_n = a > 0$. Prove $\exists\, k \in \mathbb{N}$ such that

$$\frac{a}{2} < x_n < 2a.$$

$\forall\, n \geq k$.

Since $a > 0$, $\frac{a}{2} > 0$. Since (x_n) converges to a, $\exists\, k \in \mathbb{N}$ s.t. $\forall\, n \geq k$, $|x_n - a| < \frac{a}{2}$.

That is $-\frac{a}{2} < x_n - a < \frac{a}{2}$. Equivalently,

$\frac{a}{2} < x_n < \frac{3a}{2}$. Since $\frac{3}{2} < 2$ and $a > 0$,

we have $\frac{3a}{2} < 2a$, so $\forall\, n \geq k$,

$$\frac{a}{2} < x_n < 2a.$$

3) i) Let f be defined by

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Prove that f is continuous at 0.

Let $\varepsilon > 0$. We want to find $\delta > 0$ s.t. $\forall\, x$ s.t. $0 < |x - 0| < \delta$, $|f(x) - 0| < \varepsilon$. Let $\delta = \varepsilon$.

Then, for any x s.t. $0 < |x| < \delta$, either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$. So either $f(x) = x$ or $f(x) = 0$.

in either case, $|f(x)| \leq |x|$ so

$$|f(x) - 0| = |f(x)| \leq |x| < \delta = \varepsilon$$

as desired.

3) ii) Let F be defined by

$$F(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Prove that F is discontinuous everywhere.

Let $c \in \mathbb{R}$. Let $\varepsilon = \frac{1}{2}$, we claim that $\forall \delta > 0$, $\exists x$ w/ $0 < |x - c| < \delta$ s.t. $|F(x) - F(c)| > \varepsilon$.

Well, let $\delta > 0$. In class (Ch 2) we proved that $\exists y \in \mathbb{Q}$ and $z \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$c - \delta < y, z < c + \delta$. If $c \in \mathbb{Q}$, then

$$|F(z) - F(c)| = |0 - 1| = 1 > \varepsilon$$

if $c \notin \mathbb{Q}$, then $|F(y) - F(c)| = |1 - 0| = 1 > \varepsilon$.

So F is not continuous at c.

4) Give examples of functions f, g such that f and g do not have limits at c but fg has a limit at c.

Ex: let $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$, $g(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$

Then neither f nor g has a limit at 0.

(Take $\varepsilon = \frac{1}{2}$, then every δ-nbd of 0 contains both positive and negative numbers, so each of f and g take on the values 0 and 1 in this nbd. There is no number L s.t. $|0 - L| < \frac{1}{2}$ and $|1 - L| < \frac{1}{2}$)

4) cont'd

However, $fg(x) = \begin{cases} 0 \cdot 1 & x \le 0 \\ 1 \cdot 0 & x > 0 \end{cases} = 0$

So fg is constant, and therefore has a limit at 0.

5) Suppose that for any $x \in [-1, 1]$ $|f(x)| \le 2|x|$

Prove that f is continuous at 0.

Note that, by "hypothesis" $|f(0)| \le 2|0| = 0$.

Since $|f(0)| \ge 0$, we have $|f(0)| = 0$, so $f(0) = 0$.

Let $\varepsilon > 0$. We want to find δ s.t. $\forall x$ with $0 < |x - 0| < \delta$, we have $|f(x) - 0| < \varepsilon$.

Let $\delta = \min\{\frac{\varepsilon}{2}, 1\}$. Then whenever $0 < |x - 0| < \delta$, we have $x \in [-1, 1]$ and,

$$|f(x) - 0| = |f(x)| \le 2|x| = 2|x - 0| < 2\delta \le 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

(↑ since $x \in [-1, 1]$.)

as desired.

Problem 6

Case 1 : If $f(0) = 0$ or $f(1) = 1$, then $c = 0$ or 1 satisfies $f(c) = c$.

Case 2 : If $f(0) \neq 0$ and $f(1) \neq 1$, then $f(0) > 0$ and $f(1) < 1$ because $f(x) \in [0, 1]$ for all $x \in [0,1]$.

Let $g(x) = f(x) - x$, g is a function defined on $[0,1]$. Because f and x are continuous functions on $[0,1]$, g is also a continuous function on $[0,1]$.

Note that $g(0) = f(0) - 0 > 0$ by our assumption.

$g(1) = f(1) - 1 < 0$

Then by Intermediate Value Theorem (5.3.7), there exists a point $c \in [0,1]$ such that $g(c) = 0$. In particular, $f(c) = c$.

Problem 7

1. It suffices to show that (x_n) is a Cauchy sequence, because any Cauchy sequence must be convergent. (3.5.5.)

$\forall \varepsilon > 0$, choose N such that $2^{N-1} > \frac{1}{\varepsilon}$ $(\Leftrightarrow N > \log_2 \frac{1}{\varepsilon} + 1)$

For any m, n such that $m > n > N$, we have

$$|x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \quad \text{(triangular inequality)}$$

$$< \frac{1}{2^{m-1}} + \cdots + \frac{1}{2^n} \quad \text{(property of } (x_n) \text{)}$$

Let $S = \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n}$, then

$$2S = \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n-1}}.$$

Hence $S = 2S - S = \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}$.

Therefore $|x_m - x_n| < \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} < \varepsilon$,

the last inequality follows from our choice of N.

2. The result is not true if we only assume $|x_{n+1} - x_n| < \frac{1}{n}$

Here is a counter-example. Let $x_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}$, $\forall n \in \mathbb{N}$.

Then $|x_{n+1} - x_n| = \frac{1}{2n+2} < \frac{1}{n}$, hence it satisfies the assumption.

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, hence $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges. Because x_n is the partial sum of $\sum_{n=1}^{\infty} \frac{1}{2n}$, hence (x_n) diverges.

Problem 8

Let $x \in (a, b)$, by the density of rational numbers, there exists $(x_n)_{n \in \mathbb{N}}$ such that x_n is a rational number in (a, b) and $\lim_{n \to \infty} x_n = x$. (5.1.3)

Because f, g are continuous, by the sequencial criterion,

$$\lim_{n \to \infty} f(x_n) = f(x) \quad \text{and} \quad \lim_{n \to \infty} g(x_n) = g(x).$$

Because $f(x) = g(x)$ if x is a rational number, then $f(x_n) = g(x_n) \quad \forall\, n \geq 1$

Therefore $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n)$

This means that $f(x) = g(x)$.

Problem 9

1. Since f is continuous on $[0,\infty)$ and $[0,k] \subseteq [0,\infty)$, hence f is continuous on $[0,k]$. Note that $[0,k]$ is a closed bounded interval, by Uniform Continuity Theorem (5.4.3), f is uniformly continuous on $[0,k]$.

Now I want to show that f is uniformly continuous on $[0,\infty)$.

$\forall \varepsilon > 0$, because f is uniformly continuous on $[0,k]$ and $[k,+\infty)$, $\exists\ \delta_1$ and δ_2 such that

① If $|x-y| < \delta_1$ and $x,y \in [0,k]$, then $|f(x) - f(y)| < \varepsilon/2$.

② If $|x-y| < \delta_2$ and $x,y \in [k,+\infty)$, then $|f(x) - f(y)| < \varepsilon/2$.

Then choose $\delta = \min\{\delta_1, \delta_2\}$, want to show that if $|x-y| < \delta$ and $x,y \in [0,+\infty)$, then $|f(x) - f(y)| < \varepsilon$.

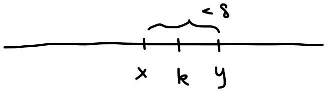
Case 1: If $|x-y| < \delta$ and $x,y \in [0,k]$, then

$|x-y| < \delta \leq \delta_1$, therefore by ①, $|f(x) - f(y)| < \varepsilon/2 < \varepsilon$

Case 2: If $|x-y| < \delta$ and $x,y \in [k,+\infty)$, then

$|x-y| < \delta \leq \delta_2$, therefore by ②, $|f(x) - f(y)| < \varepsilon/2 < \varepsilon$.

case 3. If $|x-y| < \delta$ and $x \in [0,k)$, $y \in (k,+\infty)$,



Then $|x-k| < \delta \leq \delta_1$, by ① $|f(x) - f(k)| < \varepsilon/2$.

$|y-k| < \delta \leq \delta_2$, by ② $|f(y) - f(k)| < \varepsilon/2$.

By triangular inequality,

$$|f(x) - f(y)| \leq |f(x) - f(k)| + |f(k) - f(y)|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

Therefore, all three cases together show that

if $|x-y| < \delta$ and $x, y \in [0,+\infty)$,

then $|f(x) - f(y)| < \varepsilon$.

This means that f is uniformly continuous on $[0,+\infty)$.

2. $\sqrt{x}$ is continuous on $[0,+\infty)$.

For $x,y \in [1,+\infty)$, $|\sqrt{x} - \sqrt{y}| = \frac{1}{\sqrt{x}+\sqrt{y}} \cdot |x-y| \leq \frac{1}{2}|x-y|$

because $x, y \geq 1$ so $\sqrt{x}, \sqrt{y} \geq 1$. Therefore $\sqrt{x}$ is a Lipschitz function on $[1,+\infty)$ and must be uniformly continuous on $[1,+\infty)$ by (5.4.4). Hence by 1, $\sqrt{x}$ is uniformly continuous on $[0,+\infty)$.

Problem 10

We prove by contradiction.

Suppose f is not constant, then there exists $x, y \in [0,1]$ such that $f(x) \neq f(y)$. Without loss of generality, let's assume $f(x) < f(y)$.

(2.4.9)

By the property of real numbers, there must exist z which is an irrational number and

$$f(x) < z < f(y).$$

Since f is continuous on $[0,1]$, by Intermediate Value Theorem (5.3.7), there exists $c \in [0,1]$ such that $f(c) = z$.

But we know that $f(x) \in \mathbb{Q} \ \forall x \in [0,1]$, this means $f(c) = z \in \mathbb{Q}$, which is impossible.

Therefore our assumption is not correct and f must be a constant function.

Final Exam Practice Problems

Problem 1. Let the sequence (x_n) be defined as follows: $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$ for any $n \in \mathbb{N}$. Prove that $1 \leq x_n \leq 2$ for any $n \in \mathbb{N}$.

Problem 2. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $\sup S = -\inf\{-s : s \in S\}$.

Problem 3. Find the infimum of the set $A = \{1 + \frac{(\sin n)^2}{\sqrt{n}} \mid n \in \mathbb{N}\}$.

Problem 4. Prove

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

Problem 5. Let (a_n) be a positive sequence such that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$. Prove that (a_n) is unbounded.

Problem 6. Assume that $\lim_{n\to\infty} x_n = +\infty$. Prove that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = +\infty.$$

Problem 7. Suppose $f(x)$ is a strictly increasing function on $[a, b]$ and $(x_n) \subset [a, b]$ is a sequence such that $\lim_{n\to\infty} f(x_n) = f(a)$. Prove that $\lim_{n\to\infty} x_n = a$.

Problem 8. * Let f be a function defined on $(0, 1)$ such that for any $c \in (0, 1)$, $\lim_{n\to\infty} f(\frac{c}{n}) = 0$. Can we conclude that $\lim_{x\to 0^+} f(x) = 0$?

Problem 9. Assume that the function f is continuous at 0 and $f(0) > 0$. Prove that there exists a $\delta > 0$ such that $f(x) > 0$ for any $|x| < \delta$.

Problem 10. For any function f, we define $w_a(\delta) = \sup\{|f(x) - f(y)| \mid |x - a| < \delta \text{ and } |y - a| < \delta\}$. Prove that f is continuous at a if and only if $\lim_{\delta\to 0^+} w_a(\delta) = 0$.

Problem 11. Suppose there exists a constant $L > 0$ such that for any $x, y \in [a, \infty)$ we have

$$|f(x) - f(y)| \leq L|x - y|.$$

If $a > 0$, prove that $\frac{f(x)}{x}$ is uniformly continuous on $[a, \infty)$.

Problem 12.

Let the function f be defined as

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f is differentiable at 0.

Problem 13. Suppose $|f(x)|$ is differentiable at a and $f(a) = 0$, prove that $f'(a) = 0$.

Problem 14. Assume there exist constants M and $a > 1$ such that for any $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|^a.$$

Prove that f is a constant.

Problem 15. Let $f(x)$ and $g(x)$ be convex functions and f is increasing. Prove that $f(g(x))$ is convex.

Problem 16. If f defined on $[0, 1]$ is a continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$. Prove that $f(x) = 0$ for any $x \in [0, 1]$.

Problem 1

We prove by induction on n.

① Base case : $n=1$, $x_1 = 1$ and $1 \leq x_1 \leq 2$

$n=2$, $x_2 = 2$ and $1 \leq x_2 \leq 2$

② Inductive step : assume it is true for any $1 \leq k \leq n$

then $1 \leq x_{n-1} \leq 2$ and $1 \leq x_n \leq 2$.

Hence $1 \leq x_{n+1} = \dfrac{x_n + x_{n-1}}{2} \leq 2$.

Therefore it also holds for $n+1$. #

Problem 2

Denote $A = \sup S$ and $B = \inf\{-s : s \in S\}$. We need to show that $A = -B$.

By definition of sup, $s \leq A$, $\forall s \in S$

$\Rightarrow$ $-s \geq -A$, $\forall s \in S$

$\Rightarrow$ $B = \inf\{-s : s \in S\} \geq -A$ $\Rightarrow$ $A \geq -B$ ①

By definition of inf

$-s \geq B$ $\forall s \in S$

$\Rightarrow$ $s \leq -B$ $\forall s \in S$

$\Rightarrow$ $A = \sup S \leq -B$ ②

Combining ① and ② we get $A = -B$. #

Problem 3

First we notice that $\lim_{n\to+\infty} \frac{(\sin n)^2}{\sqrt{n}} = 0$

This is because

$$0 \leqslant \frac{(\sin n)^2}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n}},$$

Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, by squeeze theorem, $\lim \frac{(\sin n)^2}{\sqrt{n}} = 0$.

$\forall n \in \mathbb{N}$, $1 + \frac{(\sin n)^2}{\sqrt{n}} \geq 1$. Therefore 1 is a lower bound of the set A. We claim that 1 is the greatest lower bound.

Suppose there exists $\varepsilon_0 > 0$ such that

$$1 + \frac{(\sin n)^2}{\sqrt{n}} \geq 1 + \varepsilon_0 \qquad \forall n \in \mathbb{N}$$

By the comparison property of limits,

$$1 = \lim_{n\to\infty} \left(1 + \frac{(\sin n)^2}{\sqrt{n}}\right) \geq 1 + \varepsilon_0$$

which is a contradiction! Therefore $\inf A = 1$. #

[Alternatively, one can argue $\forall \varepsilon > 0$, $\exists N \geq 1$ such that

$$1 \leqslant \frac{(\sin N)^2}{\sqrt{N}} + 1 < \varepsilon + 1$$

therefore by Lemma 2.3.4 (for inf), $1 = \inf A$.]

Problem 4

Notice that for each $1 \le i \le n$

$$\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+i}} \le \frac{1}{n},$$

therefore $\frac{n}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}} \le \frac{n}{n} = 1$.

Since $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = \frac{1}{\sqrt{1+0}} = 1$,

by squeeze theorem, we have

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right) = 1.$$

#

Problem 5

Since $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$, for $\varepsilon = \frac{1}{2}$, there exists N

such that for $n > N$, we have

$$\left| \frac{a_n}{a_{n+1}} \right| < \frac{1}{2}.$$

Since $a_n > 0$ for each n, we have

$$a_{n+1} > 2a_n \quad \text{for } n > N.$$

Therefore $a_n > 2^{n-N} \cdot a_N$ for $n > N$

We can conclude that a_n is unbounded because

$$\lim_{n\to\infty} 2^{n-N} \cdot a_N = +\infty.$$

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Problem 6 First proof

To show $\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = +\infty$, we need to show that for any $A>0$, there exists $N \geq 1$ such that when $n>N$,

$$\frac{a_1+\cdots+a_n}{n} > A.$$

For any $A>0$, because $\lim_{n\to\infty} a_n = +\infty$, there exists $N_1 \geq 1$ such that $a_n > 2A$ for $n>N_1$.

Therefore
$$\frac{a_1+\cdots+a_n}{n} = \frac{a_1+\cdots+a_{N_1}+a_{N_1+1}+\cdots+a_n}{n}$$
$$> \frac{a_1+\cdots+a_{N_1}+(n-N_1)\cdot 2A}{n} \quad \text{if } n>N_1.$$

Notice that
$$\lim_{n\to\infty} \frac{a_1+\cdots+a_{N_1}+(n-N_1)\cdot 2A}{n}$$
$$= \lim_{n\to\infty} 2A + \frac{a_1+\cdots+a_{N_1}-2N_1A}{n}$$
$$= 2A > A$$

Therefore by the property of limit, there exists $N_2 \geq 1$ such that when $n>N_2$,

$$\frac{a_1+\cdots+a_{N_1}+(n-N_1)2A}{n} > A$$

Hence if we choose $N=\max\{N_1,N_2\}$, for $n>N$ we have

$$\frac{a_1+\cdots+a_n}{n} \underset{n>N_1}{>} \frac{a_1+\cdots+a_{N_1}+(n-N_1)2A}{n} \underset{n>N_2}{>} A.$$

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Problem 6 Second proof

$\forall A > 0$, we need to choose N such that

when $n > N$, $\dfrac{x_1 + x_2 + \cdots + x_n}{n} > A$

① Since $\lim x_n = +\infty$, there exists N_1 such that

when $n > N_1$, $x_n > 2A$.

② Choose N_2 such that $N_2 > \dfrac{2N_1 A - (x_1 + \cdots + x_{N_1})}{A}$.

Then we choose $N = \max\{N_1, N_2\}$.

If $n > N$, then $\dfrac{x_1 + \cdots + x_n}{n} = \dfrac{(x_1 + \cdots + x_{N_1}) + (x_{N_1+1} + \cdots + x_n)}{n}$

$$> \frac{x_1 + \cdots + x_{N_1} + (n - N_1) 2A}{n}$$

$$= A + \frac{nA - (2N_1 A - (x_1 + \cdots + x_{N_1}))}{n}$$

$$> A + \frac{N_2 A - (2N_1 A - (x_1 + \cdots + x_{N_1}))}{n} > A + 0 = A$$

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[less informal proof: $\dfrac{x_1 + \cdots + x_{N_1} + (n - N_1) 2A}{n} > A$

$\Leftrightarrow \quad x_1 + \cdots + x_{N_1} + (n - N_1) 2A > nA$

$\Leftrightarrow \quad nA > 2N_1 A - (x_1 + \cdots + x_{N_1})$

$\Leftrightarrow \quad n > \dfrac{2N_1 A - (x_1 + \cdots + x_{N_1})}{A}$]

Problem 7

We will prove by contradiction.

Suppose $\lim_{n\to\infty} x_n \neq a$, then there exists $\varepsilon_0 > 0$ and a subsequence $(x_{n_k}) \subset [a,b]$ such that $|x_{n_k} - a| > \varepsilon_0$, in particular $x_{n_k} > a + \varepsilon_0$.

Since f is a strictly increasing function,

$$(*) \qquad f(x_{n_k}) > f(a+\varepsilon_0) \qquad \forall\, k \geq 1.$$

Because $\lim_{n\to\infty} f(x_n)$ exists,

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = f(a).$$

From $(*)$ we know that

$$\lim_{k\to\infty} f(x_{n_k}) \geq f(a+\varepsilon_0).$$

In particular this implies that

$$f(a) \geq f(a+\varepsilon_0).$$

But this contradicts with the fact that f is strictly increasing. Therefore our assumption is wrong and we must have $\lim_{n\to\infty} x_n = a$. #

Problem 8.

We cannot conclude that $\lim_{x \to 0^+} f(x) = 0$.

Here is the construction of a counter-example.

Because $(0,1)$ is uncountable, we can choose a number $b \in (0,1)$ such that $\forall k \geq 1$, $b^k \notin \mathbb{Q}$.

(For example b can be chosen to be any transcendental number like $\frac{1}{e}$ or $\frac{1}{\pi}$).

Let $f(x) = \begin{cases} 1 & \text{if } x = b^k \text{ for some } k \geq 1 \\ 0 & \text{else.} \end{cases}$

We will show that $f(x)$ satisfies

① $\forall c \in (0,1)$, $\lim_{n \to \infty} f(\frac{c}{n}) = 0$

② $\lim_{x \to 0^+} f(x) \neq 0$

① Let $c \in (0,1)$ be any real number, then there is at most one integer n such that $\frac{c}{n} = b^k$ for some $k \geq 1$.

Suppose this is not true and there exists n_1, n_2 such that

$\frac{c}{n_1} = b^{k_1}$, $\frac{c}{n_2} = b^{k_2}$ and $k_2 > k_1$.

$\Rightarrow$ $b^{k_1} \cdot n_1 = b^{k_2} \cdot n_2$ $\Rightarrow$ $b^{k_2 - k_1} = \frac{n_1}{n_2} \in \mathbb{Q}$

This contradicts the choice of b.

Therefore for n large enough, $\frac{c}{n}$ is not equal to any b^k.

By the definition of f, we have $f(\frac{c}{n}) = 0$.

Thus $\lim_{n\to\infty} f(\frac{c}{n}) = 0$.

② We will prove $\lim_{x\to 0^+} f(x) \neq 0$ by contradiction.

Suppose $\lim_{x\to 0^+} f(x) = 0$, then for any sequence x_n

such that $\lim_{n\to\infty} x_n = 0$, we must have $\lim_{n\to\infty} f(x_n) = 0$.

Let $x_n = b^n$, since $b < 1$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} b^n = 0$.

But $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} 1 = 1 \neq 0$

Therefore $\lim_{x\to 0^+} f(x) \neq 0$.

The conclusion is that f is a counterexample. #

9) f continuous at 0, $f(0)>0$. Want δ s.t. $f(x)>0$ $\forall$ $|x|<\delta$

Say $f(0)=a>0$. Note that $\varepsilon := \frac{a}{2} > 0$. Since f is continuous at 0, $\exists$ $\delta>0$ s.t. $\forall$ $|x-0|<\delta$, we have $|f(x)-a|<\varepsilon$. That is, $\forall$ $|x|<\delta$, $|f(x)-a|<\frac{a}{2}$

$\Rightarrow$ $-\frac{a}{2} < f(x)-a < \frac{a}{2}$ so $0<\frac{a}{2}<f(x) < \frac{3a}{2}$.

So $f(x)>0$ $\forall$ $|x|<\delta$ as desired

(we will reference this argument later in a more general setting. The content is the same, just with "0" replaced by "c")

10. $w_a(\delta) := \sup\{|f(x)-f(y)| : |x-a|\le\delta, |y-a|<\delta\}$, want f cts at a $\Leftrightarrow$ $\lim_{\delta\to0^+} w_a(\delta)=0$.

($\Rightarrow$): suppose f is cts at a. Want to show that $\forall$ $\varepsilon>0$, $\exists$ $\gamma>0$ s.t. $\forall 0<\delta-0<\gamma$, we have $w_a(\delta)<\varepsilon$. Since $w_a(\delta)$ is the sup of a set, that means we want γ s.t. $\forall$ $0<\delta<\gamma$, ε is an upper bound for $\{|f(x)-f(y)| : |x-a|\le\delta, |y-a|<\delta\} =: S_a(\delta)$

Since f is cts at a, $\exists$ $\gamma>0$ s.t. $\forall$ z s.t. $|z-a|<\gamma$, we have $|f(z)-f(a)|<\varepsilon/2$. Now, for $\delta<\gamma$, x, y s.t. $|x-a|\le\delta$, $|y-a|<\delta$. Then $|x-a|\le\delta<\gamma$, $|y-a|<\delta<\gamma$

so $|f(x)-f(a)|<\varepsilon/2$ and $|f(y)-f(a)|<\varepsilon/2$.

$\Rightarrow$ $|f(x)-f(y)| = |f(x)-f(a)+f(a)-f(y)| \le |f(x)-f(a)|+|f(y)-f(a)|$

$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. So ε is a strict upper bound for $S_a(\delta)$ and so $w_a(\delta)<\varepsilon$.

($\Leftarrow$): Suppose $\lim_{\delta\to0^+} w_a(\delta)=0$. Want: $\forall$ $\varepsilon>0$ $\exists$ $\gamma>0$ s.t. whenever $|z-a|<\gamma$, have $|f(z)-f(a)|<\varepsilon$. Fix $\varepsilon>0$. we know (hypothesis) $\exists$ $\gamma>0$ s.t. whenever $0<\delta<\gamma$ we have

ε is a strict upper bound for $S_a(\delta)$. Now for any z s.t. $\delta := |z-a| < \gamma$, we have (since $|a-a| = 0 < \delta$) that $|f(z) - f(a)| \in S_a(\delta)$, so $|f(z) - f(a)| < \varepsilon$.

11) $\exists$ $L > 0$ s.t. $\forall$ $x, y \in [a, \infty)$, we have

$$|f(x) - f(y)| \leq L|x-y| \quad , \quad w/ \ a > 0.$$

want: $\forall$ $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. whenever $|x-y| < \delta$, $\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| < \varepsilon$.

Fix $\varepsilon > 0$. Let $\delta = \frac{\varepsilon a^2}{2La + |f(a)|}$ (Note: this is > 0).

Now, $\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| = \left|\frac{y f(x) - x f(y)}{xy}\right| = \frac{|y f(x) - y f(y) + y f(y) - x f(y)|}{|xy|}$

$\leq \frac{|y(f(x) - f(y))| + |f(y)(x-y)|}{|xy|} = \frac{|y||f(x) - f(y)| + |f(y)||x-y|}{|xy|}$

$\leq \frac{|y| L|x-y| + |f(y)||x-y|}{|xy|} = |x-y|\left(\frac{L|y|}{|xy|} + \frac{|f(y)|}{|xy|}\right)$

Need to do something about $f(y)$: See how far f can move you from a.

$|f(y)| = |f(y) - f(a) + f(a)| \leq |f(y) - f(a)| + |f(a)| \leq L|y-a| + |f(a)|$

$\underset{(y \geq a)}{=} L(y-a) + |f(a)| \underset{(a>0)}{\leq} Ly + |f(a)|$

So $\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| \leq |x-y|\left(\frac{L}{x} + \frac{Ly}{xy} + \frac{|f(a)|}{xy}\right)$

($x, y \geq 0$ so $|\cdot|$ can be dropped)

$= |x-y|\left(\frac{2L}{x} + \frac{|f(a)|}{xy}\right) \underset{x,y \geq a}{\leq} |x-y|\left(\frac{2L}{a} + \frac{|f(a)|}{a^2}\right) = |x-y|\left(\frac{2La + |f(a)|}{a^2}\right)$

$< \delta\left(\frac{2La + |f(a)|}{a^2}\right) = \varepsilon.$ $\square$

12) $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Need to examine $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)-0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$

$\frac{f(x)}{x} \underset{\text{for } x\neq 0}{=} \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ This is a piecewise function whose pieces have the same limit, (0), at 0,

so $\lim_{x\to 0} \frac{f(x)}{x} = 0.$

13) Suppose $|f(x)|$ is differentiable at a. with $f(a)=0$. Show that $f'(a)=0$.

suppose $f'(a)\neq 0$, then $|f'(a)| = L > 0$. $f(a)=0$ $|\cdot|$ cts

Since $|\cdot|$ is cts, $|f'(a)| = \left|\lim_{x\to a} \frac{f(x)}{x-a}\right| \overset{\downarrow}{=} \lim_{x\to a}\left|\frac{f(x)}{x-a}\right|$

we will show that $|f|$ is not diff'ble at a. well,

$\lim_{x\to a} \frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a}\frac{|f(x)|}{x-a}$ Consider sequences $(x_n) := (a+\frac{1}{n})$ $(y_n) := (a-\frac{1}{n})$

If the above limit exists, both sequences must give the same result.

$\lim_{n\to\infty} \frac{|f(x_n)|}{x_n - a} = \lim_{n\to\infty} \frac{|f(x_n)|}{1/n} = \left|\lim_{n\to\infty}\frac{f(x_n)}{1/n}\right| = |f'(a)| = L > 0$

$\lim_{n\to\infty} \frac{|f(y_n)|}{y_n - a} = \lim_{n\to\infty} \frac{|f(y_n)|}{-\frac{1}{n}} = -\left|\lim_{n\to\infty}\frac{f(y_n)}{1/n}\right| = -|f'(a)| = -L < 0.$

So $|f|$ is not differentiable at a.

14) $\exists$ M, $a>1$ s.t $\forall$ $x, y \in \mathbb{R}$

$$|f(x)-f(y)| \le M|x-y|^a$$

Show f is const.

we will show that f is diff'ble at every $c \in \mathbb{R}$ and that $f'(c)=0$. Let $c \in \mathbb{R}$.

$$\left|\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right| = \lim_{x\to c}\left|\frac{f(x)-f(c)}{x-c}\right| \le \lim_{x\to c} \frac{M|x-c|^a}{|x-c|} = M\lim_{x\to c}\frac{|x-c|^a}{|x-c|}$$

Since $a>1$, $\lim_{x\to c} \frac{|x-c|^a}{|x-c|} = 0$ so $\left|\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right| \le M \cdot 0 = 0$

so f is differentiable at c and $f'(c)=0$. This holds $\forall$ $c \in \mathbb{R}$, so f is constant.

15) Let $f(x)$, $g(x)$ convex, f increasing. Prove $f(g(x))$ is convex.

Let $x_1, x_2 \in \mathbb{R}$, $t \in [0,1]$ want

$$f(g((1-t)x_1 + tx_2)) \le (1-t)f(g(x_1)) + f(g(x_2))$$

well, g is convex, so $g((1-t)x_1 + tx_2) \le (1-t)g(x_1) + tg(x_2)$

f is increasing so $f(g((1-t)x_1 + tx_2)) \le f((1-t)\underbrace{g(x_1)}_{y_1} + t\underbrace{g(x_2)}_{y_2})$

f is convex, so $f((1-t)y_1 + ty_2) \le (1-t)f(y_1) + tf(y_2)$

$= (1-t)f(g(x_1)) + tf(g(x_2))$ as desired.

16) $f:[0,1]\to\mathbb{R}$ is cts, $\int_0^x f = \int_x^1 f$ $\forall$ $x \in [0,1]$ want $f(x)=0$ $\forall$ $x\in[0,1]$

Note that $\forall$ $x \in [0,1]$, $\int_0^1 f = \int_0^x f + \int_x^1 f$ (Thm 7.2.13)

$= 2\int_0^x f$. Setting $x=0 \Rightarrow \int_0^1 f = 0$.

($\int_0^0 f = 0$ by def)

Now, $\int_0^x f + \int_x^1 f = 0$, so

$\int_0^x f = \int_x^1 f = -\int_0^x f \Rightarrow \int_0^x f = 0$ $\forall x \in [0,1]$ ...

For $x, y \in [0,1]$, $x < y$, we have $\int_0^x f = 0 = \int_0^y f$
and $\int_x^y f = \int_0^y f - \int_0^x f = 0$. Finally, for $c \in [0,1]$ if

$f(c) > 0$, by problem 9, $\exists$ $a, b \in [0,1]$ s.t. $f(x) > 0$ $\forall x \in [a,b]$
$\Rightarrow \int_a^b f > 0$, but this is a contradiction.

if $f(c) < 0$, then $-f(c) > 0$ so $-\int_a^b f > 0$ (where a, b are as above)
so $\int_a^b f < 0$, but this is a contradiction.

So $f(c) = 0$. This holds $\forall c \in [0,1]$, since c was arbitrary.

$\square$