Sylvain BONNOT	**MAT 319 Foundations of analysis**	

We will meet on MWF : 10:40 am to 11:35 am in Physics P112.The class will be taught in parallel with MAT 320, and then separately from Oct.8th

First day of class: Sep 5th, 2007.
Final exam : TBA.

Office hours:
every Thursday from 2:00 pm to 5:00 pm in my office, 5D-148 in the Math Tower.
My office is in the I.M.S (Institute for Math. Sciences), located on floor 5 and a half.

How to contact me?
the best way is to email me there: `bonnot at math dot sunysb dot edu`

Our textbook:
Introduction to Real analysis (Hardcover), Wiley, Third edition , by R.Bartle, D. Sherbert

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Link to Current Homework: The Homework is an important part of this class. Click here to go to the homework page.

Course notes and announcements:

- Hi everybody, this will be the last annoucement for this class... I wish an excellent break to all of you! If you miss real analysis, then you can still read the solutions of the final: part I, and part II !
- Just a reminder: our Final Exam is on Friday Dec. 21st, usual room (P-112), from 11 AM to 1:30 PM...
- Your TA Rob told me that he will give an additional practice session next week on Thursday at 2pm in room 5-127. It's certainly a good idea to attend that session...
- And here come the solutions for the practice final exam . Now the link is the correct one!
- Here is a practice final exam . You will get the solutions of it very soon (tomorrow). Prepare your questions for Wednesday and Friday.
- Here are the solutions of Midterm II .
- I am still waiting before putting online the solution of the midterm (because some of you wanted to bring back a complete solution of the last problem for their writing requirement)... I put online the new HW10, which is light, so that you can enjoy your break (this doesn't make sense to me, because I can't find a better way to spend a break than doing some math HW, so I might add some more problems...)

- Some hints for HW9 are now available.
  They basically complement what I said this morning.

- Brand New ! the solutions of the Practice MidtermII are available!
  Please try it first, then read the solutions, then ask me questions on Wednesday (if you need...) Also you might want to prepare some questions for our review session on Wednesday.

- Practice MidtermII is available!
  Midterm II will be on Friday Nov. 9th.(usual time, usual classroom) It is cumulative in the sense that you need to remember what is the limit of a sequence, or what is a real number, or a least upper bound etc... Thus it covers from chapter 2.1 to 5.3 included. However the problems will cover essentially the chapters studied after Midterm I (thus 3.3 to 5.3). More precisely it should be pretty close to the following practice exam. (Note that the actual exam will be shorter than that). Also remember that I typed for you the solutions of the HW problems, so you might want to have a look at them... There will be a review session next Wednesday (Nov. 7th). Also you can find me in my office on Monday and Tuesday afternoon, and Thursday afternoon (usual office hours) next week.

- Starting on Monday October 8th, 319 and 320 are taught separately. MAT 319 will meet in the usual classroom P-112, MWF 10:40 to 11:35 (usual time). Please make sure to bring back the form with your choice between 319 and 320, and go to the Solar system to check that you are currently registered in 319.

  HW5 has been already assigned by Prof. Phillips: p.80: 1, 3, 8a, 14, 15 and p.86: 1, 3c, 9, 13.

Quick intro: Simply put, our goal for this class is to revisit all the main theorems of a standard calculus and to provide detailed proofs for these.

Link to Current Homework: Regularly you will have to consult this homework page to know what has been assigned.

Syllabus (tentative) :

Day of	**Sections Covered**
Week 6:October 8, 10, 12	Contractive sequences, 3.6, 3.7
Week 7:Oct. 15,17,19	4.1, 4.2
Week 8:Oct. 22,24,26	4.3, 5.1, 5.2
Week 9:Oct. 29, 31, Nov. 2	5.3, 5.4
Week 10:Nov. 5,7,9	5.6, 6.1
Week 11:Nov. 12, 14, 16	End 0f 6.1, 6.2
Week 12:Nov. 19, 21	6.2, 6.3
Week 13:Nov. 26,28,30	End of 6.3, 6.4

Week 14:Dec. 3,5,7	7.1, 7.2
Week 15:Dec. 10,12,14	7.3, Review
Week 16:Dec. 17, 19, 21	Exam week, Final on 21st

Exams:

Midterm 1	Oct. 1st	Usual room
Midterm 2	Friday Nov. 9th, 10:40 to 11:35	Usual room
Final	Fr Dec. 21st 2007, 11:00 am to 1:30 pm	Usual room

Homework and grading policy: Here is how your final grade will be computed. of the following:

Exam I	25%
Exam II	25%
Final Exam	35%
Homework	15%

Late homework will not be accepted.

DSS advisory:

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services (DSS) office: ECC (Educational Communications Center) Building, room 128, telephone (631) 632-6748/TDD. DSS will determine with you what accommodations are necessary and appropriate. Arrangements should be made early in the semester (before the first exam) so that your needs can be accommodated. All information and documentation of disability is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and DSS. For procedures and information, go to the following web site http://www.ehs.sunysb.edu and search Fire safety and Evacuation and Disabilities.

MAT 319 Homework Assignments

Fall 2007

Link to main page for MAT 319.

Mathematics department

#	Problems	Due Date
HW1	Solutions of HW1	
HW2	Solutions of HW2	
HW3	Solutions of HW3	
HW4	Solutions of HW4	
HW5	p.80: 1, 3, 8a, 14, 15 and p.86: 1, 3c, 9, 13 Solutions of HW5	**Wed.** 10/10/2007
HW6	**sect. 3.6**: 1, 8d, 10 and **sect. 3.7**: 3a, 6b, 8, 11 Solutions of HW6	**Wed.** 10/17/2007
HW7	**sect. 4.1**: 2, 9d, 10b, 11c, 13 and **sect. 4.2**: 1d, 4, 5 Solutions of HW7	**Wed.** 10/24/2007
HW8	**sect. 4.3:** : 5a, 5c, 8 and **sect. 5.1**: 7, 12 and **sect. 5.2:**:1a, 9 Solutions of HW8	**FRIDAY** 11/02/2007
	no HW is due for the week of the exam	
HW9	**sect. 5.3:** : 3, 6, 11, 13 and **sect. 5.4**: 2, 4, 9 **and sect. 5.6:**:10 Solutions of HW9	**MONDAY** 11/19/2007
HW10	**sect. 6.1**: 1a (only), 4, 9, 10, 11a (only) Solutions of HW10	**WED.** 11/28/2007
HW11	**sect. 6.1**: 16 and **sect. 6.2:** 1b (only), 2d (only), 4, 12, 16 Solutions of HW11	**FR.** 12/07/2007
HW12: the Last one!	**sect. 6.3**: 7c(only), 9c(only) **sect. 6.4:** 4 (I know I did it in class), 16 Solutions of HW12	**FR.** 12/14/2007

Solutions for the Final

#1. Both $(\tan x)$ and (x) are differentiable on $(0,+\infty)$ and $\frac{d}{dx}(x) = 1 \neq 0$ so we can use L'Hospital's Rule.

Thus $\lim_{x\to 0^+} \frac{\tan x}{x} = \lim_{x\to 0^+} \frac{\tan' x}{1} = \lim_{x\to 0^+} \frac{1}{\cos^2 x} = 1.$

#2 ⓐ We know that $x \mapsto x$ is continuous on $\mathbb{R}$ and that $x \mapsto |x-2|$ is also continuous on $\mathbb{R}$ (composition of 2 cont. functions)

Thus by the product rule, $g(x)$ is continuous on $\mathbb{R}$.

ⓑ On $(-\infty,2)$, $g(x) = -x.(x-2)$ (a polynomial fc.) so it is differentiable on $(-\infty,2)$.

On $(2,+\infty)$, $g(x) = x.(x-2)$ " " " " " .

<u>At the point $x=2$:</u> From the right: $\lim_{x\to 2^+} \frac{g(x)-g(2)}{x-2} = \lim_{x\to 2^+} \frac{+x.(x-2)-0}{x-2} = +2.$

From the left: $\lim_{x\to 2^-} \frac{g(x)-g(2)}{x-2} = \lim_{x\to 2^-} \frac{-x.(x-2)-0}{x-2} = -2$

Since $-2 \neq +2$, $\lim_{x\to 2} \frac{g(x)-g(2)}{x-2}$ doesn't exist, so $g'(2)$ doesn't exist.

#3 One has $\left|f(x)-\frac{1}{2}\right| = \left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{2x^2-2x+2-x-1}{2(x+1)}\right| = \left|\frac{2x^2-3x+1}{2(x+1)}\right| = |x-1| \cdot \frac{|2x-1|}{2|x+1|}.$

Let's prove that $\frac{|2x-1|}{2|x+1|}$ is bounded on $(0,2)$ (which is a neighborhood of 1):

Indeed $0<x<2 \Rightarrow -1<2x-1<3$ so $|2x-1| \leq 3$

and $0<x<2 \Rightarrow 2<2(x+1)<6$ so $\frac{1}{2|x+1|} < \frac{1}{2}$

} Thus on $(0,2)$ one has $\left|f(x)-\frac{1}{2}\right| \leq \frac{3}{2} \cdot |x-1|.$

Therefore, for a given $\varepsilon>0$, pick $\delta = \min\left\{1, \frac{2}{3}\varepsilon\right\}$: for any $x \in (1-\delta, 1+\delta)$ one has $\left|f(x)-\frac{1}{2}\right| \leq \frac{3}{2}\cdot\frac{2}{3}\varepsilon = \varepsilon.$

#4 By the density of $\mathbb{Q}$ in $\mathbb{R}$, any $x \in \mathbb{R}$ is the limit of a sequence (r_n) where each $r_n \in \mathbb{Q}$.

By continuity of f, one has: $f(x) = \lim f(r_n) = \lim 2r_n = 2x$, and we are done.

If you take now g defined by $\begin{cases} g(r) = 2r & \text{if } r \in \mathbb{Q} \\ g(x) = 0 & \text{if } x \in \mathbb{R}-\mathbb{Q} \end{cases}$ then clearly $g(x) \neq 2x$ everywhere (this g is not continuous).

#5. Each function $x \mapsto x$ and $x \mapsto \sin x$ is continuous on $\mathbb{R}$, and the values agree at 0 $(0 = \sin 0)$ so g is cont. on $\mathbb{R}$.

~~Each fc $x \mapsto x$~~ Since $x \mapsto x$ is differentiable on $(-\infty,0)$, so is g.

" $x \mapsto \sin x$ " " " $(0,+\infty)$ " "

At the point $x=0$: $\lim_{x\to 0^-} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^-} \frac{x}{x} = 1$ and $\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$ (L'Hospital's).

Thus $\lim_{x\to 0} \frac{g(x)-g(0)}{x}$ exists and g is differentiable on $\mathbb{R}$.

#6 One has $\frac{1}{n+\sqrt{n}\cdot\sin n} = \frac{1}{n}\cdot\left(\frac{1}{1+\frac{\sin n}{\sqrt{n}}}\right)$ and $\frac{1}{1+\frac{\sin n}{\sqrt{n}}} \to 1$ (because $0 \leq \left|\frac{\sin n}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \to 0$)

So by the comparison theorem $\sum \frac{1}{n+\sqrt{n}\cdot\sin n}$ converges if and only if $\sum \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, so does our infinite series.

#7: As usual we write

$$\frac{g(x+h)-g(x)}{h} = \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\sqrt{x}-\sqrt{x+h}}{h\cdot\sqrt{x}\cdot\sqrt{x+h}} = \frac{x-(x+h)}{\sqrt{x}+\sqrt{x+h}}\cdot\frac{1}{h\sqrt{x}\cdot\sqrt{x+h}}$$

$$= \frac{-1}{\sqrt{x}+\sqrt{x+h}}\cdot\frac{1}{\sqrt{x}\cdot\sqrt{x+h}}$$

where $\frac{-1}{\sqrt{x}+\sqrt{x+h}} \to -\frac{1}{2\sqrt{x}}$ and $\frac{1}{\sqrt{x}\cdot\sqrt{x+h}} \to \frac{1}{x}$.

Thus $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \frac{-1}{2x\cdot\sqrt{x}}$.

#8 a) By induction:
- $x_0 > 0$
- if $x_n > 0$ then $6 < 6 + x_n$ and thus $0 < \sqrt{6} < \sqrt{6+x_n} = x_{n+1}$, so $x_{n+1} > 0$.

b) If $x_n \to \ell$ then necessarily $x_{n+1} \to \ell$ and $\sqrt{6+x_n} \to \sqrt{6+\ell}$ (by continuity of $x \mapsto \sqrt{6+x}$).

so necessarily one must have $\begin{cases} \ell = \sqrt{6+\ell} \\ \text{and} \\ \ell \geq 0 \end{cases} \Rightarrow \ell = 3$ (the other root $\ell = -2$ is not ≥ 0).

c) $|x_{n+2} - x_{n+1}| = |\sqrt{6+x_{n+1}} - \sqrt{6+x_n}| = \frac{|x_{n+1}-x_n|}{\sqrt{6+x_n}+\sqrt{6+x_{n+1}}} \leq \underbrace{\frac{1}{2\sqrt{6}}}_{<1}\cdot|x_{n+1}-x_n|$.

Thus (x_n) is contractive so it is convergent to $\ell = 3$ (the only possible limit).

#9 $f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(c)$ for some $c \in (0,x)$, with $f(x) = (1+x)^{1/3}$, $f'(x) = \frac{1}{3}(1+x)^{-2/3}$,

$f''(x) = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(1+x)^{-5/3}$ and $f'''(x) = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+x)^{-8/3}$, thus Taylor's thm applied to f between 0 and x gives:

$f(x) = 1 + \frac{1}{3}x + \frac{x^2}{2}\left(-\frac{2}{9}\right) + \underbrace{\frac{10}{27}\frac{1}{(1+c)^{8/3}}}_{\geq 0}$, thus the result.

#10 $f(x) = \frac{\ln x + 3\sqrt{x} + 2x}{\sin x + 2 + x\sqrt{x}} = \frac{x\left(2 + \frac{\ln x}{x} + \frac{3\sqrt{x}}{x}\right)}{x\sqrt{x}\left(1 + \frac{2}{x\sqrt{x}} + \frac{\sin x}{x\sqrt{x}}\right)}$, but $\frac{\ln x}{x} \xrightarrow[x\to+\infty]{} 0$ (by l'Hospital's), $\frac{3\sqrt{x}}{x} \xrightarrow[x\to+\infty]{} 0$, $\frac{2}{x\sqrt{x}} \xrightarrow[x\to+\infty]{} 0$

and $\left|\frac{\sin x}{x\sqrt{x}}\right| \leq \frac{1}{|x\sqrt{x}|} \xrightarrow[x\to+\infty]{} 0$ so $\frac{\sin x}{x\sqrt{x}} \to 0$. Thus $f(x) = \frac{x}{x\sqrt{x}}\cdot g(x)$ with $\lim_{x\to+\infty} g(x) = 2$.

By the comparison thm, $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{\sqrt{x}} = 0$ (You can also apply the product rule: $f(x) = \frac{1}{\sqrt{x}}\cdot g(x)$, with $\frac{1}{\sqrt{x}} \to 0$ and $g(x) \to 2$).

MAT 319

Solutions of the Practice FINAL

Problem 1. What is the limit of $(x_n) = \frac{n^3}{n!}$?

Proof. Observe that $\frac{x_{n+1}}{x_n} = \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} = \frac{(n+1)^2}{n^3} \to 0$ therefore the sequence is converging to zero by the comparison theorem.

□

Problem 2. Use the definition of the limit to prove that $\lim \frac{n^2-1}{3n^2+1} = \frac{1}{3}$.

Proof. $\left|\frac{n^2-1}{3n^2+1} - \frac{1}{3}\right| = \left|\frac{3n^2-3-3n^2-1}{(3n^2+1).3}\right| = \frac{4}{9n^2+3} \leqslant \frac{1}{n^2}$ thus for a given $\varepsilon > 0$ if we take an integer $K > \frac{1}{\sqrt{\varepsilon}}$ we have that for any $n \geqslant K$, $\left|x_n - 1/3\right| \leqslant \frac{1}{n^2} < \varepsilon$.

□

Problem 3. Prove that an increasing sequence that is bounded above is necessarily converging.

Proof. See the textbook for this one... □

Problem 4. Show that if u_n is unbounded then there is a subsequence u_{n_k} of terms that are all non zero and such that $\frac{1}{u_{n_k}} \to 0$.

Proof. Since the sequence is unbounded, for any natural number k there is a term u_{n_k} of the sequence that is strictly larger than k, therefore one has $0 < \frac{1}{u_{n_k}} < \frac{1}{k} \to 0$ and this subsequence converges to zero by the squeeze theorem.

□

Problem 5. Is the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+2}$ convergent?

Proof. As usual factor by the leading term: $x_n = \frac{1}{n^2-n+2} = \frac{1}{n^2} \cdot \frac{1}{1 - \frac{1}{n} + \frac{2}{n^2}}$

But now if you write $y_n = 1/n^2$, we know that

$x_n/y_n = \frac{1}{1-\frac{1}{n}+\frac{2}{n^2}} \to 1$, therefore by the comparison theorem for infinite series, we know that our series converges if and only if $\sum y_n$ converges, but this is the case (p-series with $p = 2 > 1$).

□

Problem 6. Evaluate the following limit, or show that it doesn't exist: $\lim_{x \to +\infty} \frac{\sqrt{x} - x^2}{\sqrt{x} + x.\sqrt{x}}$.

Proof. Factor again by the leading term:

$f(x) = \frac{\sqrt{x} - x^2}{\sqrt{x} + x.\sqrt{x}} = \frac{-x^2}{x.\sqrt{x}} \cdot \frac{-\sqrt{x}/x^2 + 1}{1/x + 1}$ so if we call $g(x) = \frac{-x}{\sqrt{x}} = -\sqrt{x}$, we have that

$f(x)/g(x) \to 1$, so by the comparison theorem, the limit of f is the same as the limit of g which is $-\infty$.

□

Problem 7. Assume that $f: \mathbb{R} \to \mathbb{R}$ is such that: for any $x \in \mathbb{R}$ there is a $\delta > 0$ such that f is bounded on $[x - \delta, x + \delta]$. Is the function f bounded on $\mathbb{R}$? (If yes, prove it; if not give a counter-example).

Proof. Of course not! Take $f(x) = x$, it is locally bounded: on $[x - \delta, x + \delta]$, the function is bounded by $x + \delta$, but it is unbounded on the entire line.

□

Problem 8. Is the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x + |x|$ differentiable everywhere? (Prove your assertion!)

Proof. On $(-\infty, 0)$ the function is equal to $3x - x = 2x$, which is differentiable (derivative is the constant function equal to 2), and similarly on $(0, +\infty)$, the function is equal to $4x$ which is differentiable.

Now it remains to study the differentiability at zero:

But $\frac{g(x) - g(0)}{x - 0} = 2$ to the left of zero, and is equal to 4 to the right of zero, therefore the function is not differentiable at zero.

□

Problem 9. If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, then show that

$\lim (n(f(c + \frac{1}{n}) - f(c))$ exists and is equal to $f'(c)$.

Proof. For a given $\varepsilon > 0$, we know the existence of $\delta > 0$ such that for any $h \in (-\delta, \delta)$ we have

$\frac{f(c+h) - f(c)}{h} \in (f'(c) - \varepsilon, f'(c) + \varepsilon)$.

Now pick any natural number $K > 1/\delta$. Then for any $n \geqslant K$ one has that $1/n$ is less than δ and therefore

$\frac{f(c + \frac{1}{n}) - f(c)}{1/n} \in (f'(c) - \varepsilon, f'(c) + \varepsilon)$, which exactly expresses the convergence of the sequence to $f'(c)$.

□

Problem 10. Show that if $x > 0$ then we have $\sqrt[3]{1+x} \leqslant 1 + \frac{1}{3}x$

Proof. Apply Taylor's theorem at the order 2 to $f(x) = \sqrt[3]{1+x}$ between the points 0 and x.

It says that $f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(c)$ for some particular $c \in (0, x)$.

Notice now that $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$, so $f'(0) = 1/3$,

and also that $f''(x) = \frac{1}{3} . \frac{-2}{3}(1+x)^{\frac{-5}{3}}$, so that $f''(c) = \frac{-2}{9}$

Therefore the remainder is $\frac{-2}{9} . \frac{x^2}{2}$ which is less than zero, and this gives the inequality we want.

□

MAT 319

Practice FINAL

Problem 1. What is the limit of $(x_n) = \frac{n^3}{n!}$?

Problem 2. Use the definition of the limit to prove that $\lim \frac{n^2-1}{3n^2+1} = \frac{1}{3}$.

Problem 3. Prove that an increasing sequence that is bounded above is necessarily converging.

Problem 4. Show that if u_n is unbounded then there is a subsequence u_{n_k} of terms that are all non zero and such that $\frac{1}{u_{n_k}} \to 0$.

Problem 5. Is the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+2}$ convergent?

Problem 6. Evaluate the following limit, or show that it doesn't exist: $\lim_{x\to+\infty} \frac{\sqrt{x}-x^2}{\sqrt{x}+x.\sqrt{x}}$.

Problem 7. Assume that $f: \mathbb{R} \to \mathbb{R}$ is such that: for any $x \in \mathbb{R}$ there is a $\delta > 0$ such that f is bounded on $[x-\delta, x+\delta]$. Is the function f bounded on $\mathbb{R}$? (If yes, prove it; if not give a counter-example).

Problem 8. Is the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x + |x|$ differentiable everywhere? (Prove your assertion!)

Problem 9. If $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, then show that
$\lim (n(f(c+\frac{1}{n}) - f(c))$ exists and is equal to $f'(c)$.

Problem 10. Show that if $x > 0$ then we have $\sqrt[3]{1+x} \leqslant 1 + \frac{1}{3}x$

Solutions of Midterm II

Name:

Student I.D:

Problem 1. (25 points) Is the infinite series $\sum_{n=1}^{+\infty} \frac{1}{-1+3n.\sqrt{n}}$ convergent? (If yes, you don't need to find the value of the limit).

Answer:

We have $\frac{1}{-1+3n.\sqrt{n}} = \frac{1}{n^{3/2}}.\frac{1}{3-\frac{1}{n^{3/2}}}$

Now $\frac{1}{3-\frac{1}{n^{3/2}}} \to \frac{1}{3}$ when $n \to +\infty$. Therefore the Comparison theorem for infinite series tells us that $\sum \frac{1}{-1+3n.\sqrt{n}}$ converges if and only if $\sum \frac{1}{n^{3/2}}$ converges.

Since the exponent $3/2 > 1$, we know that $\sum \frac{1}{n^{3/2}}$ converges and therefore our infinite series is convergent.

Problem 2. (30 points) What is $\lim_{x\to+\infty} \frac{7x^2+1}{\sqrt{2x+5}}$?

Answer:

As usual we factor by the dominant terms: $\frac{7x^2+1}{\sqrt{2x+5}} = \frac{x^2}{\sqrt{x}}.\frac{7+\frac{1}{x^2}}{\sqrt{2+\frac{5}{x}}}$

Now $\lim_{x\to+\infty} \frac{7+\frac{1}{x^2}}{\sqrt{2+\frac{5}{x}}} = \frac{7}{\sqrt{2}} > 0$, by the sum rule, the quotient rule and the square root rule.

But now the Comparison theorem for functions tells us that $f(x) = \frac{7x^2+1}{\sqrt{2x+5}}$ has a limit equal to $+\infty$ at $+\infty$ if and only if the limit of $g(x) = \frac{x^2}{\sqrt{x}} = x^{3/2}$ at $+\infty$ is equal to $+\infty$. Since this is the case, we just proved that $\lim_{x\to+\infty} \frac{7x^2+1}{\sqrt{2x+5}} = +\infty$.

Problem 3. (30 points) Use the definition of a limit (I mean use "ε, δ")

to prove that $\lim_{x\to 3} \frac{2x^2+4}{x-1} = 11$.

Answer:

As usual we study the quantity $\left|f(x) - L\right| = \left|\frac{2x^2+4}{x-1} - 11\right| = \left|\frac{2x^2+4-11x+11}{x-1}\right| = \left|\frac{2x-5}{x-1}\right|.\left|x-3\right|$

Let us prove the existence of a small neighborhood of 3 where the quantity $\left|\frac{2x-5}{x-1}\right|$ is bounded above by a constant. Consider the neighborhood $V = (2, 4)$ of the point 3:

then $x \in V \Rightarrow 2 < x < 4 \Rightarrow 4 < 2x < 8 \Rightarrow -1 < 2x - 5 < 3$ which implies that $-3 < 2x - 5 < 3$, but this exactly means that $|2x - 5| < 3$ (observe that we are only interested in an upper bound, not a lower bound).

Similarly, $x \in V \Rightarrow 2 < x < 4 \Rightarrow 1 < x - 1 < 3 \Rightarrow 1 < \left|x - 1\right| < 3 \Rightarrow \frac{1}{3} < \frac{1}{|x-1|} < 1$.

If you put things together, you get that for any $x \in V$ we have:

$\left|\frac{2x-5}{x-1}\right| < 3$.

Now given $\varepsilon > 0$, if we take $0 < \delta$ satisfying both $\delta < 1$ (because we want the δ-neighborhood of 3 to be included in V, which is the 1-neighborhood of 3) and $\delta < \frac{\varepsilon}{3}$, we will have the following: for any x such that $\left|x - 3\right| < \delta$ we have that $\left|f(x) - 11\right| < 3.\delta < 3.\frac{\varepsilon}{3} = \varepsilon$. Thus we proved that

$\lim_{x\to 3} \frac{2x^2+4}{x-1} = 11$.

Problem 4. (15 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that for any $x \in \mathbb{R}$, we have

$$\left|f(x) - f(1)\right| < 6.\sqrt{|x-1|}.$$

Show that such a function f is continuous at 1. (You will get some partial credit if you recall the definition of the continuity of a function at a point).

Answer:

For any given $\varepsilon > 0$, if we take $0 < \delta < \left(\frac{\varepsilon}{6}\right)^2$, we have the following:

$\left|x - 1\right| < \delta$ implies that $\left|f(x) - f(1)\right| < 6.\sqrt{|x-1|} < 6.\sqrt{\delta} < 6.\frac{\varepsilon}{6} = \varepsilon$, but this means exactly that the function f is continuous at 1.

Since we didn't practice too much with that material (uniform continuity, continuity on intervals), I would like to give you some remarks, hints, and treat some similar examples...

Problem 1. *Show that every polynomial of odd degree with real coefficients has at least one real root.*

Answer. We did that in class already, but here is a complete solution. Consider the dominant term in your polynomial $P(x)$: $a_n x^n$. If you factor by it, you will get $P(x) = a_n x^n(1 + \frac{a_{n-1}}{x} + \ldots + \frac{a_0}{x^n})$. Clearly $\lim_{x\to+\infty} 1 + \frac{a_{n-1}}{x} + \ldots + \frac{a_0}{x^n} = 1$, therefore by the comparison theorem, $\lim P(x) = \lim a_n x^n$ which is $\pm\infty$ depending on the sign of a_n. Assume now that $a_n > 0$: then the limit of $P(x)$ at $+\infty$ is $+\infty$, and the limit at $+\infty$ is $-\infty$, therefore there exists an α such that for any $x > \alpha$ we have $f(x) > 1$ (for example), and there is a β such that for any $x < \beta$ we have $f(x) < -1$. Thus we found two real numbers such that $f(\alpha) \geq 1 > 0$ and $f(\beta) \leq -1 < 0$. Apply now the intermediate value theorem on $[\beta, \alpha]$, and get a zero of the function. The case where the coefficient $a_n < 0$ can be treated similarly.

Remark. The same idea can be used for problem 13 in section 5.3: far away your function is bounded (for example by 1), and in the middle you have a continuous function on a closed interval.

On problem 3, section 5.3. Try to build a sequence of points x_n that is converging and such that the sequence $f(x_n)$ converges to zero. At some point you might have to use Bolzano-Weierstrass...

On problem 11. This is actually how I proved the theorems in class. The key fact is to know that if a continuous function is strictly positive at a point c then there is a small δ-neighborhood of c on which the function is still strictly positive.

Problem 2. *Show that* $f(x) = \frac{1}{x}$ *is uniformly continuous on* $[1, +\infty)$

Answer. The key is to understand that a continuous function on a closed interval is uniformly continuous, and then to notice that since the function goes to zero when x becomes large, it will be uniformly continuous "at the infinity".

More precisely: given $\epsilon > 0$: there exists an $\alpha > 1$ such that for any $x > \alpha$ we have $|f(x)| < \frac{\epsilon}{2}$. Now consider the closed interval $[1, \alpha + 2]$: since f is continuous, it is uniformly continuous on that closed interval, therefore there exists a $\delta > 0$ such that for any $x, y \in [1, \alpha + 2]$, we have $|f(x) - f(y)| < \epsilon$. By possibly taking $\delta' = min(\delta, 1)$ we can even assume that $\delta < 1$. Now pick any two real numbers u, v in $[1, \infty)$ satisfying $|u - v| < \delta$, and assume that for example $u < v$:

1. if $v \leq \alpha + 2$: then both u, v are in $[1, \alpha + 2]$, and therefore they satisfy $|f(u) - f(v)| < \epsilon$;

2. if $v \geq \alpha + 2$, then necessarily $\alpha < \alpha + 1 < u < v$, and therefore by the triangle inequality we have $|f(u) - f(v)| \leq |f(u)| + |f(v)| < \epsilon/2 + \epsilon/2 = \epsilon$.

Thus we proved the uniform continuity of the function. Notice that basically we only used the fact that the function is converging to zero at infinity. Another possible approach would be to use the particular form of the function: $|f(x) - f(y)| = |\frac{x-y}{xy} \leq |x - y|$ if x, y are larger than 1 (and then for a given ϵ just pick $\delta = \epsilon$: the same δ will now work for any pair x, y.

MAT 319/320

Solutions of Practice Midterm II

Problem 1. Define a sequence (x_n) by : $x_0 = 1$ and $x_{n+1} = \frac{1}{3}x_n + 1$. Does (x_n) have a limit? If yes, what is this limit?

Proof. First, notice that if the sequence has a limit l, then it must satisfy the equation : $l = \frac{1}{3}.l + 1$, thus the only possible limit is $l = \frac{3}{2}$.

Let's show that this sequence is monotone and bounded, and therefore it will be convergent.

1. Claim: the sequence is bounded above by $\frac{3}{2}$.
   Indeed, by induction we get that: $x_0 \leqslant 3/2$, and if $x_n \leqslant 3/2$ then we deduce that $x_{n+1} \leqslant \frac{1}{3}.\frac{3}{2} + 1 = 3/2$, so we are done.
2. Claim: the sequence is increasing: this comes from the fact that $(x \leqslant 3/2) \Rightarrow \frac{1}{3}x + 1 \geqslant x$.

Therefore the sequence is converging to 3/2.

□

Problem 2. Is the infinite series $\sum_{n=1}^{+\infty} \frac{1}{n+\sqrt{n}}$ convergent? (If yes, you do not need to find the value of the limit).

Proof. We have $a_n = \frac{1}{n+\sqrt{n}} = \frac{1}{n}.\frac{1}{1+\frac{\sqrt{n}}{n}}$. Thus $\frac{a_n}{1/n} \to 1$, which implies that $\sum a_n$ converges if and only if $\sum \frac{1}{n}$ converges, by the comparison theorem. Since $\sum \frac{1}{n}$ diverges, we deduce that $\sum a_n$ diverges. □

Problem 3. Recall the definition of the continuity of a function f at a point c.

Proof. Hehe, see the textbook! (don't write that kind of answer for the actual midterm...) □

Problem 4. What is $\lim_{x\to+\infty} \frac{x+3}{\sqrt{x+7}+1}$?

Proof. As usual we factor both numerator and denominator by the "dominant term":
$\frac{x+3}{\sqrt{x+7}+1} = \frac{x}{\sqrt{x}}.\frac{1+\frac{3}{x}}{\sqrt{1+\frac{7}{x}}+\frac{1}{\sqrt{x}}}$. Now $\lim_{x\to+\infty} 1 + \frac{3}{x} = 1$ and $\lim_{x\to+\infty} \sqrt{1+\frac{7}{x}} + \frac{1}{\sqrt{x}} = 1$ because $\frac{1}{\sqrt{x}} \to 1$ and $\sqrt{1+\frac{7}{x}} \to 1$ by the square root rule.Therefore by the comparison theorem, we know that $\frac{x+3}{\sqrt{x+7}+1}$ has a limit equal to $+\infty$ if and only if $\lim_{x\to+\infty} \sqrt{x} = +\infty$, which is the case.

Thus, $\lim_{x\to+\infty} \frac{x+3}{\sqrt{x+7}+1} = +\infty$. □

Problem 5. Use the **definition of a limit** (I mean use "ε, δ") to prove that $\lim_{x\to-1} \frac{5x^2+2x+1}{x+3} = 2$. How could you prove the same thing using an easier way?

Proof. The easy way: a rational function is continuous at any point where the denominator is not zero. Since $x+3$ is not zero at $x = -1$, we know that the limit is actually equal to $\frac{5.1+2.(-1)+1}{-1+3} = 2$.

The complicated way (using the definition):
$\left|\frac{5x^2+2x+1}{x+3} - 2\right| = \left|\frac{5x^2+2x+1-2x-6}{x+3}\right| = 5.\left|x+1\right|.\left|\frac{x-1}{x+3}\right|$. Now let's prove that on a neighborhood of -1, the function $5.\left|\frac{x-1}{x+3}\right|$ is bounded above.

One can take for example the neighborhood $V = (-2, -\frac{1}{2})$. Then on V, one has:
$-3 < x-1 < \frac{-3}{2} \Rightarrow \left|x-1\right| < 3$, and similarly on V one has : $1 < x+3 < 5/2 \Rightarrow \frac{1}{|x+3|} < 1$.

Therefore, on V we have that $5.\left|\frac{x-1}{x+3}\right| < 15$.

Now for a given $\varepsilon > 0$, if we take $x \in V$, we have that $\left|\frac{5x^2+2x+1}{x+3} - 2\right| < 15.\left|x+1\right|$, so it is enough to take $\delta > 0$ less than $\varepsilon/15$ and such that $(-1-\delta, -1+\delta) \subset V$. Clearly $\delta = \min(\varepsilon/15, 1/2)$ will work. ☐

Problem 6. Let $f: [0,3] \rightarrow \mathbb{R}$ be a continuous function. Assume that $f(1) > 0$, then prove the existence of a small δ-neighborhood of 1 on which the function f has no root (meaning there is no x in this neighborhood such that $f(x) = 0$).

Proof. Take $\varepsilon = \frac{f(1)}{2} > 0$. By continuity, there exists a $\delta > 0$ such that for any $x \in [0,3] \cap (1-\delta, 1+\delta)$,

one has $f(x) \in \left(\frac{f(1)}{2}, \frac{3}{2}f(1)\right)$, and thus on that neighborhood of 1, we have that $f(x) > 0$, so it has no zero. ☐

MAT 319/320

Practice Midterm II

Problem 1. Define a sequence (x_n) by : $x_0 = 1$ and $x_{n+1} = \frac{1}{3}x_n + 1$. Does (x_n) have a limit? If yes, what is this limit?

Problem 2. Is the infinite series $\sum_{n=1}^{+\infty} \frac{1}{n+\sqrt{n}}$ convergent? (If yes, you do not need to find the value of the limit).

Problem 3. Recall the definition of the continuity of a function f at a point c.

Problem 4. What is $\lim_{x\to+\infty} \frac{x+3}{\sqrt{x+7}+1}$?

Problem 5. Use the **definition of a limit** (I mean use "ε, δ") to prove that $\lim_{x\to-1} \frac{5x^2+2x+1}{x+3} = 2$. How could you prove the same thing using an easier way?

Problem 6. Let $f: [0,3] \to \mathbb{R}$ be a continuous function. Assume that $f(1) > 0$, then prove the existence of a small δ-neighborhood of 1 on which the function f has no root (meaning there is no x in this neighborhood such that $f(x) = 0$).

Correction of HW1

Exercise 1. Page 15, #2.

Proof. By induction:

1. The property is true for $n=1$, because $1^3=\left[\frac{1}{2}.1.2\right]^2$
2. Assume that the property is true for n, and prove that it's true for $n+1$:

$$\begin{aligned} 1^3+\ldots+n^3 &= \left[\frac{1}{2}.n.(n+1)\right]^2 \\ \Rightarrow\ 1^3+\ldots+n^3+(n+1)^3 &= \left[\frac{1}{2}.n.(n+1)\right]^2+(n+1)^3 \\ \Rightarrow\ 1^3+\ldots+(n+1)^3 &= \frac{1}{4}.(n+1)^2\left[n^2+4(n+1)\right] \\ \Rightarrow\ 1^3+\ldots+(n+1)^3 &= \frac{1}{4}.(n+1)^2[n+2]^2 \\ \Rightarrow\ 1^3+\ldots+(n+1)^3 &= \left[\frac{1}{2}.(n+1).(n+2)\right]^2 \end{aligned}$$

But this is exactly the property for (n+1).

□

Exercise 2. Page 29, #3.

Proof. a) $2x+5=8 \ \Rightarrow 2x=3$(existence of negative elements) $\ \Rightarrow x=3/2$(existence of inverse for nonzero elements).

b) add $-2x$ to both sides to get $x^2-2x=0$. Then factor (using distributivity) to get $x(x-2)=0$. Conclude with theorem 2.1.3.

c) add -3 to both sides to get $x^2-4=0$, use distributivity to factor and conclude like in b).

d) Same: apply theorem 2.1.3 ($a.b=0$ implies $a=0$ or $b=0$).

□

Exercise 3. Page 30, #8.

Proof. a) Clearly $\frac{a}{b}+\frac{c}{d}=\frac{a.d+b.c}{b.d}\in\mathbb{Q}$ and $\frac{a}{b}.\frac{c}{d}=\frac{a.c}{b.d}\in\mathbb{Q}$.

b) If x is rational and y irrational, then $x+y$ can't be rational because $y=(x+y)+(-x)$ would also be rational.

Now if in addition $x\neq 0$, then $x.y$ can't be rational because $y=(x.y).(1/x)$ would also be rational from a).

□

Exercise 4. Page 30, #18.

Proof. By contradiction: assume that $a>b$. Then take $\varepsilon=\frac{a-b}{2}$. One should have $a-b\leqslant(\frac{a-b}{2})$ which is absurd. □

Exercise 5. Page 30, #23.

Proof. Let us prove that if $a,b>0$ then $a<b$ if and only if $a^n<b^n$ for any $n\in\mathbb{N}$.

Clearly the right hand side implies the left one.

Let's prove the rest by induction:

1. The property is true for $n=1$ because $a<b\Rightarrow a^1<b^1$.
2. Assume that one has $a^n<b^n$. Then $a.a^n<a.b^n<b.b^n$ so we are done.

□

Exercise 6. Page 34, #1.

Proof. a) Clearly $\left|a\right|^2=a^2$, and since $\left|a\right|\geq 0$, it is the square root of a^2.

b) Just notice that $\left|a\right|=\left|\frac{a}{b}.b\right|=\left|\frac{a}{b}\right|.\left|b\right|$ which gives the result.

□

Exercise 7. Page 34, #6.

Proof. a) $|4.x-5|\leqslant 13$ is equivalent to

$$\begin{aligned}-13\leqslant 4.x-5\leqslant 13\\ -13+5\leqslant 4.x\leqslant 13+5\\ -2\leqslant x\leqslant 9/2\end{aligned}$$

so this is equivalent to $x\in[-2,9/2]$.

b) $\left|x^2-1\right|\leqslant 3$ is equivalent to

$$\begin{aligned}-3\leqslant x^2-1\leqslant 3\\ -2\leqslant x^2\leqslant 4\end{aligned}$$

but the last line is equivalent to $0\leqslant x^2\leqslant 4$, which is itself equivalent to $x\in[-2,2]$.

□

Exercise 8. Page 34, #15.

Proof. Assume $a<b$, then any positive real number strictly less than $(b-a)/2$ will work.

Take $\varepsilon=(b-a)/2$, then $U=(a-\frac{b-a}{2},\frac{a+b}{2})$ and $V=(\frac{a+b}{2},b+\frac{b-a}{2})$, and these two open intervals are disjoint.

It's probably cleaner to use $\varepsilon=\frac{b-a}{3}$ instead... □

Correction of HW2

Exercise 1. Page 38, #2.

Proof. The set S_2 is not empty and is bounded below (for example by 0), so it has an infimum. Let's prove that $0 = \inf(S_2)$:

1. For any x in S_2, one has $0 \leqslant x$;
2. For any $\varepsilon > 0$, one can find x in S_2 such that $0 \leqslant x < 0 + \varepsilon$ (take $\varepsilon/2$ for example).

Therefore $0 = \inf(S_2)$.

Now S_2 is not bounded above, so it doesn't have upper bounds (and therefore doesn't have a sup).

□

Exercise 2. Page 38, #3.

Proof. Supremum: Since $n \in \mathbb{N} \Rightarrow \frac{1}{n} \leqslant 1$, one knows that S_3 is bounded above, since it is also a non empty subset of $\mathbb{R}$, we know that it has a supremum. Let's prove that $1 = \sup(S_3)$:

1. For any x in S_3, one has $x \leqslant 1$;
2. For any $\varepsilon > 0$, one can find an x in S_3 such that $1 - \varepsilon < x \leqslant 1$ (just take $x = 1$!)

Infimum:

Let's prove that $0 = \inf(S_3)$:

1. For any $x = 1/n$ in S_3, one has $x \geqslant 0$;
2. For any $\varepsilon > 0$, one can find an x in S_3 such that $0 \leqslant x < \varepsilon$ (indeed by the archimedean property one knows the existence of an integer $n_\varepsilon > 1/\varepsilon$, then just take $x = 1/n_\varepsilon$).

□

Exercise 3. Page 38, #7.

Proof.

a) Assume that u is an upper bound of S non empty: this means that for any x in S one has $x \leqslant u$. Now if t is any real number such that $t > u$, we will get that $t > x$ for any $x \in S$, so $t \notin S$.

b) Conversely: Assume now that $(t > u) \Rightarrow t \notin S$. Suppose that u is not an upper bound. Since S is not empty, this would imply the existence of $y \in S$ such that $y > u$ (contradiction).

□

Exercise 4. Page 38, #9.

Proof.

a) If α is an upper bound for A, and β is an upper bound for B, then the maximum of the two numbers α, β is an upper bound for $A \cup B$. For the lower bounds, take the minimum instead.So the union of two bounded sets is a bounded set.

b) At this point we know the existence of sup $(A \cup B)$.Let's prove that sup $(A \cup B)$ = sup $\{\sup A, \sup B\}$:

   1. We already know that $Z = \sup\{\sup A, \sup B\}$ is an upper bound of $A \cup B$;
   2. For any $\varepsilon > 0$, is there an element $x \in A \cup B$ such that $Z - \varepsilon < x$?

There are two cases: if $Z = \sup A$, then we now the existence of an element y in A such that $\sup A - \varepsilon < y$ so we are done (because $y \in A \subset A \cup B$). If $Z = \sup B$, the same argument works (replace A by B).

□

Exercise 5. Page 43, #1.

Proof. Let's show that $\sup S = 1$, where $S = \left\{1 - \frac{1}{n}, n \in \mathbb{N}\right\}$:

1. For any $x = 1 - \frac{1}{n}$, one has $x \leqslant 1$;
2. For any $\varepsilon > 0$, one can find an x in S such that $1 - \varepsilon < x \leqslant 1$: indeed by the archimedean property one knows the existence of an integer $n_\varepsilon > 1/\varepsilon$. Therefore $1/n_\varepsilon < \varepsilon$ and $1 - \varepsilon < 1 - 1/n_\varepsilon$.

□

Exercise 6. Page 43, #14.

Proof. As in the textbook, let $S := \{s \in \mathbb{R}; 0 \leqslant s \text{ and } s^2 < 3\}$.

S is not empty (it contains 1 for example) and is bounded above (for example by 2, because $s \geqslant 2$ implies that $s^2 \geqslant 4$ and thus such an s is not in S). Therefore, by completeness of $\mathbb{R}$ we know the existence of $x = \sup(S)$. Let's prove now that $x^2 = 3$.

1. $\boldsymbol{x^2 > 3}$ **is impossible:**

   It is enough to find an integer $n \geqslant 1$ such that $(x - \frac{1}{n})^2 > 3$. Because then any $s \in S$ would be such that $s^2 < (x - \frac{1}{n})^2$, implying $s < (x - \frac{1}{n})$ (because $s \geqslant 0$ and $(x - \frac{1}{n}) > 0$); but that last inequality would mean that $(x - \frac{1}{n})$ is an upper bound of S (absurd).

   Let's find such an integer n:

   we notice that $(x - \frac{1}{n})^2 = x^2 - \frac{2.x}{n} + \frac{1}{n^2} \geqslant x^2 - \frac{2.x}{n}$. So we would be done if we could find n such that $x^2 - \frac{2.x}{n} > 3$, but this is equivalent to finding an n such that $\frac{x^2 - 3}{2.x} > \frac{1}{n}$ where x is given to you. But we know that this is possible, by the archimedean property of $\mathbb{R}$.

   Remark: other possible proof:

   Remark that $x^2 > 3$ implies $x > 3/x$, therefore one has that $y = \frac{1}{2}(x + \frac{3}{x}) < x$. But now $y^2 > 3$. Indeed $y^2 - 3 = \frac{1}{4}(x^2 + 6 + \frac{9}{x^2} - 12) = \left[\frac{1}{2}(x - \frac{3}{x})\right]^2 > 0$.
2. $\boldsymbol{x^2 < 3}$ **is impossible:**

   If one can finds an integer n such that $(x + \frac{1}{n})^2 \leqslant 3$, we are done (because we found an element of S strictly larger than $\sup S$,which is absurd).

   Notice that $(x + \frac{1}{n})^2 = x^2 + \frac{2.x}{n} + \frac{1}{n^2} \leqslant x^2 + \frac{2.x}{n} + \frac{1}{n}$. So we will be done if we can find an integer n such that $x^2 + \frac{2.x}{n} + \frac{1}{n} \leqslant 3$, which is equivalent to $\frac{1}{n}(2.x + 1) \leqslant 3 - x^2$, or if one prefers $n \geqslant \frac{2x+1}{3-x^2}$ (notice that $3 - x^2 \neq 0$, so I can divide by it!). But such an integer can always be found, given x, thanks to the archimedean property of $\mathbb{R}$.

□

Exercise 7. Page 43, #18.

Proof. Since $u > 0$, we know that $x < y$ implies $x/u < y/u$. Then we know the existence of a rational number $r \in \mathbb{Q}$ such that $x/u < r < y/u$. But this implies that $x < r.u < y$.

□

Correction of HW3

Exercise 1. Page 50, #2.

Proof.

1. If S is bounded then there exists a lower bound m and an upper bound M. By definition, they are such that any x in S satisfies $m \leqslant x \leqslant M$. But this means $x \in [m, M]$.Therefore $S \subset [m, M]$.
2. Conversely, $S \subset [m, M]$ exactly means that any x in S is bounded above by M, and below by m.

□

Exercise 2. Page 50, #9.

Proof. By contradiction: assume that the intersection is non empty, and therefore contains some real number x. Pick any integer K strictly larger than x (for example $1 + E(x)$, where $E(x)$ is the integral part of x): then clearly $x \notin (K, \infty)$ and thus $x \notin \bigcap_{n=1}^{\infty} (n, \infty)$, a contradiction.

□

Exercise 3. Page 50, #13.

Proof. Since 1/3 is strictly less than 1, the binary representation starts with 0.

We want to find $a, b, c, d \in \{0, 1\}$ such that the binary representation of 1/3 starts with $(0.\text{abcd}...)_2$.

We notice that $\frac{1}{2} > \frac{1}{3}$, so the first digit a must be 0 (not one). Then $\frac{1}{4} < \frac{1}{3}$, so the next digit is 1. Then $\frac{1}{4} + \frac{1}{8}$ is too large so the following digit must be 0. Similarly the fourth digit is 1 because $\frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} < 1/3$.

It seems that there is a pattern: so let's prove that the binary expansion of 1/3 is 0.010101...

Call $x := (0.0101010101...)_2$ Then notice that $2^2.x = (1.01010101...)_2$, so by subtraction one has that $(2^2 - 1).x = 1$ which means exactly that $x = 1/3$.

□

Exercise 4. Page 50, #17.

Proof. Write $x = 1.25137137...$ then $100x = 125 + 0.137137...$

But if you write $y = 0.137137...$, you see that $999y = 137$, therefore $x = \frac{125 + \frac{137}{999}}{100} = \frac{125012}{99900}$.

Similarly, if $y = 35.14653653...$ you see that $100y = 3514 + \frac{653}{999}$ therefore $y = \frac{3511139}{99900}$.

□

Exercise 5. Page 59, #3c.

Proof. We have already $z_1 = 1, z_2 = 2, z_3 = \frac{2+1}{2-1} = 3, z_4 = \frac{3+2}{3-2} = 5, z_5 = \frac{5+3}{5-2} = \frac{8}{3}$. □

Exercise 6. Page 59, #4.

Proof. Let $\varepsilon > 0$, by the archimedean property we know the existence of an integer K satisfying $K \geqslant \frac{|b|}{\varepsilon}$. Therefore, for any $n \geqslant K$ one has $n \geqslant \frac{|b|}{\varepsilon}$ and thus $\left|\frac{b}{n}\right| \leqslant \varepsilon$. Therefore, the sequence is converging to zero. □

Exercise 7. Page 59, #5c.

Proof. One has $0 \leqslant \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{6n+2-6n-15}{4n+10}\right| \leqslant \frac{13}{4n+10} \leqslant \frac{13}{4}.\frac{1}{n}$

Since we know that $1/n$ converges to zero, we deduce that x_n converges to $3/2$. □

Exercise 8. Page 59, #6c.

Proof. One has $0 \leqslant \left|\frac{\sqrt{n}}{n+1}\right| \leqslant \frac{1}{\sqrt{n}}$ for $n \geqslant 1$, so it is enough to prove that $1/(\sqrt{n})$ converges to zero.

Given any $\varepsilon > 0$, by the archimedean property one can find an integer $K \geqslant \varepsilon^2$, therefore for any $n \geqslant K$, one has $n \geqslant \varepsilon^2$ and so $0 \leqslant 1/(\sqrt{n}) \leqslant \varepsilon$, so $(1/\sqrt{n})$ converges to zero. □

Exercise 9. Page 59, #8.

Proof. The convergence of (x_n) to zero translates as follows:

for any $\varepsilon > 0$ there exists an integer K such that: for all $n \geqslant K$ one has $|x_n| < \varepsilon$.

The convergence of $(|x_n|)$ to zero translates as follows:

for any $\varepsilon > 0$ there exists an integer K such that: for all $n \geqslant K$ one has $||x_n|| < \varepsilon$.

Since $|x_n| \geqslant 0$, one has that $|x_n| = ||x_n||$ so the two propositions are equivalent.

Now if $x_n = (-1)^n$, one can see that $|x_n| = 1$ so it converges, but (x_n) doesn't converge.

□

Exercise 10. Page 67, #5b.

Proof. A convergent sequence must be bounded. Since $((-1)^n.n^2)$ is unbounded, it cannot converge.

(Remark: to be convinced that it is unbounded, use the Archimedean property. Given any $M > 0$, there exists an integer $K > \sqrt{M}$ and therefore any $n \geqslant K$ satisfies $\left|(-1)^n.n^2\right| > M$)

□

Exercise 11. Page 67, #6d.

Proof. One has $x_n = \frac{n+1}{n\sqrt{n}} = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}}$. Now we have already proved above that $(1/\sqrt{n})$ converges to zero (Archimedean property!), and since $0 \leqslant \left|\frac{1}{n\sqrt{n}}\right| \leqslant \frac{1}{n} \to 0$, we see that x_n is the sum of two sequences converging to zero, therefore it converges to zero.

□

Exercise 12. Page 67, #7.

Proof. Let $M > 0$ be an upper bound for the sequence (b_n).

Given any $\varepsilon > 0$, since (a_n) converges to zero, we know the existence of an integer K such that for all $n \geqslant K$ one has $|a_n| \leqslant \frac{\varepsilon}{M}$.

Now for any $n \geqslant K$, one has $|a_n.b_n| \leqslant |a_n|.M \leqslant \varepsilon$. But this exactly says that $(a_n b_n)$ converges to zero.

The theorem 3.2.3 cannot be applied because (b_n) is only bounded, and not necessarily convergent.

□

Exercise 13. Page 67, #17.

Proof. Let r be a real number satisfying $1 < r < L$. Since $\left(x_{n+1}/x_n\right)$ converges to L, we know the existence of an integer K such that for any $n \geqslant K$ one has $\left|\frac{x_{n+1}}{x_n} - L\right| < L - r$. But this implies that for any $n \geqslant K$ one has $\frac{x_{n+1}}{x_n} > r$.

Let's prove by induction that for any $n \geqslant K$ one has $x_n \geqslant r^{n-K}.x_K$

This is true for $n = K$ because $x_K = r^0.x_K$.

Assume it is true for n, then we have that $x_{n+1} > r.x_n > r.r^{n-K}.x_K = r^{n+1-K}.x_K$, so we are done.

Now it remains to prove that the sequence (r^n) for $r > 1$ is unbounded.

Here is one possible way: write $r = 1 + d$, and prove by induction that for any n one has $(1 + d)^n > 1 + n.d$.

Another way is to take the $\log(r^n)$ and apply the archimedean property.

□

Correction of HW4

Exercise 1. Page 67, #6a.

Proof.
By the sum rule, the limit of $(2+1/n)$ is equal to 2. By the product rule, the limit of $(2+1/n)^2$ is $2.2=4$.

□

Exercise 2. Page 67, #9.

Proof. One has $y_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ (multiply the numerator and denominator by the conjugate quantity).

But now one has $0 \leqslant y_n \leqslant \frac{1}{\sqrt{n}}$. Therefore if one proves that $(1/\sqrt{n})$ converges to zero, the squeeze theorem implies that (y_n) converges itself to zero.

Fix any $\varepsilon > 0$, then by the archimedean property there exists a natural number K such that $K > \varepsilon^2$, but this implies that for any $n \geqslant K$ one has $n > \varepsilon^2$, implying $\frac{1}{\sqrt{n}} < \varepsilon$, thus we proved that $(1/\sqrt{n})$ converges to zero, and hence (y_n) converges to zero.

Now $\sqrt{n}\, y_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n}}.\frac{1}{\sqrt{(1+1/n)}+1} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$. By the square root theorem $\sqrt{1+\frac{1}{n}}$ converges to 1. By the quotient theorem (which applies because the limit of the denominator is nonzero), one knows that $\sqrt{n}.y_n$ converges to $1/2$.

□

Exercise 3. Page 67, #21.

Proof. Pick any $\varepsilon > 0$.

Since (x_n) is convergent to a limit x, we know the existence of a natural number K such that for any $n \geqslant K$ one has $|x_n - x| < \varepsilon/2$. We also know the existence of another natural number M' such that for any $n \geqslant M'$ one has $|x_n - y_n| < \varepsilon/2$.

Now for any $n \geqslant L = \max\{K, M'\}$ one has by the triangle inequality:

$$|y_n - x| \leqslant |y_n - x_n| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus we proved that (y_n) converges, to the same limit x.

□

Exercise 4. Page 74, #1.

Proof. Let's prove that for any $n \geqslant 1$ one has $4 \leqslant x_n \leqslant 8$.(Make a drawing to guess these bounds!)

This is true for $n=1$ (because $x_1 = 8$). Assume it is true for k: then one has $x_{k+1} = \frac{1}{2}x_k + 2 \leqslant \frac{1}{2}8 + 2 = 6 \leqslant 8$, and also $\frac{1}{2}x_k + 2 \geqslant \frac{4}{2} + 2 = 4$ so this is true for x_{k+1}.

Now let us prove that the function $f(x) = \frac{1}{2}x + 2$ is such that $f(x) < x$ on the interval $(4,8]$: indeed $f(x) < x$ is equivalent to $\frac{1}{2}x + 2 < x$, which is equivalent to $2 < \frac{1}{2}x$ or simply $x > 4$.

Therefore since any x_n belongs to that interval, one has that $x_{n+1} = f(x_n) < x_n$, and thus our sequence is strictly decreasing. Since it is also bounded below, we know that it must converge to a limit x.

Now the limit x must satisfy $x = \frac{1}{2}x + 2$, which is equivalent to $x = 4$, so the limit is 4. □

Exercise 5. Page 74, #4.

Proof. Let's prove that for any $n \geqslant 1$ one has $0 \leqslant x_n \leqslant 2$. (Again make a drawing to guess this). This is true for $n = 1$ because $x_1 = 1$. Assume it is true for k:

then one has $2 \leqslant 2 + x_k \leqslant 4$ and thus $0 \leqslant \sqrt{2} \leqslant \sqrt{2 + x_k} \leqslant \sqrt{4} = 2$, so it is true for $k + 1$.

Now let us prove that on the interval $[0,2]$ the function $f(x) = \sqrt{2 + x}$ satisfies $f(x) \geqslant x$.But since $f(x) > 0$ on this interval, so is $\sqrt{2 + x} + x$, thus one has $f(x) - x = \frac{2}{\sqrt{2 + x} + x} \geqslant 0$ (multiply numerator and denominator by the conjugate quantity, which is nonzero).

Since any x_n belongs to that interval one has $x_{n+1} = f(x_n) \geqslant x_n$, so the sequence is increasing. It is also bounded by 2, so it is convergent to a real number x by our theorem on convergence of monotone bounded sequences. Now the limit x must satisfy $x = \sqrt{2 + x}$, which is equivalent to $\left(x \geqslant 0 \text{ and } x^2 = 2 + x\right)$, which is equivalent to $(x = 2)$ (notice that $x^2 - x - 2 = (x - 2).(x + 1)$ and we want the positive root).

Thus the sequence converges to 2.

□

Correction of HW5

Exercise 1. Page 80, #1.

Proof.
Take for example the following sequence: $x_{2k}=k, x_{2k+1}=1$.

□

Exercise 2. Page 80, #3.

Proof. Recall that f_n satisfies the relation $f_{n+2}=f_{n+1}+f_n$ and that all the f_n are >0.

therefore we deduce that $\frac{f_{n+2}}{f_{n+1}}=1+\frac{f_n}{f_{n+1}}$.We know that $x_n=\frac{f_{n+1}}{f_n}$ has a limit L. Notice that L cannot be zero (because then $\frac{f_n}{f_{n+1}}$ would then be unbounded, which is not the case because it is equal to $\frac{f_{n+2}}{f_{n+1}}-1$, which converges to $L-1$). Thus we can apply the quotient theorem and obtain the equality $L=1+\frac{1}{L}$. This implies that $L^2=L+1$.By solving this equation and keeping the positive root we get $L=\frac{1+\sqrt{5}}{2}$.

□

Exercise 3. Page 80, #8a.

Proof. Let's compare $x_{n+1}=(3(n+1))^{1/2(n+1)}$ and $x_n=(3n)^{1/2n}$. We have

$x_{n+1}^{2n.2(n+1)}=3(n+1)^{2n}=(3n)^{2n}.(1+\frac{1}{n})^{2n}$, whereas $x_n^{2n.2(n+1)}=(3n)^{2(n+1)}=(3n)^{2n}.(3n)^2$, therefore after some integer K, the sequence is decreasing (because $(1+\frac{1}{n})^{2n}$ is eventually strictly smaller than $(3n)^2$). Since it is bounded below by 0, it has a limit L.

Now the subsequence x_{2n} must converge to the same limit L, but we have
$x_{2n}=(3.2n)^{1/4n}=2^{1/4n}.x_n^{1/2}$ so this converges to $1.\sqrt{L}=\sqrt{L}$ so $L=\sqrt{L}$ and thus $L=1$.

□

Exercise 4. Page 80, #14.

Proof. Pick $\varepsilon_1=1$, then we know the existence of x_{n_1} such that $s-\varepsilon_1<x_{n_1}\leqslant s$ (def. of a sup), and we even know that $x_{n_1}<s$.

Pick $\varepsilon_2=\frac{1}{k_2}$, such that k_2 is at least 2 (thus $\varepsilon_2<\frac{1}{2}$), and such that $x_{n_1}<s-\varepsilon_2$, then one knows the exisence of x_{n_2} such that $s-\varepsilon_2<x_{n_2}<s$.

Pick $\varepsilon_3=\frac{1}{k_3}$, such that k_3 is at least 3 (thus $\varepsilon_3<\frac{1}{3}$), and such that $x_{n_2}<s-\varepsilon_3$, then one knows the existence of x_{n_3} such that $s-\varepsilon_3<x_{n_3}<s$.

By continuing like this one constructs an increasing subsequence x_{n_k} that has the property that $s-\frac{1}{k}<x_{n_k}<s$, therefore it converges to s (Squeeze theorem!). □

Exercise 5. Page 80, #15.

Proof. Since the I_n are nested one knows that $x_n \in I_0$ so this sequence is bounded and therefore by Bolzano-Weierstrass. it has a converging subsequence (x_{n_k}), with limit L.

Let's prove by contradiction that $L \in \cap_{n=1}^{\infty} I_n$. Indeed, if it's not the case, then say $L \notin I_N$ for some N. Pick $\varepsilon > 0$ small enough such that $(L - \varepsilon, L + \varepsilon) \cap I_N = \emptyset$. By convergence of (x_{n_k}), there is some element of this subsequence, say x_{n_K} that lands in $(L - \varepsilon, L + \varepsilon)$ and such that n_K is larger than N, thus I_{n_K} intersects $(L - \varepsilon, L + \varepsilon)$, but this is a contradiction because $I_{n_K} \subset I_N$ (and I_N doesn't intersect that interval).

□

Exercise 6. Page 86, #1.

Proof. $(-1)^n$ is bounded and not convergent so it's not a Cauchy sequence.

□

Exercise 7. Page 86, #3c.

Proof. $(\ln n)$ is not bounded so it's certainly not a Cauchy sequence. This can be proved using the definition:pick $\varepsilon = 1$, can we find K such that for any $n, m \geqslant K$ one has $|\ln n - \ln m| = |\ln \frac{n}{m}|$ less than 1? The answer is no: take $n = 5m > m \geqslant K$, then $\ln \frac{6m}{m} = \ln 6 > 1$. □

Exercise 8. Page 86, #9.

Proof. Notice that $|x_{n+p} - x_n| \leqslant |x_{n+p} - x_{n+p-1}| + \ldots + |x_{n+1} - x_n| < r^{n+p-1} + \ldots + r^n$.

But this last sum is also $r^n.(r^{p-1} + \ldots + 1) = r^n.\frac{1 - r^p}{1 - r} < \frac{1}{1 - r}.r^n$.

Given any $\varepsilon > 0$, then one can find a natural number K such that for any $n \geqslant K$ one has $\frac{1}{1 - r}.r^n \leqslant \frac{1}{1 - r}.r^K < \varepsilon$, and thus the sequence is a Cauchy sequence.

□

Exercise 9. Page 86, #13.

Proof. First we notice that x_n is never zero so the sequence is well-defined.

Then one has $\left|x_{n+2} - x_{n+1}\right| = \left|2 + \frac{1}{x_{n+1}} - 2 - \frac{1}{x_n}\right| = \left|\frac{x_n - x_{n+1}}{x_n.x_{n+1}}\right| \leqslant \frac{1}{4}.\left|x_{n+1} - x_n\right|$ because every x_n is > 2.So the sequence is contractive, and therefore converges to a limit x.This limit must satisfy $x = 2 + \frac{1}{x}$ and be positive, thus one must have $x^2 = 2x + 1$, and then one gets that $x = 1 + \frac{\sqrt{5}}{2}$.

□

Correction of HW6

Exercise 1. Section 3.6, #1.

Proof. Pick $\alpha_1 = 1$. Since the sequence is unbounded, one can find $x_{n_1} > \alpha_1$.

Pick $\alpha_2 = \max(2, x_{n_1} + 1)$. For the same reason, one can find $x_{n_2} > \alpha_2$.

Continue like this: by construction, the subsequence is increasing and satisfies $x_{n_k} > k$, therefore it is properly divergent. □

Exercise 2. Section 3.6, #8d.

Proof. Read Example 3.4.6(c).In it, they construct two sequences $(n_k)_{k\geqslant 1}$ and $(m_k)_{k\geqslant 1}$ of natural numbers, such that $\sin(n_k) \in [1/2, 1]$ and $\sin(m_k) \in [-1, -1/2]$. By taking subsequences and using Bolzano-Weierstrass, one can even assume that $\sin(n_k)$ converges to $c_1 \in [1/2, 1]$ and that $\sin(m_k) \to c_2 \in [-1, -1/2]$. Now if you consider the subsequences $(n'_k) = (n_k^2)$ and $(m'_k) = (m_k^2)$, they are such that $\sin(\sqrt{n'_k})$ and $\sin(\sqrt{m'_k})$ converge towards two distinct numbers, therefore the sequence $\sin(\sqrt{n})$ cannot converge. □

Exercise 3. Section 3.6, #10.

Proof. Since $(a_n/n) \to L \neq 0$ one knows that $\lim(a_n) = +\infty$ if and only if $\lim(n) = +\infty$, which is the case, therefore $\lim(a_n) = +\infty$. □

Exercise 4. Section 3.7, #3a.

Proof. For any natural number $n \geqslant 0$ one has:

$\frac{1}{(n+1).(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$. Therefore by adding these equalities for $n = 0$ to $n = N$, one gets:

$\sum_{n=0}^{n=N} \frac{1}{(n+1).(n+2)} = \sum_{n=0}^{n=N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{1} - \frac{1}{N+2} \to 1$ when $N \to +\infty$, so

$\sum_{n=0}^{+\infty} \frac{1}{(n+1).(n+2)} = 1$. □

Exercise 5. Section 3.7, #6b.

Proof. In class, using the Cauchy convergence criterion, I proved that if $\sum |x_n|$ converges then so does $\sum x_n$.

Here, since $0 \leqslant \frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}$, and we know that $\sum \frac{1}{n^2}$ converges, we deduce by the comparison theorem that $\sum \frac{|\cos n|}{n^2}$ converges, and therefore $\sum \frac{\cos n}{n^2}$ converges, by the above argument. □

Exercise 6. Section 3.7, #8.

Proof. Since $\sum a_n$ is convergent, we know that necessarily $(a_n) \to 0$, therefore for $n \geqslant K$ for some large natural number K, we have that $0 < a_n < 1$. Thus for $n \geqslant K$ one has $0 < a_n^2 < a_n$, and then by the comparison theorem one deduces that $\sum a_n^2$ converges. □

Exercise 7. Section 3.7, #11.

Proof. Notice that $\frac{b_n}{1/n} = (a_1 + ... + a_n)$ converges to $\sum a_n > 0$. Therefore by our comparison theorem, $\sum b_n$ converges if and only if $\sum \frac{1}{n}$ converges. Since it isn't the case, we deduce that $\sum b_n$ diverges. □

Solutions for HW7

Exercise 1. Section 4.1, #2.

Proof. Notice that $\left|\sqrt{x}-2\right| = \frac{|x-4|}{\sqrt{x}+2} \leqslant \frac{1}{2}\left|x-4\right|$, therefore if you take $\left|x-4\right| < 1$ you will get that $\left|\sqrt{x}-2\right| < 1/2$ and for the second inequality it is sufficient to take $\left|x-4\right| < 2.10^{-2}$. □

Exercise 2. Section 4.1, #9d.

Proof. One has $\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| = \left|\frac{2x^2-x+1-x-1}{2x+2}\right| = \left|\frac{2x.(x-1)}{2x+2}\right| = \left|\frac{2x}{2x+2}\right|.\left|x-1\right|$.

Now, on the neighborhood $[0,+\infty)$ of the point 1, we have that $\left|\frac{2x}{2x+2}\right| \leqslant 1$. Therefore for a given $\varepsilon > 0$, for any $x \in [0,+\infty)$ satisfying $\left|x-1\right| < \varepsilon$ one has that $\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| < \varepsilon$. □

Exercise 3. Section 4.1, #10b.

Proof. One has $\left|\frac{x+5}{2x+3}-4\right| = \left|\frac{x+5-8x-12}{2x+3}\right| = \frac{7}{|2x+3|}.\left|x+1\right|$

Now it will be sufficient to prove that $\frac{7}{|2x+3|}$ is bounded in a neighborhood of -1. (Notice that it is NOT bounded everywhere! More precisely that function is large when you are too close to $-3/2$). Let's consider the following neighborhood of -1 given by $V = (-5/4, 0)$.

Since $(x > -5/4) \Rightarrow 2x+3 > \frac{-5}{2}+3 = \frac{1}{2}$ we get the following inequality:

on V, $0 < \frac{7}{|2x+3|} < 2.7 = 14$. Therefore, for a given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{14}$. Then for any $x \in V$ satisfying $\left|x+1\right| < \frac{\varepsilon}{14}$, one has $\left|\frac{x+5}{2x+3}-4\right| < \varepsilon$. Thus $\lim_{x\to -1} f(x) = 4$. □

Exercise 4. Section 4.1, #11c.

Proof. The functions $x \mapsto x + \mathrm{sgn}(x)$ has a left-hand limit at zero equal to -1 and a right-hand limit equal to $+1$, therefore it has no limit at zero. □

Exercise 5. Section 4.1, #13.

Proof. When $x \to c$, to say that $\left|f(x)\right|^2 \to 0$ implies that $\left|f(x)\right| \to 0$ (square root rule) which is equivalent to the following phrase: $\lim_{x\to c} f(x) = 0$.

Now take $c = 0$, and the function $f(x) = \mathrm{signe}(x)$, and declare by convention that $f(0) = 1$. The square of this function is a constant function equal to 1, but the function itself has no limit at 0.

□

Exercise 6. Section 4.2, #1d.

Proof. The sum rule implies that the numerator has a limit equal to 1, and by the sum and product rules the denominator has a limit equal to 2. Then the Quotient rule implies that the limit is 1/2.(Notice that the denominator is never zero!)

□

Exercise 7. Section 4.2, #4.

Proof. We proved in class the non-existence of $\lim_{x\to 0}\ \cos(1/x)$ (take the two sequences converging to zero given by $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{(2n+1)\pi}$ the function takes constant values equal to 1 on the first one, and constant values equal to -1 on the other).

Now the function $x.\cos(1/x)$ has a limit equal to zero at zero because of the squeeze theorem applied to the following inequality:

$0 \leqslant |x.\cos(1/x)| \leqslant |x| \to 0$, when $x \to 0$. □

Exercise 8. Section 4.2, #5.

Proof. On a neighborhood V of c one has an inequality of the form $0 \leqslant |f(x).g(x)| \leqslant M.|g(x)|$, where M is an upper bound for f on V. Now apply the squeze theorem to that inequality to get the result. □

Solutions for HW8

Exercise 1. Section 4.3, #5a.

Proof. We have that $\lim_{x\to 1+}\frac{1}{x-1}=+\infty$. Indeed, for any given $\alpha>0$ if we take $x\in(1,1+\frac{1}{\alpha})$, then we have that $\frac{1}{x-1}>\alpha$. Now since $\lim_{x\to 1+} x=1$, we know that by the comparison theorem, $\lim_{x\to 1+}\frac{x}{x-1}=+\infty$ if and only if $\lim_{x\to 1+}\frac{1}{x-1}=+\infty$, which is the case. In conclusion, $\lim_{x\to 1+}\frac{x}{x-1}=+\infty$.

□

Exercise 2. Section 4.3, #5c.

Proof. $\lim_{x\to 0+} x+2=2$ and $\lim_{x\to 0+}\frac{1}{\sqrt{x}}=+\infty$: indeed for any given $\alpha>0$ if we take $x\in(0,\frac{1}{\alpha^2})$, then we have that $\frac{1}{\sqrt{x}}>\alpha$.Thus again by the comparison theorem we know that $\lim_{x\to 0+}\frac{x+2}{\sqrt{x}}=+\infty$.

□

Exercise 3. Section 4.3, #8.

Proof. $\lim_{x\to+\infty} f(x)=L$ means the following:

for any given $\varepsilon>0$ there exists an $\alpha>0$ such that $x>\alpha\Rightarrow f(x)\in(L-\varepsilon,L+\varepsilon)$.

If one writes $\delta=1/\alpha$ the condition $x>\alpha$ is equivalent to $y=1/x<\delta$, so the condition is now equivalent to:

for any given $\varepsilon>0$ there exists a $\delta>0$ such that $0<y<\delta\Rightarrow f(1/y)\in(L-\varepsilon,L+\varepsilon)$. But this exactly means that $\lim_{x\to 0+} f(1/y)=L$. □

Exercise 4. Section 5.1, #7.

Proof. See the solutions for the Practice midterm II, where this is proved... □

Exercise 5. Section 5.1, #12.

Proof. Since the rational numbers are dense in $\mathbb{R}$, we know that any real number x is the limit of a sequence of rational numbers (r_n). By continuity of f we know that $\lim f(r_n)=f(x)$, but this is zero since f is zero on any rational number.

□

Exercise 6. Section 5.2, #1a.

Proof. The rational function $\frac{x^2+2x+1}{x^2+1}$ is continuous at every point where the denominator is not zero. Since that denominator is everywhere >1 we deduce that the function is continuous on $\mathbb{R}$.

□

Exercise 7. Section 5.2, #9.

Proof. Any real number can be written in base 2 (see binary representations in chapter 2.5). The sequence of truncated binary representations converge to the number you started with, therefore you get x as a limit of rational numbers of the form $\frac{m}{2^n}$. By continuity, as in exercise 5, you deduce that $h(x)=0$. □

Solutions for HW9

Exercise 1. Section 5.3, #3.

Proof. Start with any point, $x_0 = 1/2$ for example. Then we know that there exists x_1 such that $\left|f(x_1)\right| \leqslant \frac{1}{2}\left|f(x_0)\right|$. If $\left|f(x_1)\right| = 0$, then we are done; if not then necessarily $x_1 \neq x_0$, and there is then another point x_2, such that $\left|f(x_2)\right| \leqslant \frac{1}{2}\left|f(x_1)\right|$. Continue this process and build a sequence (x_n) satisfying $\left|f(x_{n+1})\right| \leqslant \frac{1}{2}\left|f(x_n)\right|$. Since all these points are in a bounded interval, the sequence is bounded and by Bolzano-Weierstrass we can extract a sequence (x_{k_n}) converging to some $c \in [a, b]$. If we write $k_{n+1} = k_n + \alpha(n)$, we get $\left|f(x_{k_{n+1}})\right| \leqslant \left(\frac{1}{2}\right)^{\alpha(n)}\left|f(x_{k_n})\right| \leqslant \frac{1}{2}\left|f(x_{k_n})\right|$.

(This just means that the extracted sequence satisfies the same inequality).

Since $(x_{k_n}) \to c$, and f is continuous, then we deduce that $f(x_{k_n}) \to f(c)$, but then the previous inequality says that: $\left|f(c)\right| \leqslant \frac{1}{2}\left|f(c)\right|$ so $f(c) = 0$ and we are done.

□

Exercise 2. Section 5.3, #6.

Proof. The hint says it all: $g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1) = -g(0)$. Now if $g(0) = 0$ we are done. If not, then apply the intermediate value theorem to g: you will get a zero for g between 0 and 1/2. But $g(x) = 0 \Rightarrow f(x) = f(x + \frac{1}{2})$.

□

Exercise 3. Section 5.3, #11.

Proof. This has been proved in class.Let me quickly give the argument:

if $f(w) < 0$, then necessarily $w < b$ (because $f(b) > 0$). By continuity of f there will be a small neighborhood of w on which the function is strictly negative: this contradicts the definition of w.

If $f(w) > 0$, then by continuity of f there is a small $\delta -$ neighborhood where the function is strictly positive: this is a contradiction, because between $w - \delta$ and w there should be a point in W. □

Exercise 4. Section 5.3, #13.

Proof. Take $\varepsilon = 1$: then there exists an $\alpha > 0$ such that for any $x > \alpha$ we have $|f(x)| < 1$, and there is a $\beta < 0$ such that for any $x < \beta$ we have $|f(x)| < 1$. But now f is continuous on $[\beta, \alpha]$ so it is bounded, say by $M > 0$. Putting everything together, we get that f is bounded on the entire line by $\max(1, M)$.

Since it is bounded, we can consider $L = \sup_{x \in \mathbb{R}} (f(x))$ and $l = \inf_{x \in \mathbb{R}} (f(x))$. If $L = l$ then the function is constant. If $L \neq l$, then one of them is nonzero. Assume it is L: pick $\varepsilon = \frac{1}{2}\left|L\right|$. Then there is an $\alpha > 0$ such that for any $x > \alpha$ we have $\left|f(x)\right| < \varepsilon$, and there is a $\beta < 0$ such that for any $x < \beta$ we have $\left|f(x)\right| < \varepsilon$. Now since again f is continuous on $[\beta, \alpha]$, it reaches a maximum at a point X in that closed interval. I claim that $f(X) = L$. Indeed if we have $f(X) < L$ then this would contradict the definition of L (because we know we can find points $y \in \mathbb{R}$, such that $f(y)$ is arbitrarily close to L, and such points are necessarily in $[\beta, \alpha]$ because outside of this closed interval everybody has an image less than $\frac{1}{2}.L$).

If $\inf f < 0$, then similarly we obtain a global minimum for the function.

Now it can happen that one of the two values $\inf f, \sup f$ is zero, in which case it is possible that the extremum is not reached: take for example $f(x) = \frac{1}{x^2+1}$ (max is reached at zero, but infimum is zero, not reached).

□

Exercise 5. Section 5.4, #2.

Proof. Just notice that $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|x - y\right|.\left|\frac{x+y}{x^2.y^2}\right|$.

But observe now that for $x \geqslant 1, y \geqslant 1$ we have $\left|\frac{x+y}{x^2.y^2}\right| \leqslant \frac{|x|+|y|}{x^2.y^2} \leqslant \frac{x^2+y^2}{x^2y^2} \leqslant \frac{1}{y^2} + \frac{1}{x^2} \leqslant 2$.

So now given $\varepsilon > 0$, just pick $\delta = \frac{\varepsilon}{2}$: then for any x, y such that $\left|x - y\right| < \delta$ we get: $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| < 2\left|x - y\right| < 2\frac{\varepsilon}{2} = \varepsilon$.

The key thing is that the first inequality is true for anybody in $[1, +\infty)$ (it is "uniformly true"!).

It is not uniformly continuous on $(0, \infty)$: pick the sequence $(x_n) = \frac{1}{2^n}$, then notice that $(x_{n+1} - x_n) \to 0$ but that $\left|f(x_n) - f(x_{n+1})\right| = 2^{2(n+1)} - 2^{2n} = 2^{2n}.(3) > 3$.

□

Exercise 6. Section 5.4, #4.

Proof. Same stuff: $\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| = \left|x - y\right|.\left|\frac{x+y}{(1+x^2).(1+y^2)}\right| \leqslant \left|x - y\right|.\frac{|x|}{1+x^2}.\frac{|y|}{1+y^2} \leqslant \left|x - y\right|$.

The last inequality comes from the fact that for any $x \in \mathbb{R}$ we have $\left|x\right| \leqslant 1 + \left|x\right|^2$. Indeed either $\left|x\right| \leq 1$ and then $\left|x\right| \leqslant 1 + \left|x\right|^2$, or $\left|x\right| > 1$ but then $\left|x\right| < x^2 \leqslant 1 + x^2$.

Now for a given $\varepsilon > 0$, just pick $\delta = \varepsilon$: then for any pair x, y satisfying $|x - y| < \delta$ we get that $\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| \leqslant \left|x - y\right| < \delta = \varepsilon$.

□

Exercise 7. Section 5.4, #9.

Proof. Just notice $\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \left|f(x) - f(y)\right|.\left|\frac{1}{f(x).f(y)}\right| \leqslant \frac{1}{k^2}\left|f(x) - f(y)\right|$.

Now : f is uniformly continuous so for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any x, y satisfying $\left|x - y\right| < \delta$ we have $\left|f(x) - f(y)\right| < k^2.\varepsilon$.

This will imply that for that given $\varepsilon > 0$, if we have $|x - y| < \delta$ we deduce $\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| < \frac{1}{k^2}k^2.\varepsilon = \varepsilon$. □

Exercise 8. Section 5.6, #10.

Proof. Let c be the interior point where f attains a max. We have $a < c < b$. If either $f(a) = f(c)$ or $f(b) = f(c)$ then we are done (f will not be injective). If not, then pick any k between

$\max\left(f(a),\, f(b)\right)$ and $f(c)$. Then by the intermediate value theorem, you know that f will reach that value k once in $[a,\, c]$, and once in $[c,\, b]$ (observe that k is not reached at c, so we really get two different points having the same value), and this proves that f is not injective. $\square$

Solutions for HW10

Exercise 1. Section 6.1, #1a.

Proof. As usual we have to go back to the definition, so we write the quotient
$\frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + h.(3x + h^2) \rightarrow 3x^2$ when $h \rightarrow 0$

□

Exercise 2. Section 6.1, #4.

Proof. The quotient $\frac{f(h) - f(0)}{h} = \frac{f(h)}{h}$ is either equal to 0 or to $\frac{h^2}{h} = h$, thus in any case we have
$0 \leqslant \left|\frac{f(h) - f(0)}{h}\right| \leqslant \left|h\right| \rightarrow 0$ when $h \rightarrow 0$, so by the squeeze theorem we deduce that $f'(0)$ exists and is equal to zero.

□

Exercise 3. Section 6.1, #9.

Proof. Assume that f is differentiable everywhere and that we have for any $x \in \mathbb{R}$
$f(-x) = f(x)$. By taking the derivative of both sides we get (using the Chain rule for the left hand side):
$f'(-x).(-1) = f'(x)$ thus $f'(-x) = -f(x)$ but this means exactly that f' is an odd function.

The same proof works if we differentiate an odd function. □

Exercise 4. Section 6.1, #10.

Proof. We have $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $g(0) = 0$.

For $x \neq 0$, say $x > 0$, we have that $1/x^2$ is differentiable, and since sin is differentiable everywhere, the Chain rule tells us that $\sin(1/x^2)$ is differentiable on $(0, +\infty)$, and so is g (by the product rule). For the same reason, the function g is differentiable on $(-\infty, 0)$.

Now let's study the differentiability at 0:

As usual we write for $h \neq 0$ the quotient $\frac{f(0+h) - f(0)}{h} = \frac{h^2 \sin(1/h^2)}{h} = h.\sin(1/h^2) \rightarrow 0$ when $h \rightarrow 0$ because h goes to zero and $\sin(1/h^2)$ is bounded. So we proved that g is differentiable everywhere and that $g'(0) = 0$.

Let's prove that g' is not bounded on $[-1, 1]$. Indeed one has for any $x \neq 0$, by the Chain rule
$g'(x) = 2x.\sin(1/x^2) + x^2.\cos(1/x^2).(\frac{-2}{x^3}) = 2x.\sin(1/x^2) + \cos(1/x^2).(\frac{-2}{x})$. In order to show that this function is unbounded it is enough to find a sequence of points (x_n) in $[-1, 1]$ such that the sequence $(g'(x_n))$ is unbounded: just take $x_n = \frac{1}{\sqrt{2n\pi}}$, then one has
$g'(x_n) = (2x_n).0 + \cos(2n\pi).(-2\sqrt{2n\pi}) = -2\sqrt{2n\pi}$ which is unbounded.

□

Exercise 5. Section 6.1, #11a.

Proof. Just apply the Chain rule: $f'(x) = L'(2x+3).2 = \frac{2}{2x+3}$.

$\square$

Solutions for HW11

Exercise 1. Section 6.1, #16.

Proof. Since the function tan is continuous and strictly increasing from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$, we know that the inverse function arctan exists, and is continuous and strictly increasing from $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover $(\tan)'(x) = \frac{(\cos x).(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{1}{(\cos x)^2}$ which is $\neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Therefore we know that arctan is differentiable on $\mathbb{R}$ and that $(\arctan x)' = \frac{1}{(\tan)'(\arctan x)} = \frac{1}{1 + (\tan(\arctan x))^2} = \frac{1}{1 + x^2}$, because $\frac{1}{(\cos x)^2} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1$ □

Exercise 2. Section 6.2, #1b.

Proof. We have $g(x) = 3x - 4x^2$ and $g'(x) = 3 - 8x$ which is < 0 on $(-\infty, 3/8)$, equal to 0 at $x = 3/8$ and > 0 on $(3/8, +\infty)$, therefore g is strictly decreasing before $3/8$ and then strictly increasing after. Thus it has a single global minimum at $3/8$ and this is also the only local extremum.

□

Exercise 3. Section 6.2, #2d.

Proof. We have $f(x) = 2x + \frac{1}{x^2}$ for $x \neq 0$. So f is differentiable everywhere except at 0, and

$f'(x) = 2 - \frac{2}{x^3}$, which is strictly positive when $x < 0$, and then < 0 on $(0, 1)$, equal to zero at 1 and strictly positive after, so we deduce that f is strictly increasing on $(-\infty, 0)$ and $(1, +\infty)$ and strictly decreasing on $(0, 1)$.

So this function has a single local extremum which is equal to 3 and obtained at $x = 1$. □

Exercise 4. Section 6.2, #4.

Proof. f is a polynomial function so it is differentiable everywhere and its derivative is given by $f'(x) = 2\sum (x - a_i) = 2nx - (2\sum a_i)$. So this function has a single local (and actually global) minimum obtained at $x = \frac{\sum a_i}{n}$ (that's the mean of the a_i).

□

Exercise 5. Section 6.2, #12.

Proof. If there is such a function then it must necessarily be continuous, constant on $\mathbb{R}^-$ and then linear with a slope $= 1$. But such a function is not differentiable at 0 (there is a limit slope equal to zero on the left, and a limit slope equal to 1 on the right).

□

Exercise 6. Section 6.2, #16.

Proof. a) By the mean value thm, we know that $f(x+h) - f(x) = h.f'(c)$ for some c between x and $x+h$. Therefore $(f(x+h) - f(x))/h = f'(c)$ and when $x \to +\infty$, so does c (which depends on x and h), so the quantity has a limit equal to b.

b) If we fix h and pass to the limit when $x \to +\infty$ the quantity $(f(x+h) - f(x))/h$ has limit equal to $(0-0)/h = 0$. So b must be zero.

c) Given $\varepsilon > 0$, first find $\alpha > 0$ such that for any $c > \alpha$ one has $f'(c) \in (b - \varepsilon/2, b + \varepsilon/2)$. Then just write

$$\frac{f(x)}{x} - b = \frac{f(x) - f(\alpha)}{x - \alpha}.\frac{x-\alpha}{x} + \frac{f(\alpha)}{x} - b = \frac{1}{x}.(f'(c).(x-\alpha) - b.(x-\alpha) - b.\alpha + f(\alpha)).$$

Therefore $\left|\frac{f(x)}{x} - b\right| \leqslant \left|f'(c) - b\right|.\left|\frac{x-\alpha}{x}\right| + \frac{|f(\alpha) - b\alpha|}{|x|} \leqslant \frac{\varepsilon}{2} + \frac{|f(\alpha) - b\alpha|}{|x|}$.Now for α fixed, observe that there is a $\beta > 0$ (that can be taken $> \alpha$) such that $x > \beta$ implies $\frac{|f(\alpha) - b\alpha|}{|x|} \leqslant \frac{\varepsilon}{2}$.

Thus for $x > \beta$ one has that $\left|\frac{f(x)}{x} - b\right| \leqslant \varepsilon$.

□

SOLUTIONS FOR HW12

Exercise 1. Section 6.3, #7c.

Proof. Just write $x^3 \ln x = \frac{\ln x}{1/x^3} = \frac{f(x)}{g(x)}$ and apply L'Hospital's rule on $(0, +\infty)$ (notice that $g'(x) = -3x^{-4}$ is well-defined and non-zero on that interval and that both functions are differentiable).

Thus one gets $\frac{f'(x)}{g'(x)} = \frac{1/x}{-3x^{-4}} = (-1/3).x^3 \to 0$ when $x \to 0+$, so the limit is zero. □

Exercise 2. Section 6.3, #9c.

Proof. As usual write $(1+3/x)^x = e^{x.\ln(1+\frac{3}{x})} = \exp(\frac{\ln(1+3/x)}{1/x})$. On the interval $(0, +\infty)$ both functions $f(x) = \ln(1+3/x)$, $g(x) = 1/x$ are differentiable and $\frac{d}{dx}(1/x)$ is nonzero on that interval therefore we can apply L'Hospital's rule:

$\frac{f'(x)}{g'(x)} = (\frac{-3}{x^2}.\frac{1}{1+3/x}).(\frac{1}{-1/x^2}) = \frac{3}{1+3/x} \to 3$ when $x \to +\infty$. So the limit of our function is e^3. □

Exercise 3. Section 6.4, #4.

Proof. Write Taylor's formula at the order one and two for $f(x) = \sqrt{1+x}$ between 0 and x:

- $f(x) = 1 + x.f'(0) + \frac{x^2}{2}.f''(c) = 1 + \frac{1}{2}x + \frac{x^2}{2}.(\frac{1}{2}).(\frac{-1}{2}).(1+c)^{-3/2}$, and since the remainder is $\leqslant 0$, one gets $\sqrt{1+x} \leqslant 1 + \frac{1}{2}x$.
- $f(x) = 1 + x.f'(0) + \frac{x^2}{2}.f''(x) + \frac{x^3}{6}.f'''(c) = 1 + \frac{1}{2}x + \frac{x^2}{2}.(\frac{1}{2}).(\frac{-1}{2}).(1+x)^{-3/2} + \frac{x^3}{6}(\frac{-1}{4}).(\frac{-3}{2})(1+c)^{-5/2}$, and since the remainder is now larger than zero, one gets the left inequality.

□

Exercise 4. Section 6.4, #16.

Proof. I found this problem much harder than I wanted!

I'll give two proofs, the first one assuming some extra hypothesis:

In the first one I assume that f'' exist everywhere (to simplify):

By Taylor's thm applied to f at the order two between a and $a+h$, one has:
$f(a+h) = f(a) + h.f'(a) + \frac{h^2}{2}f''(c)$ for some c between a and $a+h$,

Now Taylor's thm applied to f at the order two between a and $a-h$, one has:

$f(a-h) = f(a) - h.f'(a) + \frac{h^2}{2}f''(d)$ for some d between a and $a-h$.

But this implies
$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \frac{1}{2}.(f''(c) + f''(d))$ and when $h \to 0$ this quantity converges to $f''(a)$.

Second proof (without assuming any extra hypothesis):

Write $g(h) = f(a+h) - f(a-h)$. Then apply the Second Fancy Version of the mean value theorem to the quotient

$\frac{g(h) - g(0)}{h^2} = \frac{g'(c)}{2c} = \frac{f'(a+c) - f'(a-c)}{2c} = \frac{f'(a+c) - f'(a)}{2c} + \frac{f'(a-c) - f'(a)}{-2c}$ and this goes to $f''(a)$ when $c \to 0$.

Now an example of a function for which the limit makes sense but that doesn't have a second derivative at zero:

Take $f(x) = -\frac{x^2}{2}$ if $x \leqslant 0$ and $f(x) = \frac{x^2}{2}$ if $x \geqslant 0$, so that the derivative is $f'(x) = \left|x\right|$ which doesn't have a derivative at zero. But now one has:

$\frac{f(0+h) - 2f(0) + f(0-h)}{h^2} = \frac{h^2/2 + (-h^2/2)}{h^2} = 0.$ □