MAT 319 "Foundations of Analysis" Fall 2004

GENERAL INFORMATION

Description and goals: Somehow this file got eaten by my computer... So some of the information about office hours etc was lost.

Professor: Professor Dusa McDuff.
Office Hours: Thursday 1:00-2:00pm. Or by appointment.
Office: 3-111 Mathematics Department. SUNY at Stony Brook.
e-mail: dusa at math.sunysb.edu.

TA: Tanvir Prince.
Office Hours:

Schedule for the first few weeks Both classes will meet on **Tuesday and Thursday 2:20-3:40 in HH112.** The recitations/workshops meet on **Monday and Wednesday 11:45--12:40**, initially in **HH 104** . There will be a **workshop on Wednesday Sept 1**, no class Sept 6, and a **recitation on Wednesday Sept 8**. For the rest of September, recitations will be on Mondays and workshops on Wednesdays.

Textbook: Introduction to Real Analysis, by R. Bartle and D. Sherbert, Third ed, Wiley.

Grading Policy for MAT 319: Homework 25%. Project 15%. Midterms (two) 15% each. Final Exam 30%.

Exam Schedule: Midterm 1: Thu Sept 30 (in class). Midterm 2: November 11 (in class). Final: Tuesday Dec 14, 5:00--7:30pm.

Recitations and Workshops: Students are expected to go to both the Monday and Wednesday class until the first Midterm. The workshops are required: students will be working in groups on a problem set that will be handed in and graded. These workshops are intended for the most part to help students learn and review concepts taught in previous classes. However they will count a little towards the homework part of the grade: they will each be worth approximately 1/3 of a homework. The recitations will be more traditional --- problems similar to the homework problems will be discussed. After the class splits, there will be one recitation per week for each class: the 319 recitation on Monday and the 320 class on Wednesday. Professor McDuff will have an office hour during the Wednesday time.

Scheduling conflicts: Any student with a scheduling conflict should make appropriate arrangements with their professor. For example, the recitation conflicts with another MAT class. Students with this conflict will be excused from the workshops, provided we are told in advance. Also, we have scheduled the first exam on a Jewish holiday: Sept 30. There will be a make up for any student for whom this is a problem. Please tell us about any such conflicts as soon as possible.

Homework This is an essential part of the class and is worth a considerable amount of the grade. The homework sets will be posted on the web in PDF format and will be due at 5pm on the due date. Solutions can be handed in to the TA or to the appropriate professor. (Put it under their office door if they are not there.) Late work will receive reduced credit, and will not be accepted after solutions are posted. You may work on your homework with other people (in fact, this is often a good idea), but the work you hand in must be your own, not copied directly from others. You should also list your working partners on the homework you hand in. The first homework will be due on **Thursday Sept 9** (because of Labor Day). Usually homework will be due on Tuesdays.

MAT 319 Project Each student in MAT 319 will work on a project (typically with one other student). The exact form of this project will depend on how many students are in the class and will be announced later. If it is feasible, it will involve an oral presentation.

Lecturing Schedule

Tuesday Oct 19	sec 3.4

Oct 21, 26	Sec 4.1, 4.2
Oct 28, Nov 2,4	Sec 5.1,5.2, 5.3
Nov 9	review
Nov 11	Midterm II (in class)
Nov 16,18,23	background material for projects
Nov 30, Dec 2,7,9	project presentations, exam review

Homeworks etc These are posted below in pdf format.

Homework 1
Worksheet 1
Worksheet 2
Worksheet 3
Worksheet 4

Homework 2
Homework 3 (due Thurs 9/23)
Homework 4 (due Tues 9/28)
Midterm 1 revision (due 10/12)
Homework 5 (due Tues 10/12)
Homework 6 (typos corrected) (due Wed 10/20)
Homework 7 (due Wed 10/27)
Homework 8 (due Wed 11/3)
Homework 9 (due Tues 11/30)
Midterm 2 (needed for HW9)
Homework 10 (due Wed 12/8)

ANNOUNCEMENTS

- More on Writing requirement and projects. I was not intending to give you back your projects unless you ask me for them. If you want to get them back soon, you could give me a stamped addressed envelope (that is large enough) and I will mail them back to you when they are graded. Otherwise you can pick them up next semester. (or at the end of exam week -- they should be done by then.) If you want to use them for the writing requirement please write a note to that effect at the end, and I will certify them (provided they are sufficiently well written.) You can use at most two pieces of writing from each class.
- Here is the review for the final.
- Here is the updated list of theorems and definitions for the final exam. It is much as before except that I added two theorems at the end about continuous functions on the interval, and also the definitions of boundary point etc. The format of the final will be like that of Midterm II. This sheet will be distributed to you during the exam. You can use these results in your answers but when you do this you should refer to the number of the theorem you are using. You should also understand the proofs of these theorems. I will ask you to explain some of the shorter proofs (either of these results, or of some of the results we have worked with in class.)
- I hope to post the review sheet very soon.
- I will hold a review session on TUESDAY DEC 14 at 12--2:30 in my office.
- Tanvir Prince will hold a Review session on MONDAY DEC 13 at 7--9 pm in Physics P112. NOTE CHANGE OF PLACE.
- The final exam is on Dec 14 in HH 112 (our usual classroom) at 5--7:30 pm.
- Here is the schedule for the presentations.

FOR PEOPLE WITH DISABILITIES

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.ehs.stonybrook.edu/fire/disabilities.asp

Last modified: 11/04/2004

Math 319/320 Homework 1

Due Thursday, September 9, 2004

Problem 1. Prove the identity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Problem 2. (i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Is it true that

$$f(A \cap B) = f(A) \cap f(B)$$

for all subsets A, B of $\mathbb{R}$? Give a brief proof or a counterexample.

(ii) Do the same question using the function $h : \mathbb{R} \to \mathbb{R}$ where $h(x) = x^3$.

Problem 3. Let $f : A \to B$ be a function and suppose that $C \subseteq A$ and $D \subseteq B$. Are the following statements true or false? Justify your answers by a brief proof or a counterexample.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

Hint: as in question 2, try some examples. You can try functions $f : \mathbb{R} \to \mathbb{R}$ or you can try functions $f : A \to B$ where A and B are finite sets.

Problem 4. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composite $g \circ f$ is injective. Is f necessarily injective? What about g? Give brief proofs or counterexamples.

Problem 5. Prove by mathematical induction: $2^n < n!$ for all $n \geq 4$. (Note: $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$.

Math 319/320 Worksheet 1

Name: **School ID:**

Problem 1. Negate the following statement:

"If your glass is half-empty, you are a pessimist."

Give your answer in words.

Problem 2. The context in this problem is the set of all human beings. Recall the symbols

$$\forall = \text{ for all}, \quad \exists = \text{ there exists}, \quad \wedge = \text{ and}, \quad \vee = \text{ or}, \quad \sim = \text{ not}.$$

Let $P(x)$ be "x is educated," $Q(x)$ be "x is female" and $R(x)$ be "x is older than 30." Then the statement "every uneducated male is older than 30" can be expressed as

$$\forall x : (\sim P(x) \wedge \sim Q(x)) \implies R(x)$$

Express the following statements in a similar way:

(i) Some educated people are younger than 30.

(ii) Every female who is older than 30 is educated.

(iii) No uneducated person is both female and older than 30.

Problem 3. Consider the statement

"For every natural number n, if n^2 is even, then n is even."

Prove this statement in two different ways: (i) by showing that its contrapositive is true; (ii) by showing that its negation is false.

Problem 4. On a bumper sticker, I saw the statement

"For every real number x, there is a real number t such that $t(1-t) > x$."

After some thought, I decided that this statement must be __________. To prove this carefully, I found a real number ____ such that for every real number ____ the inequality ____________________ held.

Math 319/320 Worksheet 2

Problem 1. Fill in the blanks in the following proof of

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

If $x \in A \cup (B \cap C)$, then either $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in$ __________, so $x \in$ ____________________. On the other hand, if $x \in B \cap C$, then $x \in$ __________ and $x \in$ __________, so again $x \in$ ____________________. Hence $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Now suppose $x \in$ ____________________. Then $x \in$ __________ and $x \in$ __________. Consider two possibilities: If $x \in A$, then

On the other hand, if $x \notin A$, then

Therefore

Problem 2. Let $\{A_j : j \in \mathbb{N}\}$ and $\{B_j : j \in \mathbb{N}\}$ be two families of sets indexed by the set of positive integers $\mathbb{N}$. Is it true that

$$\bigcup_{j \in \mathbb{N}} (A_j \setminus B_j) = (\bigcup_{j \in \mathbb{N}} A_j) \setminus (\bigcup_{j \in \mathbb{N}} B_j)?$$

Give a proof or counterexample.

Problem 3. Let $f : A \to B$ be a function, $C \subset A$ and $D \subset B$. Show that
$$C \subset f^{-1}(f(C)) \qquad \text{and} \qquad f(f^{-1}D) \subset D.$$

If f is injective, do either of these inclusions become equalities?

What if f is surjective?

Problem 4. Let A and B be subsets of a universal set U. Simplify the following expressions. It might be helpful to draw Venn diagrams.

(i) $(A \cap B) \cup (U \setminus A)$

(ii) $A \cup [B \cap (U \setminus A)]$

Math 319/320 Worksheet 3

Name: **School ID:**

Problem 1. Consider a non-empty set $S \subset \mathbb{R}$. When should we call S "bounded below"? Formulate the definition of a "lower bound" and the "greatest lower bound" for S. The greatest lower bound of S, if exists, is called the *infimum* of S and is denoted by $\inf(S)$.

Problem 2. Find $\sup(S)$ and $\inf(S)$ in each case. Justify your answers.

- $S = \left\{ \dfrac{(-1)^n}{n} : n \in \mathbb{N} \right\}$

- $S = (-1, 0] \cup [1, 2] \cup \{3\}$.

Problem 3. The real number system has the *Archimedean property (AP)*, which can be formulated as follows:

"For every real number $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$."

Below we show that (AP) is a consequence of the Completeness Axiom for $\mathbb{R}$. Fill in the blanks:

Suppose (AP) fails. Then we can find some __________ such that _______________ for all $n \in \mathbb{N}$. In other words, the real number $1/x$ should be an _______________ for $\mathbb{N}$. By ______________________, $\mathbb{N}$ must have a least upper bound $b \in \mathbb{R}$. Since _______________, $b - 1$ cannot be an upper bound for $\mathbb{N}$, so there must be an $n \in \mathbb{N}$ such that _______________. But this means _______________, which contradicts the definition of $b = \sup(\mathbb{N})$. The contradiction shows that (AP) must hold.

Problem 4. Using the Archimedean property, carefully show that if

$$S = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\},$$

then $\sup(S) = 1$.

Math 319/320 Worksheet 4

Name: **School ID:**

Problem 1. Find $\text{int}(S)$, S', $\text{cl}(S)$ and $\text{bd}(S)$ if

- $S = [0, 1] \cup (1, 2)$

- $S = \bigcap_{n=1}^{\infty} \left[0, 1 + \frac{2}{n}\right)$

- $S = [0, +\infty) \cap \mathbb{Q}$

Problem 2. True or false? Give a short proof or counterexample.

- If S is open and T is closed, then $S \setminus T$ is open.

- If S is not open, then S is closed.

- If S is closed, then $S' \subset S$.

Problem 3. Let $S \subset \mathbb{R}$ be non-empty. Show that $\text{int}(S)$ is an open set.

Math 319/320 Homework 2

Due Tuesday, September 14, 2004

Problem 1. (i) Show that the set S_{odd} of odd (positive and negative) integers is denumberable by

(a) enumerating them and

(b) giving an explicit formula for the corresponding bijection $f : S_{odd} \to \mathbb{N}$.

Hint: Imitate the following example. An **enumeration** of the set $\mathbb{Z}$ of integers is given by $\{0, 1, -1, 2, -2, 3 \dots\}$. The corresponding explicit bijection $f : \mathbb{Z} \to \mathbb{N}$ is

$$f(0) = 1, \qquad f(k) = 2k, \text{ if } k > 0, \qquad f(k) = -2k + 1, \text{ if } k < 0.$$

Problem 2. (i) Show that for all $n \geq 1$ there is no injection of $\mathbb{N}_n$ onto a *proper* subset of $\mathbb{N}_n$. In other words, any injection $f : \mathbb{N}_n \to \mathbb{N}_n$ is surjective.

(ii) Deduce from (i) that if S is any finite set, there is no injection of S onto a *proper* subset of S.

(iii) Show that (ii) does not hold for the infinite set $S = \mathbb{N}$.

Hint for (i): You can try to prove this by induction (cf. proof of Thm B.1 in Appendix B of the textbook.) Alternatively, you can deduce it from Thm B.1 by supposing that there is a nonsurjective map $f : \mathbb{N}_n \to \mathbb{N}_n$ and looking at the composite $g \circ f$ for a suitable map $g : \mathbb{N}_n \to \mathbb{N}_{n-1}$. (Actually, these different approaches boil down to basically the same argument.)

Problem 3. Prove that if a set S has n elements, then $\mathcal{P}(S)$ has 2^n elements. **Hint:** Use induction.

Problem 4. Prove that $\sqrt{3}$ is irrational.

Bonus Problem 5. Show that if S is a subset of $\mathbb{N}$ that is not contained in any of the sets $\mathbb{N}_n$ then S is denumerable.

Math 319/320 Homework 3

Due Thursday, September 23, 2004

Problem 1. Prove that in any ordered field $\mathbb{F}$, a^2 is positive for all a in $\mathbb{F}$ except $a = 0$.

Problem 2. Let $\mathbb{P}$ be the positive set in an ordered field $\mathbb{F}$. Assume that for all p in $\mathbb{P}$, $a \leq p$. Show that a is not positive. **Hint:** Use the fact, proven in class, that there is no smallest positive element of an ordered field.

Problem 3. Let ϵ and δ be positive real numbers. Recall that $V_\epsilon(a)$ denotes the interval $(a-\epsilon, a+\epsilon)$. Find γ such that

$$V_\epsilon(a) \cap V_\delta(a) = V_\gamma(a).$$

Find β such that

$$V_\epsilon(a) \cup V_\delta(a) = V_\beta(a).$$

Note that the numbers γ and β depend on ϵ and δ, i.e. you should be able to find an (easy!) formula for γ in terms of ϵ and δ, and similarly for β.

Problem 4. Use induction to prove that for all n and all real numbers $a_1, \ldots, a_n$:

$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|.$$

(cf. Corollary 2.2.5 in the textbook.)

Problem 5. Complete the argument given in class to show that there is a real number whose square is 2.

Math 319/320 Homework 4

Due Tuesday, September 28, 2004

Problem 1. (a) Is the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with formula $f(x) = 1/x$ bounded?
(b) Is f bounded when restricted to the domain $D := [1, \infty)$?
(c) Find the supremum and infimum of $f(D)$ or explain why they do not exist.

Problem 2. State the Archimedean property for rational numbers and prove that it holds.
Note: Do not use the Completeness Property of the real numbers, i.e. do not imitate the proof of 2.4.3. There is a different short argument that applies when x is rational.

Problem 3. Let $S \subset \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound for S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.

Problem 4. (a) Use induction to prove that a nonempty finite subset of $\mathbb{R}$ contains its supremum.
Hint: Argue by induction on the number of elements in the finite set.
(b) Give an example to show that this statement does not hold for every infinite subset of $\mathbb{R}$.

Problem 5. Prove Theorem 2.5.1 case (iii).

Math 319 Midterm 1: extra credit

Due Tuesday, October 12, 2004

Extra credit for students in MAT 319

Rewrite solutions to all the questions in Midterm 1, handing your solutions to Professor McDuff by 5 pm on Tuesday, October 12 (either in lecture or under her office door Math 3-111.) If there are any of these questions that you do not want to rework, please hand in your original exam script and you will be credited with your original grade for the problem. The final grade for Midterm I will be made up of your original grade out of 50 added to the new grade out of 25.

Hints In general, in each question you must look to see where a detailed proof is needed. Many of you wrote down various somewhat relevant thoughts without making a coherent argument.

1. Make sure your algebra is correct. Explain clearly where you use the inductive hypothesis.

2. It is obvious that 2 is included in this infinite intersection. You must find an argument to show that nothing else is in the intersection.

3. Give the precise definition and then use it in (ii).

4. Many of you were very confused here. The set S is the set of solutions to the equation $\sin x = 1$. The fact that the function $\sin x$ is bounded is irrelevant because you are asked about points in the domain of sin not points in its range (ie values of the function.)

5. Many of you wrote things like "S is bijective". But this is nonsense: S is a set and only functions can be bijective. In the second part, divide the argument into two: first consider the case when S and T are disjoint, and then look at the general case.

Math 319/320 Homework 5

Due Tuesday, October 12, 2004

Writing requirement

If you have not already satisfied the Mathematics Department writing requirement, you can do this by handing in correct and well written solutions to Midterm I. This can be done anytime, but it might be a good idea to do it now. Students in MAT 320 should hand in their work to Prof Ebin; those in MAT 319 can use the rewrite due Oct 12 as a first attempt.

MAT 319 students should do problems 1 through 4 and MAT 320 students problems 2 through 5.

Problem 1. (i) Let $x_n = \dfrac{1}{3n-1}, n \geq 1$. Find an integer K such that

$$x_n < \frac{1}{35}$$

for all $n \geq K$. Explain your reasoning.

(ii) Same as (i) with $x_n = \dfrac{\sqrt{n}}{n+3}$.

Problem 2. Use the definition of limit to prove that:

(i) $\lim \dfrac{n}{n-1} = 1$.

(ii) $\lim \dfrac{1}{3^n} = 0$.

Problem 3. (i) Show that $\lim \dfrac{n}{3^n} = 0$.

Hint: use the Binomial theorem as in 3.1.11 (d).

(ii) Show that the sequence $n^2 - \sin n$ has no limit.

Problem 4. (i) Suppose that $\lim z_n = z$ where $z \neq 0$. Show that there is $K \in \mathbb{N}$ such that $z_n \neq 0$ for all $n \geq K$.

(ii) Suppose that $\lim z_n = z$ where $z \neq 0$ and $z_n \neq 0$ for all n. Show there is $\delta > 0$ such that $|z_n| > \delta$ for all n.

Problem 5 Give an example of two divergent sequences X and Y such that

(a) the sum $X + Y$ converges, and

(b) their product XY converges.

Prove all your assertions.

Math 319 Homework 6

Due Wednesday, October 20, 2004

New due date: After talking to one or two of you I decided to compromise on the Homework due date. It will now be Wednesday at 5pm, and so after my office hours on Wed morning at 11:45. My reason for wanting you to hand this in before Thursday is that I want you to have some space in your brain each week for the new material presented in lecture. Ideally you would have some time each week to think about Tuesday's lecture before coming to the Thursday lecture. That's why I wanted HW in on Tuesday. This is a compromise...

Problem 1: Further practice with finding the integer $K(\epsilon)$.
(i) Let $x_n = \frac{3n+1}{n+1}$. Find K so that $|x_n - 3| < \frac{1}{10}$ for all $n \geq K$.
(ii) Let $x_n = \frac{1}{\sqrt{2n+1}}$. Find K so that $|x_n| < \frac{1}{10}$ for all $n \geq K$.

Problem 2. Do the following sequences converge? You can use any method, and any theorem from secs 3.1 and 3.2, but explain which theorems you are using.
(i) $x_n = \frac{n+1}{2^n}$;
(ii) $x_n = (-1)^n n$;
(iii) $x_n = \frac{\sqrt{n+1}}{n^2}$.

Problem 3. Let $s_1 = 2$ and define s_n inductively by the equation

$$s_{n+1} = \frac{1}{2}\big(s_n + \frac{2}{s_n}\big).$$

(i) Write down the first 10 terms of this sequence. (Use a calculator!) Verify that it is decreasing.
(ii) By what we did in class this sequence should converge to $\sqrt{2}$. Calculate the three quantities $s = \sqrt{2}$, $s_5 - \sqrt{2}$ and $\frac{s_5^2-2}{s_5}$. Check that $0 < s_5 - \sqrt{2} < \frac{s_5^2-2}{s_5}$.
(iii) Same as (ii) for the term s_{10}. i.e. Calculate the three quantities $s_{10} - \sqrt{2}$ and $\frac{s_{10}^2-2}{s_{10}}$. Check that $0 < s_{10} - \sqrt{2} < \frac{s_{10}^2-2}{s_{10}}$.
Note: I have not done this and so I do not know how many decimal places you need to keep to see a difference between $\sqrt{2}$ and s_{10}. Use your judgement.

Bonus Show that this sequence s_n is monotonic decreasing. (cf example 3.3.5). The book also gives the error estimate that you verified in (ii) for the case $n = 5$.

Problem 4. Let $x_1 = 3$ and $x_{n+1} = \frac{1}{3}(2 + x_n)$ for all n.
(i) Write down the first 5 terms of this sequence.
(ii) Prove by induction on n that $x_n > 1$ for all $n \geq 1$.

(iii) Show that this sequence is monotonic decreasing.
Hint: Find an expression for $x_n - x_{n+1}$ and use (ii).
(iv) Deduce that this sequence has a limit.
(v) Use the defining equation $x_{n+1} = \frac{1}{3}(2 + x_n)$ to show that this limit is 1.

Problem 5. Give a detailed proof that a monotonic sequence (x_n) that is bounded below converges to the greatest lower bound $\inf S$ of the set $S = \{x_n : n \geq 1\}$.
Note: You are not allowed to use the fact that a bounded monotonic increasing sequence (x_n) converges to $\sup S$. i.e. do not use the proof on the top of p 70. Rather redo the work at the bottom of p.69 changing everything to the decreasing case.

Math 319 Homework 7

Due Wednesday, October 27, 2004

Problem 1. (a) Give an example of an unbounded sequence that has a convergent subsequence.

(b) Explain how to construct a monotonic decreasing sequence of rational numbers that converges to $\sqrt{2}$.

Problem 2. Consider the sequences

$$x_n = \sin\left(\frac{n\pi}{4}\right), \quad y_n = \frac{1}{\sqrt{n}}, \quad z_n = x_n y_n.$$

(a) Write down the first 10 terms of (x_n). Does it converge? Find two different monotonic subsequences of (x_n).

(b) Write down (in decimals) the first 10 terms of the product sequence (z_n). How many peaks does this sequence have? Find a monotonic decreasing subsequence of (z_n).

(c) Does (z_n) converge? (Explain which theorems you use in your argument.)

Problem 3. (a) Suppose that $x_n \geq 0$ for all n and that $\lim x_n = 2$. Find a subsequence of $((-1)^n x_n)$ that converges to 2 and another that converges to -2. Does $((-1)^n x_n)$ converge?

(b) Suppose that $x_n \geq 0$ for all n and you are told that $((-1)^n x_n)$ converges. Show that (x_n) converges. What is its limit?

Problem 4. (a) Consider the intervals $I_k = (1 + \frac{1}{3k}, 1 + \frac{1}{3k+1})$ for $k = 1, 2, 3, 4, \ldots$. Write down I_1, I_2 and I_3 explicitly.

(b) Make an accurate sketch of the set $A = \cup_{k \geq 1} I_k$.

(c) Describe all cluster points of A.

Problem 5. (a) Show that a point $c \in \mathbb{R}$ is a cluster point for a subset A of $\mathbb{R}$ if and only if there is a sequence (x_n) that converges to c and whose elements x_n all lie in $A \setminus \{c\}$. (Write the proof in your own words.)

(b) Can you choose this sequence to be monotonic? monotonic increasing?

Math 319 Homework 8

Due Wednesday, November 3, 2004

Problem 1. (a) Let $f(x) = x + x^2$. Prove from the definition that $\lim_{x\to 3} f = 12$.

(b) Let $f(x) = \frac{1}{2x+3}$. Prove from the definition that $\lim_{x\to 1} f = 1/5$.

Problem 2. Let A be the subset $\{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. Let (x_n) be any sequence of real numbers. Define a function $f : A \to \mathbb{R}$ by $f(\frac{1}{n}) = x_n$.

(a) If $x_n = n - 3$, write down the values of f at the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and sketch the graph of f.

(b) Prove that if $\lim x_n = L$ then $\lim_{x\to 0} f = L$.

(c) Prove that if $\lim_{x\to 0} f = L$ then $\lim x_n = L$.

Note: This follows from Thm 4.1.8, but I want you to prove it from the definitions.

Problem 3. Properly divergent sequences Let (x_n) be a sequence of real numbers. Definition 3.6.1 in the book says that

(i) $\lim x_n = \infty$ if for all $\alpha \in \mathbb{R}$ there is a number $K(\alpha)$ such that $x_n > \alpha$ whenever $n \geq K(\alpha)$. (You should think of α as a large positive number.) In this case we say that (x_n) **tends to** ∞.

(ii) $\lim x_n = -\infty$ if for all $\beta \in \mathbb{R}$ there is a number $K(\beta)$ such that $x_n < \beta$ whenever $n \geq K(\beta)$. (Here you should think of β as a very negative number.) In this case we say that (x_n) **tends to** $-\infty$.

(x_n) is called **properly divergent** if it tends either to ∞ or to $-\infty$. (To avoid confusion we do NOT call such a sequence convergent.)

(a) Show from the definition that $\lim(n^2 - n) = \infty$.

(b) Give an example of sequences (x_n) and (y_n) such that $\lim x_n = 0, \lim y_n = \infty$ and $\lim x_n y_n = \infty$.

(c) Is it possible to find sequences (x_n) and (y_n) such that $\lim x_n = 0, \lim y_n = \infty$ and $\lim x_n + y_n = 3$?

Justify your answers.

Problem 4. Let $f : (-1, 1) \to \mathbb{R}$ be a function and suppose that for all sequences (x_n) with $\lim x_n = 0$ we have $\lim f(x_n) = \infty$. Show that for all α there is $\delta > 0$ such that

$$0 < |x| < \delta \implies f(x) > \alpha.$$

Note: this condition on f means that $\lim\limits_{x \to 0} f = \infty$. (see def 4.3.5). So I am asking you to prove a version of (ii) implies (i) in the sequential convergence theorem 4.1.8.)

Problem 5. Suppose that $\lim_{x\to 3} f = 4$. Suppose that $x_n = 3 - \dfrac{1}{n}$. Show that the sequence $(f(x_n))$ converges to 4.

Note: This is an explicit example of (i) implies (ii) in the sequential convergence theorem 4.1.8.

Math 319 Homework 9

1. Redo all questions on Midterm 2 for which you got less than 7/10. I have a note of how many questions this is, so you do not need to hand back your original exam. This work will be incorporated into your final grade in some way, but not as part of the exam grade.

I have posted a copy of exam 2. I have written some comments on the problems below.

This is due at 5pm on Tuesday Nov 30.

2. Work on the project. During the next two lectures (Thursday Nov 18 and Tuesday Nov 23) I will be asking people to sign up for presentation times. People who volunteer to go early (specially on Nov 30) will get special consideration when I grade them.

I will have extra office hours during the next few days to give you a chance to discuss things with me. You can also email me your questions. There should also be time on Tuesday Nov 23 for me to answer your questions.

Comments on Midterm 2.

Typically people found questions 1,2,3 hard and questions 4 and 5 relatively easy.

1. Only a few of you could write out this proof well. Look up an argument in the book and try to write it in your own words.

2. Many of you found this hard (i.e. confusing?) but it really is an easy question. Remember to explain why points such as $3\frac{1}{2}$ or 5 are not cluster points, as well as explaining why 3 is.

3. It is easiest here to argue by contradiction. ie suppose that $L \notin [0, 2]$ and get a contradiction. The idea is very close to the proof that if $x_n \geq 0$ for all n and $\lim x_n = x$ then $x \geq 0$.

4. Almost all of you got this one. Often I took off a point because, although you did the steps and got the right answer, you did not explain clearly what you are doing.

5. Again, quite a few of you did this question well. I was lenient about asking for proofs here, giving almsot full marks to people who gave correct answers together with basically correct intuitive arguments (rather than a formal proof.) Note that the ratio test never works for sequences of this kind (ie where x_n is a rational function of n.) Also you have to be careful with the comparison test when the terms are both positive and negative; it is better to use the squeeze theorem (or put in absolute values).

The final exam will be very much like this one. So work on rewriting this should pay off.

Math 319/320 Second Midterm

November 11, 2004

Name: **School ID:**

Answer all the following questions, justifying all your statements. Each question is worth 10 points. There are five questions. Good luck!

Problem 1. Prove ONE of the following statements.
EITHER: (i) Prove that a monotonic increasing sequence that is bounded above converges.
OR: (ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that $\lim f(x_n) = f(c)$ for all sequences (x_n) such that $\lim x_n = c$. Prove that f is continuous at the point c.

1	
2	
3	
4	
5	
Total	

Problem 2. Let $A = \left\{ 3 + \frac{1}{n} : n \geq 1 \right\}$. Which points in $\mathbb{R}$ are cluster points of A? Prove all your claims from the definitions.

Problem 3. Suppose that (x_n) is a convergent sequence with limit L and that $x_n \in [0, 2]$ for all n. Prove from the definitions that $L \in [0, 2]$.

Problem 4. Show from the definition that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is continuous at the point $x = 2$.

Problem 5. Which of the following sequences (x_n) are convergent? If they are convergent, what are their limits? Prove your claims.

$$\text{(i)} \qquad x_n = \frac{(-1)^n n}{n^2 - 3}, \qquad\qquad \text{(ii)} \qquad x_n = \frac{(-1)^n n^2}{n^2 - 3}.$$

Math 319 Homework 10

due: Wednesday Dec 8, 2004

This problem set concerns the following concepts.

Let A be a subset of $\mathbb{R}$. A point $c \in \mathbb{R}$ is called a **boundary point of** A if *every* ϵ-neighborhood of c contains a point of A and a point of its complement $\mathbb{R} \setminus A$. (Note that c does NOT have to be in A.)

A point $c \in A$ is called an **interior point of** A if there is $\epsilon > 0$ such that the ϵ-neighborhood of x is entirely contained in A.

The subset A of $\mathbb{R}$ is said to be **open** if for every $x \in A$ there is $\epsilon > 0$ such that the ϵ-neighborhood of x is entirely contained in A.

The subset B of $\mathbb{R}$ is said to be **closed** iff its complement $\mathbb{R} \setminus B$ is open.

Problem 1. Let $A = [1, 3)$ (a half open interval).
(i) Show that the points 1 and 3 are boundary points of A.
(ii) Show that every point $x \in (1, 3)$ is an interior point of A.
(iii) Which of the following sets are closed?
$A = [2, 3), \quad B = \{0, 1/n : n \geq 1\}, \quad C = (2, 4).$

Problem 2. Let B be any subset of $\mathbb{R}$. Show that each point $x \in B$ is either a boundary point of B or an interior point of B, but cannot be both.

Problem 3. Let B be any subset of $\mathbb{R}$.
(i) Show that if $c \in B$ is a boundary point of B then it is a cluster point for the complement $\mathbb{R} \setminus B$ of B.
(ii) Prove that if c is a cluster point both for B and for its complement $\mathbb{R} \setminus B$ then c is a boundary point of B.
(iii) Take $B = \{1, 2\}$, a set containing just 2 points. What are its boundary points? What are its cluster points?

NOTE: The first version of Question 3(i) was wrong: it is not true that a boundary point c of B has to be a cluster point of B. The trouble is that c might be an isolated point of B, ie. there might be $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \cap B = \{c\}$. Then c would be a boundary point but not a cluster point.

Problem 4. Let $A \subset \mathbb{R}$ be open.
(i) Show that every point of A is an interior point of A.
(ii) Use the result of Problem 2 to show that A contains NONE of its boundary points.
The next two parts are **bonus problems.**
(iii) Problem 4(ii) implies that all the boundary points of A lie in its complement $B := \mathbb{R} \setminus A$. Use this together with the definition of a closed set to show that a set is closed iff it contains all its boundary points.
(iv) Finally combine what you have just proved (Problem 4(iii)) with Problem 3(ii) to deduce that a set is closed iff it contains all its cluster points.

Math 319 Review sheet for Final, Dec 2004

This exam will be out of 90 points: each question will be worth 15 points. The exam will be much like Midterm II. One question will ask you to prove a part of one of the theorems on the sheet. There will be no questions about countable/uncountable sets or using the Well-ordering principle for $\mathbb{N}$.

1: Prove ONE of the following results:
EITHER: **(i)** Let the subset A of $\mathbb{R}$ be open and suppose that c is a cluster point of the complement $\mathbb{R} \setminus A$. Prove that $c \notin A$.

OR: **(ii)** Let $f : A \to \mathbb{R}$ be any function and let $s := \sup\{f(x) : x \in A\}$. Show that there is a sequence $x_n \in A$ such that $\lim f(x_n) = s$.

2: (i) Let $x > 0$. Prove that $\lim_{n\to\infty} \frac{1}{1+nx} = 0$.
(ii) Let $0 < b < 1$. Prove that $\lim b^n = 0$.

(Use Archimedes Principle and Bernoulli's inequality, both of which are now on the sheet)

3: Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \frac{3x}{x^2+2}$. Prove from the definition that $\lim_{x\to 1} f(x) = 1$.

4: Describe examples satisfying the following conditions. Justify your answers.
(i) A bounded set A and a continuous function $f : A \to \mathbb{R}$ that is not bounded.

(ii) An infinite bounded subset A of $\mathbb{R}$ such that every point of A is a boundary point.

(iii) A convergent sequence that is not monotonic.

5: Define the sequence (x_n) recursively by setting $x_1 = 1$ and $x_n = 1 + \frac{x_{n-1}}{2}$.
(i) Show that (x_n) is monotonic increasing.
(ii) Show that it converges and find its limit.

6: Which of the following sequences are convergent? Justify your answers.

$$\textbf{(i)} \quad x_n = \frac{(-1)^n \sin n}{n+1}; \qquad \textbf{(ii)} \quad x_n = \frac{(-2)^n}{n+1}; \qquad \textbf{(iii)} \quad x_n = \frac{n-3}{2n-1}.$$

Math 319: Definitions and Theorems for the Final

Def 3.1.3 A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms x_n satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called **divergent**.

Def 3.4.1 Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \ldots)$ is called a **subsequence** of X.

Def 4.1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a **cluster point** of A if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Let A be a subset of $\mathbb{R}$. A point $c \in \mathbb{R}$ is called a **boundary point of** A if *every* ϵ-neighborhood of c contains a point of A and a point of its complement $\mathbb{R} \setminus A$. (c does NOT have to be in A.) A point $c \in A$ is called an **interior point of** A if there is $\epsilon > 0$ such that the ϵ-neighborhood of x is entirely contained in A.

The subset A of $\mathbb{R}$ is said to be **open** if for every $x \in A$ there is $\epsilon > 0$ such that the ϵ-neighborhood of x is entirely contained in A. The subset B of $\mathbb{R}$ is said to be **closed** iff its complement $\mathbb{R} \setminus B$ is open.

Def 4.1.4. Let $A \subset \mathbb{R}$ and let c be a cluster point of A. A function $f : A \to \mathbb{R}$ is said to have **limit** L **at** c if for all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - c| < \delta,\ x \in A \Longrightarrow |f(x) - L| < \epsilon$.

Def 5.1.1. Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then f is **continuous at** c if for every $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta,\ x \in A \Longrightarrow |f(x) - f(c)| < \epsilon$. If B is a subset of A we say that f is **continous on** B if it is continuous at all points $b \in B$.

Archimedes Principle: For all $x \in \mathbb{R}$ there is an integer $n > x$.

Bernoulli inequality: For all $x \geq 0$ and $n \geq 1$ $(1 + x)^n \geq 1 + nx$.

Thm 3.1.10 Comparison theorem for limits. Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ca_n$ for all $n \geq m$, then $\lim x_n = 0$.

Thm 3.2.2 A convergent sequence of real numbers is bounded.

Thm 3.3.2 Monotone Convergence Theorem. A monotone sequence of real numbers is convergent if and only if it is bounded.

Thm 3.4.2 If $X = (x_n)$ converges to $x \in \mathbb{R}$ then every subsequence X' of X also converges to x.

3.4.7: Monotone subsequence theorem. Every sequence has a monotone subsequence.

3.4.8: Bolzano–Weierstrass theorem. A bounded sequence of real numbers has a convergent subsequence.

Thm 4.1.8. Sequential criterion Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then $\lim_{x \to c} f = L$ iff for every (x_n) in $A \setminus \{c\}$ that converges to c the sequence $(f(x_n))$ converges to L.

Thm 5.1.3. Sequential criterion for continuity: $f : A \to \mathbb{R}$ is continuous at $c \in A$ iff for every (x_n) in A that converges to c the sequence $(f(x_n))$ converges to $f(c)$.

Thm 5.3.2 Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I, i.e. there is M such that $|f(x)| \leq M$ for all $x \in I$.

Thm 5.3.4 Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I, i.e. there re points c, d in I such that $f(c) \leq f(x) \leq f(d)$ for all $x \in I$.

I will have my usual office hours Nov 24 at 11:45am, and extra office hours next week Tuesday Nov 30 1-2.

Here is the schedule for the projects: Each group should decide on the best order for their presentations. I have written down your names in random order here.

Tuesday Nov 30:

1. Felix Grucci, Mike La Barbera, Roseline Paraisy on #13
2. Adam Bernstein, Oriana Bogler #3
3. Joanne Black, Zhen Huang, Jessica Aguirre #1.

Thurs Dec 2:

1. Theresa Albergo, Allan Martin, Meg Smith, Sinan Karasulu #2
2. Nikki-Ann Gerardi, Christina Gjorgjioski #12
3. Elvia Avila, Jeanne O'Leary, Linda Pavlica #10

Tuesd Dec 7:

1. David Cotter, Li-Min Wu, XiuXia Ou #11
2. Miguel Sanchez, Jason Ferrara, Kristen Pagnano #5
3. Munjal Subodh #?
4. Lindsay Rosenthal, Michael Sarro # 14

Thurs Dec 9:

1. Sheena Kurian, Remy Casado # 11
2. Cindy Chang, Travis Lee # 8 (?)
3. Kristen Garvey, Erica Zinober, Robert Miller # 15.