MAT308: Differential Equations with Linear Algebra

Syllabus

Tentative Schedule for MAT308, Spring 2012

Week	Monday	Wednesday	Homework
1-23	10.1 Differential Equations and Direction Fields 10.2 Separation of Variables	10.3 Linear Equations, Integrating Factors	1.1A 2, 3, 7, 8, 11, 20, 25 10.2 4, 9, 18, 19, 23, 26, 28
1-30	3.1 Linear Functions on Real Vector Spaces 3.2 Vector Spaces	3.2 Vector Spaces 3.3 Linear Functions	10.3 6, 7, 9, 16 3.1 2, 4, 6, 8, 12, 17, 18, 19, 20, 33
2-6	3.4 Image and Null Space	3.5AB Coordinates	3.2 12, 16, 17, 18, 20, 24, 25, 26, 31 3.3 6, 7, 8, 11, 12, 17, 20, 22, 24 3.4 8, 9, 10, 11, 12
2-13	3.5C Dimension	3.6 Eigenvalues and Eigenvectors	3.4 16, 18, 19, 21 3.5AB 4, 8, 15, 16, 17, 24, 27, 41 3.5C 3, 4, 5, 10, 11
2-20	3.7AB Inner Products	3.7BC Inner Products	3.6A 2, 4, 6, 13, 14 3.6BC 4, 7, 8, 9, 14 3.7A 1, 2, 8 3.7B 2, 4 3.7C 2, 3
2-27	Review	Review	Midterm
3-5	11.1 Differential Operators 11.2 Complex Solutions, Higher Order Equations	11.3 Nonhomogeneous Equations	11.1 8, 9, 14, 36, 39 11.2A 15, 18, 20 11.2BC 1, 4, 8, 49
3-12	11.4 Oscillation 11.5 Laplace Transform	11.6 Convolution	11.3AB 2, 5, 6, 10, 43 11.4 16, 17, 18 11.5 19, 20, 21, 23 11.6 1, 4, 7
3-19	Review	Review	Midterm
3-26	12.1 Vector Fields 12.2 Linear Systems	13.1 Eigenvalues and Eigenvectors	Enjoy your break!
4-2	Spring Break	Spring Break	
4-9	13.2 Matrix Exponentials	13.2 Cayley-Hamilton Theorem, Linear Independence	12.1AB 4, 7, 8, 24, 34 12.2 5, 36
4-16	13.3 Nonhomogeneous Systems	13.4 Equilibria and Stability	13.1 7, 10, 11 13.2ABC 2, 11, 22 13.2D 7, 16

			13.3 1, 2, 4, 11
4-23	13.4 Nonlinear Systems	14.6 Differential Equations 14.7 Power Series Solutions	13.4A 2, 4, 6, 13, 21 13.4B 2, 5, 8 14.6 5, 8
4-30	Review	Final (due May 6 in my office)	Final (due May 6 in my office)

MAT 308, Spring 2012
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COURSE DESCRIPTION

Linear Algebra and differential equations are intimately related subjects. While some linear algebra was used in multivariable calculus, we will study the subject more fully and then use this to examine solutions to differential equations.
MAT307 and MAT308 together cover the same material as MAT203, MAT211, and MAT303, at a slightly more theoretical level. This means that this course is going to move *quickly* and will be a significant amount of work. Since about half of the linear algebra material was covered in MAT307, you should have taken either MAT307 or MAT211 before taking MAT308.

TEXTBOOK

The official textbook for this class is *Multivariable Mathematics* by Williamson and Trotter. I strongly recommend that you read the assigned chapter of the textbook before lecture. The text is rather densely written, so you may not understand it on the first or even second reading. Keep trying!

HOMEWORK

Weekly problem sets will be assigned and collected in class on Mondays. You are encouraged to discuss the homework problems with others, but your write-up must be your own work.

GRADING

There will be two midterm exams and a final exam. The exams will be take-home, assigned on Wednesday and due the following Monday. In addition to the exams, weekly homework will be assigned and collected. Final course grades will be based on this breakdown:

30 % final exam, 20 % each midterm exam, 30 % homework

Please note that homework is worth 30 % of your grade – the only way to earn a good grade in this class is to work hard on the homework each week. The exam dates are as follows:

First Midterm: Feb 29 - Mar 5
Second Midterm: Mar 21 - Mar 26
Final Exam: May 14, 5:15-7:45

DISABILITIES

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services at http://studentaffairs.stonybrook.edu/dss/ or (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

http://www.stonybrook.edu/ehs/fire/disabilities.shtml

ACADEMIC INTEGRITY

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

http://www.stonybrook.edu/uaa/academicjudiciary/

CRITICAL INCIDENCE MANAGEMENT

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

MAT308 Exam 1

While taking this exam, you may use your book and class notes, but no other sources of information. This includes other books, other people, and the internet.

1. Find all solutions to the differential equation.

   (a) $y' = x + xy^2$.

   (b) $y' - 3y = x^2$.

2. Determine whether each of the sets below is a vector space. If it is a vector space, prove it. If not, explain why not.

   (a) The set of all polynomials $p(x)$ of degree at most 7 such that $p(2) = p'(1)$.

   (b) $\{(x, y) \in \mathbb{R}^2 | x \geq y\}$.

   (c) The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. (A function $f(x)$ has *compact support* if there is an interval $[a, b]$ such that $f(x) = 0$ whenever $x < a$ or $x > b$.)

3. Determine whether each of the maps between vector spaces below is linear. If it is linear, prove it. If not, explain why not.

   (a) The map $B : \mathbb{R} \to \mathbb{R}$ given by $B(x) = \frac{1}{x}$.

   (b) The map $r : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the x-axis.

   (c) The map $A : \mathbb{R}^n \to \mathbb{R}$ given by $A(\vec{x}) = |\vec{x}|$.

4. Describe the kernel and image of each of the following linear maps.

   (a) The map
   $$\begin{pmatrix} 1 & 7 & 4 \\ 4 & 3 & 6 \\ 1 & 2 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

   (b) The map $ev_3 : \mathcal{P}_4 \to \mathbb{R}$ given by $ev_3(p) = p(3)$.

   (c) The map $m : \mathcal{P}_3 \to \mathcal{P}_4$ given by $m(p) = (x - 3)p$.

5. Find the dimension of each of the following vector spaces.

   (a) The subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 2, 3, 4)$, $(5, 6, 7, 8)$, $(9, 10, 11, 12)$, and $(13, 14, 15, 16)$.

   (b) The set of polynomials of degree at most 8 whose second derivative is identically zero.

   (c) The set of solutions to the differential equation $y' = e^{x^2} y$. (Hint: you do not have to solve the differential equation!)

6. Find the eigenvalues and corresponding eigenvectors of each of the following linear maps.

   (a)
$$\begin{pmatrix} 2 & -1 \\ 3 & 6 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

   (b)
$$\begin{pmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 5 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

   (c) The map $L : \mathcal{P}_5 \to \mathcal{P}_5$ given by $L(p) = p - p'$.

7. Let ℓ^2 be the subspace of $\mathbb{R}^\infty$ consisting of sequences $(a_1, a_2, a_3, \ldots)$ such that $\sum_{k=1}^{\infty} a_k^2$ is finite.

   (a) Show that the pairing $\langle (a_1, a_2, a_3, \ldots,), (b_1, b_2, b_3, \ldots) \rangle = \sum_{k=1}^{\infty} a_k b_k$ is an inner product on ℓ^2.

   (b) Let e_i be the sequence with a 1 in the i^{th} position and 0's in every other position. Show that the e_i's form an orthonormal set of vectors in ℓ^2.

   (c) Show that the e_i's form a *complete* set of orthonormal vectors.

8. Let V and W be subspaces of $\mathbb{R}^n$. Show that, if $dimV + dimW > n$, then V and W both contain a common line through the origin.

9. Let $a_1, a_2, \ldots, a_{n+1}, b_1, b_2, \ldots, b_{n+1}$ be real numbers. Prove that there is a unique polynomial $p(x)$ of degree at most n such that $p(a_i) = b_i$ for all $i = 1, 2, \ldots, n+1$. (In other words, a polynomial of degree at most n is uniquely determined by its value at $n+1$ points.)

10. The function
$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$
is known as the *bump function.*

    (a) Show that the n^{th} derivative of $f(x)$ can be written in the form $f^{(n)}(x) = p_n(\frac{1}{x})e^{-\frac{1}{x}}$, where p_n is a polynomial.

    (b) Use this to prove that the bump function is smooth. (Hint: consider the substitution $y = \frac{1}{x}$.)

    (c) Conclude that, for any real number $\epsilon > 0$, the linear map $L : \mathcal{C}^\infty(-\epsilon, \epsilon) \to \mathbb{R}^\infty$ given by
$$L(g) = (g(0), g'(0), g''(0), g'''(0), \ldots)$$
is not injective.

    (d) Is L surjective?

MAT308 Exam 2

While taking this exam, you may use your book and class notes, but no other sources of information. This includes other books, other people, and the internet.

1. Find all solutions to the given differential equation.
   (a) $y'' + y' + y = 0$.
   (b) $y''' - y' = 0$.
   (c) $y''' - 3y'' + 3y' - y = 0$.

2. Use the method of undetermined coefficients to find the general solution to the given differential equation.
   (a) $y'' - 4y' = x$.
   (b) $y'' - 10y' + 21y = e^{5x}$.

3. Use the method of variation of parameters to find the general solution to the differential equation $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = e^{4t}$.

4. Compute the Laplace transform of each of the following functions.
   (a) 2^t.
   (b) $\chi_{[a,b]}(t)$, where

$$\chi_{[a,b]}(t) = \begin{cases} 1 & \text{if } a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}$$

5. Use the Laplace transform to find the solution to the given initial value problem.
   (a) $y'' - 10y' + 24y = e^{3t}$, $y(0) = 2$, $y'(0) = 1$.
   (b) $y'' + y = cos(t)$, $y(0) = 0$, $y'(0) = 1$.

6. Compute the following:
   (a) $sin(t) * cos(t)$.
   (b) $\frac{1}{t^2} * t$.

7. Let $B : \mathcal{C}^\infty(0,\infty) \to \mathcal{C}^\infty(0,\infty)$ denote the linear map given by $B(y) = y' - \frac{y}{x}$. Find the general solution to the differential equation $B^n(y) = 0$. (Hint: mimic the proof of Theorem 11.1.1 in your book.)

8. The goal of this problem is to show that the complex roots of a polynomial with real coefficients occur in complex conjugate pairs. If $z = a + bi$ is a complex number, we define the *complex conjugate* of z to be $\overline{z} := a - bi$.

   (a) Let z_1 and z_2 be arbitrary complex numbers. Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

   (b) If f is a polynomial with real coefficients, show that $f(\overline{z}) = \overline{f(z)}$ for any complex number z.

   (c) Conclude that, if z is a root of f, then $\overline{z}$ is also a root of f.

   (d) Use the above to show that any homogeneous linear differential equation with constant coefficients of odd order admits at least one solution of the form e^{ax}, where a is a real number.

9. For a given smooth function $f(x)$, let $A_f : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ denote the linear map given by $A_f(y) = y' - fy$.

   (a) Show that the set of solutions to the differential equation $A_f(y) = 0$ is a 1-dimensional vector space. (Hint: one approach might be to use the Existence and Uniqueness Theorem, which is Theorem 10.1.1 in your book.)

   (b) Let $f_1, \dots, f_n$ be smooth functions. Show that the set of solutions to the differential equation

$$A_{f_1} A_{f_2} \cdots A_{f_n}(y) = 0$$

   is an n-dimensional vector space.

10. For every positive integer k, define the function $f_k : \mathbb{R}^{>0} \to \mathbb{R}$ as follows:

$$f_k(t) = \begin{cases} k & \text{if } t \leq \frac{1}{k} \\ 0 & \text{if } t > \frac{1}{k} \end{cases}$$

   (a) Compute $\lim_{k\to\infty} L[f_k(t)]$.

   (b) Compute $L[\lim_{k\to\infty} f_k(t)]$.

   (c) Show that, for any continuous function $g : \mathbb{R}^{>0} \to \mathbb{R}$,

$$\lim_{k\to\infty} (f_k * g) = g.$$