Stony Brook University

Yu Li (李宇)

Search this site

Yu Li
CV
Teaching
MAT 303: Calculus IV with Applications
MAT 319: Foundations of Analysis
Sitemap

Teaching >

MAT 303: Calculus IV with Applications

Spring 2019

Course Description: This course will introduce basic methods for solving ordinary differential equations, with a particular emphasis on linear differential equations with constant coefficients and systems of differential equations. Differential equations are the language in which the laws of physics are expressed, and have numerous applications in the physical, biological, and social sciences. We will discuss many standard applications. We will also briefly discuss some numerical methods for solving differential equations.

Textbook: Edwards & Penney, **Differential Equations and Boundary Value Problems: Computing and Modeling, 5th Edition.** You can use other editions, but be aware that numeration of the exercises might be different.

Instructor: Yu Li, Math Tower 4-101B. Office Hours: MW 11:30am-12:30am. Email: yu.li.4@stonybrook.edu.

Course Assistant: Jaroslaw Jaracz, Math Tower 5-125B, Office Hours: W 3:00pm-4:00pm. Email: jjaracz@math.stonybrook.edu.

Class schedule: MWF 10:00am-10:53am Library E4320.

Homework: Homework is a fundamental part of this course. Assignments will be posted on the course website at the beginning of each week and will be due to your TA (at the start of recitation) of the following week. Late homework will not be accepted.

Week	**Lectures**	**Homework**
1/28	1.1,1.2	1.1:#8,15,29,31,36; 1.2:#10,16,27
2/4	1.3,1.4 A web app on slope fields	1.3:#21,22,28,32; 1.4:#5,17,21,27,33,37,44,64
2/11	1.5	1.5:#5,9,14,16,20,24,29,34,37,39,40
2/18	1.5, 1.6	1.6:#3,4,7,8,11,13,20,25,27,30,31,33,38,71
2/25	1.6, Midterm 1 on 3/1	No Homework
3/4	3.1	3.1:#1,3,8,13,15,19,21,23,25
3/11	3.1, 3.2	3.1:#31,33,36,39,42,51; 3.2:#1,3,5,8,9,11,13,16,20,22,24
3/18	Spring Recess	No Homework

3/25	3.3, 3.5	3.3:#1,3,7,10,16,18,21,23,26,27,30,37,48,49
4/1	3.5, Midterm 2 on 4/5	No Homework
4/8	4.1,4.2,5.1	4.1:#1,5,7,18,20,23,25;4.2:#1,5,6,9,12,17,21,24,29
4/15	5.1,5.2	5.1:#3,5,6,7,11,14,25,27,34,36,41
4/22	5.2,5.5	5.2:#1,3,6,11,15,17,12,26,38,40,41
4/29	5.5,5.6	5.5:#1,3,7,11,13,16,25,27; 5.6:#1,4,7
5/6	5.7, Review	

Exams: There will be two in-class midterms on **Friday, March 1** and **Friday, April 5**. The final exam is on **Monday, May 20, 8:00am-10:45am** and the room is Library E4320.

If you register for this course you must make sure that you are available at these times, as there will be **no make-ups** for missed exams.

Course grade is computed by the following scheme:

Homework: 20%

Midterm Test I: 20%

Midterm Test II: 20%

Final Exam: 40%

Help: The **Math Learning Center** (MLC) is located in Math Tower S-235, and offers free help to any student requesting it. It also provides a locale for students wishing to form study groups. The MLC is open 9am-7pm Monday through Friday. A list of graduate students available for hire as private tutors is maintained by the Undergraduate Mathematics Office, Math Tower P-143.

Disability Support Services (DSS)

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.stonybrook.edu/ehs/fire/disabilities

Academic Integrity

Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/commcms/academic_integrity/index.html

Critical Incident Management Statement

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive

behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures.

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Ċ	Final practice exam ans...	Yu Li, May 12, 2019, 10::	v.1	ď
Ċ	Midterm 1 Practice Pro...	Yu Li, Feb 17, 2019, 6:09	v.1	ď
Ċ	Midterm 1 practice ans...	Yu Li, Feb 24, 2019, 1:44	v.1	ď
Ċ	Midterm 2 Practice Pro...	Yu Li, Mar 31, 2019, 1:55	v.1	ď
Ċ	Practice final exam.pdf (1	Yu Li, May 12, 2019, 10::	v.1	ď
Ċ	Solutions of the ODE in...	Yu Li, Feb 11, 2019, 1:47	v.1	ď
Ċ	midterm 2 practice ans...	Yu Li, Mar 31, 2019, 1:55	v.1	ď

Problem: Solve the following differential equation.

$$\frac{dy}{dx} = y\sqrt{y^2-1}, \quad y(a) = b. \tag{1}$$

Proof. By separating the variables, we have

$$\int \frac{1}{y\sqrt{y^2-1}}\,dy = \int 1\,dx. \tag{2}$$

We use the substitution $y = \sec\theta$, then $y^2 - 1 = \tan^2\theta$ and $dy = \tan\theta\sec\theta\,d\theta$. The equation (2) becomes

$$\int 1\,d\theta = \int 1\,dx, \tag{3}$$

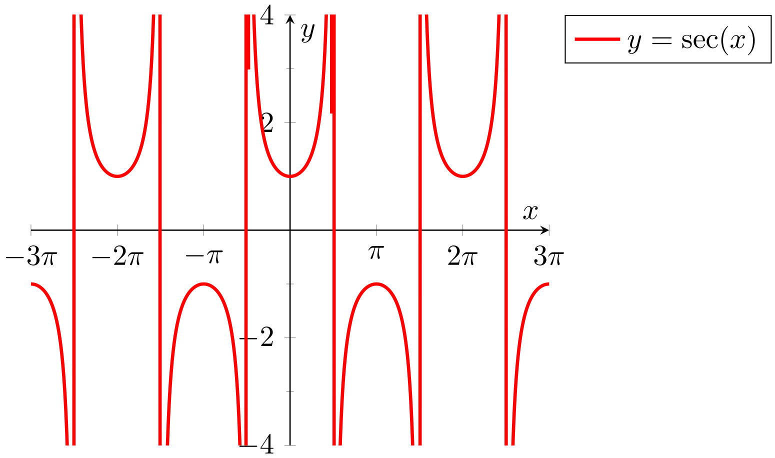
which is equivalent to

$$\theta = x + C. \tag{4}$$

Therefore we have

$$y = \sec\theta = \sec(x + C) \tag{5}$$

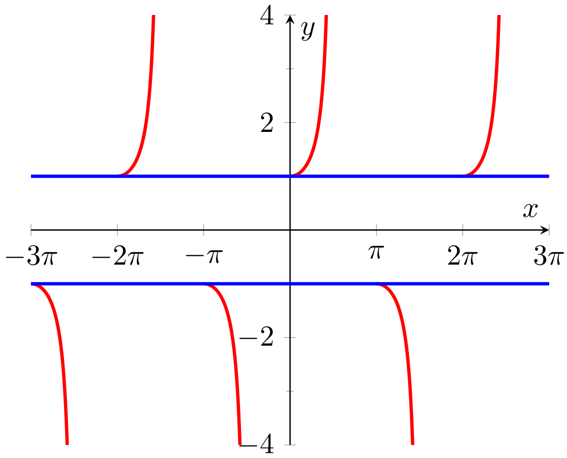
is the general solution. Notice that an equation like (5) is a translation of the function $y = \sec x$, whose graph is below.



However, from (1), we observe that $\frac{dy}{dx} = y\sqrt{y^2-1} \geq 0$ if $y \geq 1$ and $\frac{dy}{dx} = y\sqrt{y^2-1} \leq 0$ if $y \leq 1$. Therefore, for the graph above the x-axis, we only keep the increasing parts and for the graph below the x-axix, we keep the decreasing part. More precisely, the general solution of (1) is

$$y = \sec(x + C), \quad x \in [-C + k\pi, -C + k\pi + \frac{\pi}{2}) \quad \text{for any} \quad k \in \mathbb{Z}. \tag{6}$$

In addition, we have two singular solutions $y \equiv 1$ and $y \equiv -1$. See the following graph of the solution curve when $C = 0$ and two singular solution curves.



In summary, for the initial point (a, b) on the plane, we have the following result.

(1) If $b > 1$ or $b < -1$, then there is a unique solution.

(2) If $-1 < b < 1$, then there is no solution.

(3) If $b = 1$ or $b = -1$, then there are two solutions.

$\square$

Midterm 1 Practice Problems

Problem 1. Solve the following initial value problems:

(a) $y' = 6e^{2x-y}, \quad y(0) = 0.$

(b) $y' = -\frac{2}{x}y + \frac{1}{x^2}, \quad y(1) = 2.$

(c) $xy' = y + x^2, \quad y(1) = 0.$

(d) $y' - \frac{y}{x} = xy^5, \quad y(1) = 1.$

(e) $(1+x)y' = 3y, \quad y(0) = 1.$

(f) $2xyy' = x^2 + y^2, \quad y(1) = 2.$

Problem 2. Find the general solution to each of the following differential equations.

(a) $9y' = xy^2 + 5xy - 14x.$

(b) $y' + \frac{2}{3x}y + \frac{3}{y^2} = 0.$

(c) $y' + y\cot x = \cos x.$

(d) $x^2y' + \frac{y^3}{x} = 2y^2.$

(e) $y' = \left(1 + \frac{4}{x}\right) y.$

(f) $(xy + y^2)dx + x^2dy = 0.$

(g) $(2x - \frac{\ln y}{x^2})dx + \frac{1}{xy}dy = 0.$

(h) $y' = \sqrt{x + y + 1}.$

(i) $y' = \frac{x+3y}{y-3x}.$

(j) $xy' = 6y + 12x^4y^{2/3}.$

Problem 3. Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x^2 - 1}.$$

(a) Find all values a and b such that this equation with the initial condition $y(a) = b$ has a unique local solution.

(b) Find the general solution to the differential equation.

Problem 4. Determine whether each of the following equations is exact. If it is exact, find its solutions.

(a) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0.$

(b) $2xy^2 + 4 = 2(3 - x^2y)y'.$

(c) $\dfrac{2xy}{x^2+1} - 2x - (2 - \ln(x^2+1))y' = 0.$

1. (a) $\frac{dy}{dx} = 6e^{2x-y} \iff \int e^{y}\,dy = \int 6e^{2x}\,dx$

$\iff e^{y} = 3e^{2x} + C$

Since $y(0) = 0$, $\quad 1 = 3 + C$ and $C = -2$

So. $y = \ln(3e^{2x} - 2)$

1 (b) $m = e^{\int \frac{2}{x}dx} = x^2 \qquad y = \frac{1}{x^2}\left(\int x^2 \cdot \frac{1}{x^2}\,dx\right) = \frac{x+C}{x^2}$

Since $y(1) = 2$, $\quad 2 = \frac{1+C}{1}$. So $C = 1$ and $y = \frac{x+1}{x^2}$

1 (c). $y' - \frac{1}{x}y = x \qquad m = e^{\int -\frac{1}{x}dx} = \frac{1}{x} \qquad y = x\left(\int \frac{1}{x}\cdot x\,dx\right)$

$= x(x+C)$

since $y(1) = 0$, $\quad 0 = 1(1+C)$. So $C = -1$ and $y = x(x-1)$

1 (d) $u = y^{-4} \qquad u' = -4y^{-5}y' = -4y^{-5}\left(xy^5 + \frac{y}{x}\right) = -\frac{4}{x}y^{-4} - 4x$

$= -\frac{4}{x}u - 4x$

$m = e^{\int \frac{4}{x}dx} = x^4 \qquad u = \frac{1}{x^4}\left(\int x^4(-4x)\,dx\right) = \frac{1}{x^4}\left(\frac{-4x^6}{6} + C\right)$

$= \frac{1}{x^4}\left(-\frac{2}{3}x^6 + C\right)$

Since $y(1)=1$. $u(1)=1$. From $1=(-\frac{2}{3}+C)$ we know

$C=\frac{5}{3}$ and hence $u=\frac{1}{x^4}(-\frac{2}{3}x^6+\frac{5}{3})$

$$y=u^{-\frac{1}{4}}=x(-\frac{2}{3}x^6+\frac{5}{3})^{-\frac{1}{4}}$$

e) $\int \frac{1}{y}\,dy=\int \frac{3}{1+x}\,dx \iff \ln|y|=3\ln|1+x|+C$

$y=A(1+x)^3$ Since $y(0)=1$, $A=1$ and $y=(1+x)^3$

f) $y'=\frac{1}{2}(\frac{x}{y}+\frac{y}{x})=\frac{1}{2}(\frac{1}{u}+u)$ where $u=\frac{y}{x}$.

$$y'=(xu)'=u+xu'=\frac{1}{2}(u+\frac{1}{u})$$

$\Rightarrow xu'=\frac{1}{2}(\frac{1}{u}-u) \iff \int \frac{1}{\frac{1}{u}-u}\,du=\int \frac{1}{2x}\,dx$

$\iff \int \frac{1}{\frac{1}{u}-u}\,du=\int \frac{u}{1-u^2}\,du=\int \frac{1}{2}(\frac{1}{1-u}-\frac{1}{1+u})\,du$

$$=\frac{1}{2}\left(-\ln|1-u|-\ln|1+u|\right)=-\frac{1}{2}\cdot\ln|1-u^2|$$

therefore $-\ln|1-u^2|=\ln|x|+C$

and $1-u^2=A\cdot\frac{1}{x}$ Since $y(1)=2$, $u(1)=\frac{y(1)}{1}=2$

From $1-2^2 = A\cdot\frac{1}{1}$ $\quad A=-3$ $\quad$ and $1-u^2 = -\frac{3}{x}$

$u = \sqrt{1+\frac{3}{x}}$ and $y = ux = x\sqrt{1+\frac{3}{x}}$.

2. (a) $9\cdot\frac{dy}{dx} = x(y^2+5y-14)$

$$\Longleftrightarrow \int \frac{9}{y^2+5y-14}\,dy = \int x\,dx = \frac{x^2}{2}+C$$

$$\int \frac{9}{y^2+5y-14}\,dy = 9\cdot\int \frac{1}{(y+7)(y-2)}\,dy = \int \frac{1}{y-2} - \frac{1}{y+7}\,dy = \ln\left|\frac{y-2}{y+7}\right|$$

So. $\frac{y-2}{y+7} = Ae^{\frac{x^2}{2}}$ and $y = \frac{2+7Ae^{\frac{x^2}{2}}}{1-Ae^{\frac{x^2}{2}}}$

(b) $u = y^3$ $\quad u' = 3y^2y' = 3y^2\left(-\frac{3}{y^2} - \frac{2}{3x}y\right) = -9-\frac{2}{x}y^3$

$$= -9-\frac{2}{x}u.$$

$m = e^{\int \frac{2}{x}dx} = x^2$ $\quad u = \frac{1}{x^2}\left(\int x^2(-9)\,dx\right)$

$$= \frac{1}{x^2}\left(-3x^3+C\right)$$

$$y = \left(\frac{1}{x^2}\left(-3x^3+C\right)\right)^{\frac{1}{3}}.$$

2 (c). $m = e^{\int \cot x \, dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln \sin x} = \sin x$

$$y = \frac{1}{\sin x}\left(\int \sin x \cdot \cos x \, dx\right) = \frac{1}{\sin x}\left(\int \frac{\sin 2x}{2} dx\right) = \frac{1}{\sin x}\left(\frac{-\cos 2x}{4} + C\right)$$

(d) $y' + \left(\frac{y}{x}\right)^3 = 2\left(\frac{y}{x}\right)^2$ $\qquad u = \frac{y}{x}$

$$y' = u + xu' = 2u^2 - u^3$$

$$\frac{du}{dx} = \frac{2u^2 - u^3 - u}{x} \iff \int \frac{1}{2u^2 - u^3 - u} du = \int \frac{1}{x} dx = \ln|x| + C$$

Now $\int \frac{1}{2u^2 - u^3 - u} du = -\int \frac{1}{u(u-1)^2} du = -\int \frac{1}{u} - \frac{1}{u-1} + \frac{1}{(u-1)^2} du$

$$= -\left(\ln|u| - \ln|u-1| - \frac{1}{u-1}\right)$$

Then. $\frac{u-1}{u} e^{\frac{1}{u-1}} = A \cdot x \iff (y-x)\, e^{\frac{x}{y-x}} = Axy$

(e). $\int \frac{1}{y} dy = \int 1 + \frac{4}{x} dx \iff \ln|y| = x + 4\ln|x| + C$

$$y = Cx^4 e^x$$

2 (f). $xy+y^2+x^2\frac{dy}{dx}=0$

$\Longleftrightarrow \quad \frac{dy}{dx} = -\frac{y}{x}-\left(\frac{y}{x}\right)^2 = -u-u^2 \quad u=\frac{y}{x}$

$y' = xu'+u = -u-u^2 \Longleftrightarrow -\int\frac{1}{2u+u^2}\,du = \int\frac{1}{x}\,dx = \ln|x|+C$

$-\int\frac{1}{2u+u^2}\,du = -\int\frac{1}{u(u+2)}\,du = \frac{1}{2}\left(\int\frac{1}{u+2}-\frac{1}{u}\,du\right) \pm$

$= \frac{1}{2}\ln\left|\frac{u+2}{u}\right|$ So. $\frac{u+2}{u} = Ax^2 \Leftrightarrow u = \frac{2}{Ax^2-1}$

and $y = \frac{2x}{Ax^2-1}$

(g) It is an exact differential equation, so we solve

$\begin{cases} F_x = 2x-\frac{\ln y}{x^2} & \text{--- ①} \\ F_y = \frac{1}{xy} & \text{--- ②} \end{cases}$ From ①. $F = x^2+\frac{1}{x}\ln y+g(y)$

$F_y = \frac{1}{xy}+g'(y) = \frac{1}{xy}$

So. $g'(y)=0$ and $g(y)=c$.

The solution is $x^2+\frac{1}{x}\ln y = C$

2 (h) $u = x+y+1$

$u' = 1+y' = 1+\sqrt{u}$ $\qquad \int \frac{1}{1+\sqrt{u}}\,du = \int 1\,dx = x+C$

$$\int \frac{1}{1+\sqrt{u}}\,du \overset{t=\sqrt{u}}{=} \int \frac{1}{1+t}\cdot 2t\cdot dt = \int 2-\frac{2}{1+t}\,dt = 2t-2\ln|1+t|$$

$$= 2\sqrt{u}-2\ln(1+\sqrt{u})$$ So,

$$2\sqrt{x+y+1}-2\ln(1+\sqrt{x+y+1}) = x+C$$

2 (i) $y' = \dfrac{1+3\frac{y}{x}}{\frac{y}{x}-3} = \dfrac{1+3u}{u-3}$ $\quad (u = \frac{y}{x})$

$$y' = u+xu' = \frac{1+3u}{u-3} \iff xu' = \frac{1+6u-u^2}{u-3}$$

$$\int \frac{u-3}{1+6u-u^2}\,du = \int \frac{1}{x}\,dx = \ln|x|+C$$

$$\int \frac{u-3}{1+6u-u^2}\,du = -\int \frac{u-3}{(u-3)^2-10}\,du = -\frac{1}{2}\ln|(u-3)^2-10|$$

so $(u-3)^2-10 = \dfrac{A}{x^2}$ $\qquad u = 3\pm\sqrt{\dfrac{A}{x^2}+10}$

$$y = 3x \pm x\sqrt{\frac{A}{x^2}+10}$$

2 (j) $y' = 6\frac{y}{x} + 12x^3 y^{\frac{2}{3}}$

$u = y^{1-\frac{2}{3}} = y^{\frac{1}{3}}$

$u' = \frac{1}{3} y^{-\frac{2}{3}} y' = \frac{1}{3} y^{-\frac{2}{3}} \left(6\frac{y}{x} + 12x^3 y^{\frac{2}{3}} \right) = \frac{2y^{\frac{1}{3}}}{x} + 4x^3$

$= \frac{2u}{x} + 4x^3$

$m = e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}$

$u = x^2 \left(\int \frac{1}{x^2} \cdot 4x^3 dx \right) = x^2 (2x^2 + c)$

$y = u^3 = x^6 (2x^2 + c)^3$

3. (a) $f(x, y) = \frac{y}{x^2-1}$, $f_y = \frac{1}{x^2-1}$

If $x^2 - 1 \neq 0$ or $x \neq \pm 1$, then the differential equation locally has a unique solution with $y(a) = b$ provided that $a \neq \pm 1$

3 (b). $\int \frac{1}{y} dy = \int \frac{1}{x^2-1} dx = \int \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx$

$\ln|y| = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$

$y = A \sqrt{\frac{x-1}{x+1}}$

4. (a) Exact

$$\begin{cases} F_x = 2xy - 9x^2 \\ F_y = 2y + x^2 + 1 \end{cases} \Longrightarrow \quad F = x^2y - 3x^3 + g(y)$$

$$F_y = x^2 + g'(y) = 2y + x^2 + 1$$

so $g'(y) = 2y + 1$ and $g(y) = y^2 + y$

Solution: $x^2y - 3x^3 + y^2 + y = C$

~~(b)~~ $\cancel{(2xy^2+4)_x = 2y^2 \neq (-2(3-x^2y))_y = 2x^2}$

4 (b) Exact

$$\begin{cases} F_x = 2xy^2 + 4 \\ F_y = 2x^2y - 6 \end{cases} \qquad F = x^2y^2 + 4x + g(y)$$

$$F_y = 2x^2y + g'(y) = 2x^2y - 6$$

$g'(y) = -6 \quad g(y) = -6y$

Solution: $x^2y^2 + 4x - 6y = C$

4 (c). Exact.

$$F_x = \frac{2xy}{x^2+1} - 2x \qquad F = \int \frac{2xy}{x^2+1} - 2x \, dx$$

$$\int$$

$$F_y = \ln(x^2+1) - 2 \qquad = \ln(x^2+1)\cdot y - x^2 + g(y)$$

$$g'(y) = \ln(x^2+1) + g'(y)$$

$$g'(y) = -2 \qquad g(y) = -2y$$

Solution $\quad \ln(x^2+1)\cdot y - x^2 - 2y = C.$

Midterm 2 Practice Problems

Problem 1. Verify that the functions $y_1 = x$, $y_2 = x^3$ are solutions of the differential equation

$$x^2y'' - 3xy' + 3y = 0.$$

Solve the initial value problem

$$y(-1) = 1, y'(-1) = -2.$$

Problem 2. Verify that the functions $y_1 = x^2$, $y_2 = x^{-1}$ are solutions of the differential equation

$$x^2y'' - 2y = 0.$$

Solve the initial value problem

$$y(1) = 5, y'(1) = -3.$$

Problem 3. Show that the functions

$$2 + e^2 - 3\sin x,\ 1 + 2e^2 - 3\sin x,\ e^2 - \sin x$$

are linearly dependent on the real line.

Problem 4. Using Wronskian to show that the functions $y_1 = x^2$, $y_2 = \sin x, y_3 = \cos x$ are linearly independent on the real line.

Problem 5. Solve the following initial value problems.

1. $y^{(3)} + 2y'' + y' = 0$, $y(0) = 2$, $y'(0) = -1$, $y''(0) = 0$.
2. $y^{(3)} - 3y'' + 3y' - y = 0$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 2$.

Problem 6. Find the general solution of the following equations.

1. $y^{(3)} - 8y = 0$.
2. $y^{(5)} - y' = 0$.
3. $y^{(3)} - 5y'' + 8y' - 4y = 0$.
4. $y^{(4)} + 2y'' + y = 0$.

Problem 7. Find the general solution of the following nonhomogeneous equations.

1. $y'' - y' - 2y = 3x + 4.$
2. $y'' - 4y = 2e^{2x}.$
3. $y'' + y = \sin x + x \cos x.$

Problem 8. Solve the initial value problem

$$y'' - 7y' + 6y = \sin(3x),\ y(0) = 3,\ y'(0) = 2.$$

Problem 1. $y = c_1 x + c_2 x^3$ $\qquad$ $y' = c_1 + 3c_2 x^2$

$$\begin{cases} c_1(-1) + c_2(-1)^3 = 1 \\ c_1 + 3c_2 = -2 \end{cases} \iff \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = -\frac{1}{2} \end{cases}$$

So. $y = -\frac{1}{2}(x + x^3)$

Problem 2. $y = c_1 x^2 + c_2 x^{-1}$ $\qquad$ $y' = 2c_1 x - c_2 x^{-2}$

$$\begin{cases} c_1 + c_2 = 5 \\ 2c_1 - c_2 = -3 \end{cases} \iff \begin{cases} c_1 = \frac{2}{3} \\ c_2 = \frac{13}{3} \end{cases}$$

$$y = \frac{2}{3}x^2 + \frac{13}{3}x^{-1}$$

Problem 3. $(2 + e^2 - 3\sin x) + (-2)(1 + 2e^2 - 3\sin x) + 3(e^2 - \sin x) = 0$

Problem 4. $W = \begin{vmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x \end{vmatrix} = x^2 \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} - \sin x \begin{vmatrix} 2x & -\sin x \\ 2 & -\cos x \end{vmatrix}$

$$+ \cos x \begin{vmatrix} 2x & \cos x \\ 2 & -\sin x \end{vmatrix}$$

$$= -x^2 - \sin x(-2x\cos x + 2\sin x) + \cos x(-2x\sin x - 2\cos$$

$$= -x^2 - 2 \neq 0$$

Problem 5. 1. $r^3+2r^2+r=0$

$\Leftrightarrow r(r^2+2r+1)=0$

$\Leftrightarrow r(r+1)^2=0$

$y_1 = 1$

$y_2 = e^{-x}$

$y_3 = xe^{-x}$

$y = c_1 + c_2 e^{-x} + c_3 x e^{-x}$

$y' = -c_2 e^{-x} + c_3 e^{-x} - c_3 x e^{-x}$

$y'' = c_2 e^{-x} - c_3 e^{-x} - c_3 e^{-x} + c_3 x e^{-x}$

$$\begin{cases} c_1 + c_2 = 2 \\ -c_2 + c_3 = -1 \\ c_2 - 2c_3 = 0. \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 2 \\ c_3 = 1 \end{cases}$$

so $y = 2e^{-x} + xe^{-x}$

2. $r^3 - 3r^2 + 3r - 1 = 0.$

$\Leftrightarrow (r-1)^3 = 0$

$$\begin{cases} y_1 = e^x \\ y_2 = xe^x \\ y_3 = x^2 e^x \end{cases}$$

$y = (c_1 + c_2 x + c_3 x^2) e^x$

$y' = (c_2 + 2c_3 x) e^x + (c_1 + c_2 x + c_3 x^2) e^x$

$= (c_1 + c_2 + (2c_3 + c_2)x + c_3 x^2) e^x$

$y'' = (2c_3 + c_2 + 2c_3 x) e^x + (c_1 + c_2 + (2c_3 + c_2)x + c_3 x^2) e^x$

$= (c_1 + 2c_2 + 2c_3 + (4c_3 + c_2)x + c_3 x^2) e^x$

$$\begin{cases} c_1 = 1 \\ c_1 + c_2 = 1 \\ c_1 + 2c_2 + 2c_3 = 2 \end{cases} \iff \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = \frac{1}{2} \end{cases} \qquad y = (1 + \tfrac{x^2}{2}) e^x$$

Problem 6. 1. $r^3 - 8 = 0$

$\iff (r-2)(r^2+2r+4) = 0$

$\iff (r-2)((r+1)^2+3) = 0$

$r_1 = 2$

$r_2 = -1 + \sqrt{3}i$

$r_3 = -1 - \sqrt{3}i$

$y_1 = e^{2x} \qquad y_2 = e^{-x}\cos\sqrt{3}x \qquad y_3 = e^{-x}\sin\sqrt{3}x$

$y = c_1 e^{2x} + c_2 e^{-x}\cos\sqrt{3}x + c_3 e^{-x}\sin\sqrt{3}x$.

2. $r^5 - r = 0 \iff r(r^4-1) = 0 \iff r(r-1)(r+1)(r^2+1) = 0$

$r_1 = 0 \quad r_2 = 1 \quad r_3 = -1 \quad r_4 = i \quad r_5 = -i$

$y_1 = 1 \quad y_2 = e^x \quad y_3 = e^{-x} \quad y_4 = \cos x \quad y_5 = \sin x$

$y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$.

3. $r^3-5r^2+8r-4=0 \iff (r-1)(r-2)^2=0$

$y_1=e^x \qquad y_2=e^{2x} \qquad y_3=xe^{2x}$

$y=c_1e^x+c_2e^{2x}+c_3xe^{2x}$

4. $r^4+2r^2+1=0 \iff (r^2+1)^2=0$

$y_1=\cos x \qquad y_2=\sin x \qquad y_3=x\cos x \qquad y_4=x\sin x$

$y=c_1\cos x+c_2\sin x+c_3x\cos x+c_4x\sin x$

Problem 7. 1. $r^2-r-2=0 \iff (r-2)(r+1)=0$ $\qquad \begin{cases} y_1=e^{2x} \\ y_2=e^{-x}\end{cases}$

$y_c=c_1e^{2x}+c_2e^{-x}$

$W(y_1,y_2)=\begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x}, & -e^{-x}\end{vmatrix} = -3e^x$

$$y_p=-e^{2x}\int\frac{e^{-x}(3x+4)}{-3e^x}\,dx + e^{-x}\int\frac{e^{2x}(3x+4)}{-3e^x}\,dx$$

$$= -\frac{3x}{2}-\frac{5}{4}$$

$y=y_p+y_c$

(5)

2. $r^2-4=0 \iff (r-2)(r+2)=0$

$y_1 = e^{2x} \qquad y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$$y_p = -e^{2x}\int \frac{e^{-2x}(2e^{2x})}{-4}dx + e^{-2x}\int \frac{e^{2x}(2e^{2x})}{-4}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{e^{2x}}{8}$$

$y_c = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x \qquad y = y_p + y_c.$

3. $r^2+1=0 \qquad y_1 = \cos x \qquad y_2 = \sin x \qquad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$$y_p = -\cos x \int \frac{\sin x(\sin x + x\cos x)}{1}dx + \sin x \int \frac{\cos x(\sin x + x\cos x)}{1}dx$$

$$= -\cos x\left(\frac{x}{2} - \frac{\sin 2x}{8} - \frac{x\cos 2x}{4}\right) + \sin x\left(x\frac{\sin 2x}{4} - \frac{\cos 2x}{8} + \frac{x^2}{4}\right)$$

$y_c = c_1\cos x + c_2\sin x \qquad y = y_p + y_c$

Problem 8.

(6)

$$r^2 - 7r + 6 = 0 \qquad y_1 = e^x \qquad y_2 = e^{6x}$$

$$\Leftrightarrow (r-1)(r-6) = 0$$

$$W = \begin{vmatrix} e^x & e^{6x} \\ e^x & 6e^{6x} \end{vmatrix} = 5e^{7x}$$

$$y_p = -e^x \int \frac{e^{6x}\sin 3x}{5e^{7x}}\,dx + e^{6x}\int \frac{e^x \sin(3x)}{5e^{7x}}\,dx$$

$$= -\frac{e^x}{5}\left(-\frac{3}{4}\right)\left(\frac{e^{-x}}{3}\sin 3x + e^{-x}\cos 3x\right)$$

$$+ \frac{e^{6x}}{5}\left(-\frac{2}{5}\right)\left(\frac{e^{-6x}}{3}\sin 3x + \frac{e^{-6x}}{6}\cos 3x\right) = \frac{17}{500}\sin 3x + \frac{41}{300}\cos 3x$$

$$y_c = c_1 e^x + c_2 e^{6x}$$

$$y = c_1 e^x + c_2 e^{6x} + \frac{17}{500}\sin 3x + \frac{41}{300}\cos 3x$$

$$y' = c_1 e^x + 6c_2 e^{6x} + \frac{51}{500}\cos 3x - \frac{41}{100}\sin 3x$$

$$\begin{cases} c_1 + c_2 + \frac{41}{300} = 3 \\ c_1 + 6c_2 + \frac{51}{500} = 2 \end{cases} \qquad \begin{cases} c_1 = \frac{362}{1875} + 3 - \frac{41}{300} \\ c_2 = -\frac{362}{1875} \end{cases}$$

Practice Final Exam

Problem 1. Find the general solution of the following differential equations.

1. $yy' = x(1 + y^2)$
2. $(1 + x)y' + y = \cos x$.
3. $x(x + y)y' + y(3x + y) = 0$.
4. $y' = \sqrt{x + y + 2}$.
5. $xy' + 6y = 3xy^{\frac{4}{3}}$.
6. $(\cos x + \ln y)\, dx + (\frac{x}{y} + e^y)\, dy = 0$.

Problem 2. A 400-gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at the rate of 5 gal/s, and the well-mixed brine in the tank flows out at the rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Problem 3. Find the general solution of the following higher-order differential equations.

1. $y'' - 3y' + 2y = 0$.
2. $4y'' + 4y' + y = 0$.
3. $y'' + 6y' + 10y = 0$.
4. $y^{(3)} + 2y'' - y' - 2y = 0$.
5. $y^{(3)} + 3y'' + 3y + y = 0$.

Problem 4. Solve the following initial value problems.

1. $y^{(3)} = y$; $y(0) = 1$, $y'(0) = y''(0) = 0$.
2. $y'' + 2y' + 2y = e^{-x}$; $y(0) = 1$, $y'(0) = 2$.

Problem 5. Let A and B be two 2×2 matrice. Prove that $\det(AB) = \det(A) \cdot \det(B)$.

Problem 6. Let A and B be two $n \times n$ matrice. Prove that $(AB)^T = B^T A^T$.

Problem 7. Solve the following systems of linear equations.

1. $$\begin{cases} 2x + 3y + 2z = 3 \\ 4x - 5y + 5z = -7 \\ -3x + 7y - 2z = 5 \end{cases}.$$

2. $$\begin{cases} 2x + 3y + 2z = 1 \\ x + 0y + 3z = -7 \\ 2x + 2y + 3z = 3 \end{cases}.$$

Problem 8. Consider the following system of linear equations

$$\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}.$$

For what value(s) of k does this have (i) a unique solution? (ii) no solution? (iii) infinitely many solutions?

Problem 9. For the matrix A given below, compute $\exp(A)$.

1. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

2. $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ for some constants a, b, c.

3. $A = \begin{pmatrix} 3 & -10 \\ 1 & -4 \end{pmatrix}$.

4. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Problem 10. Solve the following homogeneous systems.

1. $$\begin{cases} x' = 3x + z \\ y' = 9x - y + 2z \\ z' = -9x + 4y - z \end{cases}.$$

2. $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \mathbf{x}$.

Problem 11. Solve the following initial value problem.

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x} + e^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 1

ϕ

1. $\frac{y}{1+y^2} y' = x \iff \int \frac{y}{1+y^2}\, dy = \int x\, dx$

$\iff \frac{1}{2} \ln(1+y^2) = \frac{x^2}{2} + c$

$\iff 1+y^2 = e^{2\left(\frac{x^2}{2}+c\right)} = e^{x^2+2c}.$

2. $y' + \frac{1}{1+x} y = \frac{\cos x}{1+x}.$

$m = e^{\int \frac{1}{1+x} dx} = e^{\ln(1+x)} = 1+x$

$y = \frac{1}{m}\left(\int m \frac{\cos x}{1+x} dx \right) = \frac{1}{1+x} \int \cos x\, dx = \frac{1}{1+x}(\sin x + c)$

3. $y' = -\frac{y(3x+y)}{x(x+y)}$ set $v = \frac{y}{x}.$

$y' = v + xv' = -v \cdot \frac{3+v}{1+v} \iff xv' = -\frac{2v^2+4v}{1+v}.$

$\int \frac{1+v}{2v^2+4v} dv = \int \frac{1}{x} dx = \ln|x| + c.$

On the other hand, $\int \frac{1+v}{2v^2+4v} dv = \frac{1}{4} \int \frac{1}{v} + \frac{1}{v+2}\, dv = \frac{1}{4}\left(\ln|v| + \ln|v+2|\right)$

Therefore. $\ln|v^2+2v| = \ln|x|^4 + 4c \iff v^2+2v = c_1 x^4$

$(v+1)^2 = 1 + C_1 x^4 \qquad y = vx = x\left(-1 \pm \sqrt{1+C_1x^4}\right)$

4. $v = x + y + z$

$v' = y' + 1 = \sqrt{v} + 1 \iff \int \frac{1}{1+\sqrt{v}}\, dv = \int 1\, dx = x + C$

$\int \frac{1}{1+\sqrt{v}}\, dv \overset{z=\sqrt{v}}{=\!=} \int \frac{2z\, dz}{1+z} = \int 2 - \frac{2}{1+z}\, dz = 2z - 2\ln|1+z|$

$= 2\sqrt{v} - 2\ln(1+\sqrt{v})$

Therefore. $2\left(\sqrt{x+y+z} - \ln(1+\sqrt{x+y+z})\right) = x + C.$

5. $y' + \frac{6}{x} y = 3y^{\frac{4}{3}} \qquad v = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$

$v' = -\frac{1}{3} y^{-\frac{4}{3}} y' = -\frac{1}{3} y^{-\frac{4}{3}}\left(3y^{\frac{4}{3}} - \frac{6}{x} y\right) = -1 + \frac{2}{x} y^{-\frac{1}{3}} = \frac{2}{x} v - 1$

It is easy to see $v = x^2 \int -x^{-2}\, dx = x^2(x^{-1} + c) = x + cx^2$

So $y = v^{-3} = (x + cx^2)^{-3}$

6. $M = \cos x + \ln y$ $\qquad N = \frac{x}{y} + e^y$

$M_y = N_x = \frac{1}{y}$ Therefore the ODE is exact.

Then we solve $\begin{cases} F_x = \cos x + \ln y & \text{--- ①} \\ F_y = \frac{x}{y} + e^y & \text{--- ②} \end{cases}$

From ①. $F = \sin x + x \ln y + g(y)$

From ② $F_y = \frac{x}{y} + g'(y) = \frac{x}{y} + e^y$ so. $g = e^y$.

Therefope: the solution is $\sin x + x \cdot \ln y + e^y = C$.

Problem 2. we ~~denote~~ set $m(t)$ to be the amount of salt at t

and $V(t) = 100 + 2t$ to be the volumn of the brine at t.

$$\begin{cases} m(0) = 50 \\ m'(t) = 1 \times 5 - \frac{m(t)}{V(t)} \times 3 = 5 - \frac{3m(t)}{100+2t} \end{cases}$$

It is easy to see $m(t) = (100+2t)^{-\frac{3}{2}} \int 5(100+2t)^{\frac{3}{2}} dt$

$$= (100+2t) + C(100+2t)^{-\frac{3}{2}}$$

Since $m(0) = 50$, $C = -50000$ and

$$m(t) = 100 + 2t - 50000\,(100+2t)^{-\frac{3}{2}}$$

If t_0 is the time when the tank is full, then.

$100 + 2t_0 = 400$ so $t_0 = 150$.

therefore $m(t_0) = 100 + 150\times 2 - 50000\,(100 + 150\times 2)^{-\frac{3}{2}}$

$$= 400 - \frac{25}{4} \ (lb).$$

Problem 3.

1. $r^2 - 3r + 2 = 0 \iff (r-1)(r-2) = 0$ ~~(2r+1)(r~~

$y = c_1 e^{x} + c_2 e^{2x}$

2. $4r^2 + 4r + 1 = 0 \iff (2r+1)^2 = 0$

$y = (c_1 + c_2 x)\, e^{-\frac{x}{2}}$

3. $r^2 + 6r + 10 = 0 \iff (r+3)^2 = -1$ $\quad r = -3 \pm i$

$y = c_1 e^{-3x} \cos x + c_2 e^{-3x} \sin x$

5. $r^3+3r^2+3r+1=0 \iff (r+1)^3=0$

$y=(c_1+c_2x+c_3x^2)e^{-x}$

Problem 4. 1. $r^3=1 \iff (r-1)(r^2+r+1)=0$

$\iff (r-1)\left((r+\frac{1}{2})^2+\frac{3}{4}\right)=0$

$r_1=1 \qquad r_2=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
r_3 //

$\cancel{y=}$

$y=c_1e^x+c_2e^{-\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x+c_3e^{-\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x$

Since $y(0)=1, \quad y'(0)=y''(0)=0$

$$\begin{cases} c_1+c_2=1 \\ c_1-\frac{c_2}{2}+\frac{\sqrt{3}}{2}c_3=0 \\ c_1-\frac{1}{2}\left(\frac{\sqrt{3}}{2}c_3-\frac{c_2}{2}\right)+\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}c_2-\frac{c_3}{2}\right)=0 \end{cases} \Longrightarrow \begin{cases} c_1=\frac{1}{3} \\ c_2=\frac{2}{3} \\ c_3=0 \end{cases}$$

2. $r^2+2r+2=0 \iff (r+1)^2=-1 \qquad r=-1\pm i$

$y_c=c_1e^{-x}\cos x+c_2e^{-x}\sin x$

$=c_1y_1+c_2y_2$

$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=e^{-2x}$

⑥

$$y_p = -e^{-x}\cos x \int \frac{e^{-x}\sin x \cancel{\sin x}\, e^{-x}}{e^{-2x}}\,dx + e^{-x}\sin x \int \frac{e^{-x}\cos x \cdot e^{-x}}{\cancel{e^{-2x}}}\,dx$$

$$= -e^{-x}\cos x(-\cos x) + e^{-x}\sin^2 x = e^{-x}$$

$$y = e^{-x} + c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$$

$$\begin{cases} y(0)=1 \\ y'(0)=2 \end{cases} \Longrightarrow \begin{cases} 1+c_1=1 \\ -1-c_1+c_2=2 \end{cases} \Longrightarrow \begin{cases} c_1=0 \\ c_2=3. \end{cases}$$

problem 5. Set $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$

$$\det(AB) = \det\begin{pmatrix} aa_1+bc_1 & ab_1+bd_1 \\ ca_1+dc_1 & cb_1+dd_1 \end{pmatrix} = (aa_1+bc_1)(cb_1+dd_1) - (ca_1+dc_1)(ab_1+bd_1)$$

$$= aca_1b_1 + ada_1d_1 + bcb_1c_1 + bdc_1d_1$$

$$-(aca_1b_1 + bca_1d_1 + adc_1b_1 + bdc_1d_1)$$

$$= (ad-bc)(a_1d_1-b_1c_1) = \det A \cdot \det B$$

Problem 6. proof: Set $A=(a_{ij})_{1\le i,j\le n}$ and $B=(b_{ij})_{1\le i,j\le n}$.

$$AB=\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{1\le i,j\le n}.$$

$$(AB)^T_{ij} = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = B^TA^T$$

Problem 7. 1.

$$\begin{pmatrix} 2 & 3 & 2 & 3 \\ 4 & -5 & 5 & -7 \\ -3 & 7 & -2 & 5 \end{pmatrix} \xrightarrow[③+\frac{3}{2}①]{②-①} \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & -11 & 1 & -13 \\ 0 & \frac{23}{2} & 1 & \frac{13}{2} \end{pmatrix}$$

$$\xrightarrow{③\times 2} \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & -11 & 1 & -13 \\ 0 & 23 & 1 & 13 \end{pmatrix} \xrightarrow{③+\frac{23}{11}②} \begin{pmatrix} 2 & 3 & 2 & 3 \\ 0 & -11 & 1 & -13 \\ 0 & 0 & \frac{34}{11} & -\frac{156}{11} \end{pmatrix}$$

$$\begin{cases} 2x+3y+2z=3 \\ -11y+z=-13 \\ \frac{34}{11}z=-\frac{156}{11} \end{cases} \Longrightarrow \begin{cases} x=\frac{84}{17} \\ y=\frac{13}{17} \\ z=-\frac{78}{17} \end{cases}$$

2. $\begin{pmatrix} 2 & 3 & 2 & 1 \\ 1 & 0 & 3 & -7 \\ 2 & 2 & 3 & 3 \end{pmatrix} \xrightarrow{2 \Leftrightarrow 1} \begin{pmatrix} 1 & 0 & 3 & -7 \\ 2 & 3 & 2 & 1 \\ 2 & 2 & 3 & 3 \end{pmatrix}$

$\xrightarrow[3-1]{2-1} \begin{pmatrix} 1 & 0 & 3 & -7 \\ 0 & 3 & -4 & 15 \\ 0 & 2 & -3 & 17 \end{pmatrix} \xrightarrow{3 - \frac{2}{3} 2} \begin{pmatrix} 1 & 0 & 3 & -7 \\ 0 & 3 & -4 & 15 \\ 0 & 0 & -\frac{1}{3} & 7. \end{pmatrix}$

$\begin{cases} x + 3z = -7 \\ 3y - 4z = 15 \\ -\frac{1}{3} z = 7 \end{cases} \Longrightarrow \begin{cases} x = 56. \\ y = -23 \\ z = -21 \end{cases}$

Problem 8. $\begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \xrightarrow{2 \Leftrightarrow 1} \begin{pmatrix} 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \xrightarrow[3-1]{2-k1}$

$\longrightarrow \begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1-k^2 & 1-k & 1-k \\ 0 & 1-k & k-1 & 0 \end{pmatrix} \circledast$

Case 1. $k=1$ $\circledast = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow x+y+z=1$

(infinitely many solutions)

Case 2 $k \neq 1$ $\circledast \xrightarrow[\frac{1}{1-k} \times 3]{\frac{1}{1-k} \times 2} \begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1+k & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{2 \Leftrightarrow 3} \begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1+k & 1 & 1 \end{pmatrix} \xrightarrow{3-(1+k)2}$

$$\begin{pmatrix} 1 & k & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k+2 & 1 \end{pmatrix} \longrightarrow \begin{cases} x + ky + z = 1 \\ y - z = 0 \\ (k+2)z = 1 \end{cases}$$

9

If $k = -2$, there is no solution

Otherwise. $z = \frac{1}{k+2} = y$ $\quad x = \frac{1}{k+2}$ (unique solution)

Problem 9. 1. $e^A = e^{I + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} = I \cdot e^{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

2. $e^A = \begin{pmatrix} 1 & a & b + \frac{ac}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

3. Consider $x' = \begin{pmatrix} 3 & -10 \\ 1 & -4 \end{pmatrix} x$. $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -10 \\ 1 & -4-\lambda \end{vmatrix} = 0$

$$\Longleftrightarrow (\lambda+2)(\lambda-1) = 0$$

$\lambda_1 = -2 \qquad \begin{pmatrix} 5 & -10 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 1 \qquad \begin{pmatrix} 2 & -10 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

10

$$\Phi(t) = \begin{pmatrix} 2e^{-2t} & 5e^{t} \\ e^{-2t} & e^{t} \end{pmatrix} \qquad \Phi(0) = \begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix} \qquad \Phi^{-1}(0) = \frac{1}{-3}\begin{pmatrix} 1 & -1 \\ -5 & 2 \end{pmatrix}$$

$$e^{A} = \Phi(\cancel{t}) \cdot \Phi^{-1}(0) = -\frac{1}{3}\begin{pmatrix} 2e^{-2} & 5e \\ e^{-2} & e \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 2e^{-2} - 25e & -2e^{-2} + 10e \\ e^{-2} - 5e & -e^{-2} + 2e \end{pmatrix}$$

4. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^2 = I \qquad A^3 = A \; - - - - -$

$$A^{2n} = I \quad \text{and} \quad A^{2n+1} = A.$$

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{I}{(2k)!} + \sum_{k=0}^{\infty} \frac{A}{(2k+1)!}$$

since $e^x \; \cancel{\cos 1} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $e^{-x} = \sum_{k=0}^{\infty} \frac{x^k(-1)^k}{k!}$

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{e^{\cancel{1}} + e^{-\cancel{1}}}{2} = \cosh \cancel{1} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{e^{\cancel{1}} - e^{-\cancel{1}}}{2} = \sinh \cancel{1}$$

So $e^{A} = \cancel{\cosh 1} \; (\cosh 1) I + (\sinh 1) A$

$$= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

(11)

Problem 10

1. $A=\begin{pmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{pmatrix}$ $\quad \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 9 & -1-\lambda & 2 \\ -9 & 4 & -1-\lambda \end{vmatrix} = (3-\lambda)\left((\lambda+1)^2+1\right)$

$\lambda_1 = 3.$ $\quad \begin{pmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} z=0 \\ 9x-4y+2z=0 \\ -9x+4y-4z=0 \end{cases} \Rightarrow \begin{cases} 9x-4y=0 \\ z=0 \end{cases}$

$V=\begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$ $\qquad X_1 = e^{3t}\begin{pmatrix} 4 \\ 9 \\ 0 \end{pmatrix}$

$\lambda_2 = -1+i$ $\qquad \begin{pmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} (4-i)x+z=0 \\ 9x-iy+2z=0 \\ -9x+4y-iz=0 \end{cases}$

$V=\begin{pmatrix} 1 \\ 2-i \\ i-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\tilde{X}_2 = e^{(-1+i)t}\left(\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right) = e^{-t}(\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right)$

$= e^{-t}\left(\cos t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right) + i\left(e^{-t}\sin t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + e^{-t}\cos t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right)$

$$X_2(t) = e^{-t}\begin{pmatrix} \cos t - \sin t \\ 2\cos t + \sin t \\ -4\cos t - \sin t \end{pmatrix} \qquad X_3(t) = e^{-t}\begin{pmatrix} \sin t \\ 2\sin t - \cos t \\ -4\sin t + \cos t \end{pmatrix}$$

$$X = C_1X_1(t) + C_2X_2(t) + C_3X_3(t)$$

2. $tA = 2tI + B$ $\qquad B = \begin{pmatrix} 0 & t & 0 & t \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 0 & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B^n = 0 \quad (n \geq 4)$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{tA} = e^{2t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad X(t) = e^{tA}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}.$$

Problem 11

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad At = tI + B$$

where $B = \begin{pmatrix} 0 & 2t & 3t & 4t \\ 0 & 0 & 6t & 3t \\ 0 & 0 & 0 & 2t \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad B^2 = \begin{pmatrix} 0 & 0 & 12t^2 & 12t^2 \\ 0 & 0 & 0 & 12t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad B^3 = \begin{pmatrix} 0 & 0 & 0 & 24t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$B^n = 0$ for $n \geq 4$

$$e^B = \begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad e^{At} = e^t \begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e^{-B} = \begin{pmatrix} 1 & -2t & -3t+6t^2 & -4t+6t^2-24t^3 \\ 0 & 1 & -6t & -3t+6t^2 \\ 0 & 0 & 1 & -2t \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad e^{-At} = e^{-t} \begin{pmatrix} 1 & -2t & -3t+6t^2 & -4t+6t^2-24t^3 \\ 0 & 1 & -6t & -3t+6t^2 \\ 0 & 0 & 1 & -2t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_p = e^{tA} \int e^{-tA} e^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} dt = e^{tA} \int \begin{pmatrix} -4t+6t^2-24t^3 \\ -3t+6t^2 \\ -2t \\ 1 \end{pmatrix} dt$$

$$= e^{tA} \begin{pmatrix} -2t^2 + 2t^3 - 6t^4 \\ -\frac{3t^2}{2} + 2t^3 \\ -t^2 \\ t \end{pmatrix} =$$

$$= e^{t}\begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2t^2+2t^3-6t^4 \\ -\frac{3t^2}{2}+2t^3 \\ -t^2 \\ t \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} -2t^2+2t^3-6t^4-3t^3+4t^4-3t^3-6t^4+4t^2+6t^3+4t^4 \\ -\frac{3t^2}{2}+2t^3-6t^3+3t^2+6t^3 \\ -t^2+2t^2 \\ t \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} -4t^4+2t^3+2t^2 \\ 2t^3-\frac{3}{2}t^2 \\ t^2 \\ t \end{pmatrix}$$

$$X = X_p + X_c = X_p + e^{-t}\begin{pmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Since $X(0)=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$