Math 303 - Calculus IV: Ordinary Differential Equations (Fall 2017)

Instructor: Ben McMillan
Email: bmcmillan@math.stonybrook.edu
Locataion: MWF 12:00pm--12:53pm in Harriman Hall 137

Announcements

- The final is in Harriman Hall 137, where lecture is.

Course Information:

The course syllabus is here. Some critical points:

- The midterm will be in class on October 11.
- My office hours this term are Mondays 1-2pm and Fridays 3-4pm in the math tower, office 2-116. I will also be available in the MLC (S-235) on Wednesdays 1:30-2:30pm.

Schedule and Homework:

The following is a tentative schedule for the course. As homework is assigned it will be posted here. You are encouraged to work with others, but please make sure to write up solutions in your own words (this will help you on the exams and quizzes!).

Unless stated otherwise, the homework is from the corresponding section of the book.

All of the problems assigned in week n are due to your TA (at the start of recitation) in week n+1.

Week	Date	Topic(s) Covered	Reading	Homework
1	8/28	What's a differential equation, and why?	Chapter 1.1	2, 3, 13, 27, 29, 34
	8/30	Integrals as solutions, initial value problems	1.2	1, 2, 4, 12, 18, 31
	9/1	Separable equations	1.4	3, 8, 11, 22
2	9/4	Labor Day	None	None
	9/6	Slope fields	1.3	[1, 8 do but don't turn in], 11, 13, 14, 27
	9/8	Linear first order equations	1.5	1, 3, 16, 20, 29, 33
3 Quiz Week	9/11	Change of variables	1.6	7, 12, 29, 57, 58
	9/13	More substitutions	1.6	20, 24, 43, 45, 48, 53
	9/15	Population models	2.1	9, 10, 12, 31
4	9/18	Logistic Model of populations	2.1	1, 21
	9/20	Equilibrium Solutions	2.2	1, 3, 14, 19
	9/22	Second Order Linear Equations	3.1	1, 9, 13, 15
5 Quiz Week	9/25	Superposition principle, existence and uniqueness of solutions	3.1	17, 19, 29
	9/27	Solving constant coefficient 2nd order homogeneous equations. Linear independence	3.1	33, 37, 51, 52
	9/29	The Wronskian	3.1	21, 24, 26
6	10/2	Inhomogeneous equations Higher order linear equations	3.1 3.2	3.1 - 27 3.2 - 13, 17, 20, 21, 23, 25
	10/4	Variation of parameters	3.5	3, 4, 13, 20, 30, 32
	10/6	Homogeneous constant coefficient equations	3.3	3, 5, 12, 21, 27
7	10/9	Return to higher order equations	3.2	None
	10/11	Midterm	.	Review
	10/13	Resonance phenomena	3.6	None
8	10/16	Systems of differential equations	4.1	1, 3, 5, 17 , 18, 24, 30
	10/18	The method of elimination	4.2	1, 4, 7, 13, 39
	10/20	Linear Algebra	5.1	Read 5.1 and do problems 1, 3, 4, 6
9 Quiz Week	10/23	More Linear Algebra	5.1	10, 11, 12, 13
	10/25	Linear algebra for systems of equations	5.1	14, 19, 20
	10/27	Continued	5.1	23, 24, 32, 33, 42
10	10/30	Eigenvectors and solutions of homogeneous systems	5.2	1, 2, 5, 9
	11/1	Eigenvectors and solutions of homogeneous systems	5.2	17, use wolfram alpha to do 42, 45
	11/3	Solution Curves to linear systems	5.3	1, 2, 9, 11

11 Quiz Week	11/6	Generalized Eigenvectors	5.5	1, 3, 7, 13
	11/8	Generalized Eigenvectors	5.5	20, 25, 33
	11/10	Matrix Exponentials	5.6	1, 3, 21, 23, 25, 27
12	11/13	Computing Matrix Exponentials	5.6	7, 9, 12, 17, 29, 32
	11/15	Nonlinear Systems of Equations	6.1	1, 4, 6, 13, 14
	11/17	Nonlinear Systems of Equations	6.1	17, 18, 28, 29
13	11/20	Linear Algebra Review	.	Here
	11/22	Thanksgiving Break	None	None
	11/24	Thanksgiving Break	None	None
14 Quiz Week	11/27	Almost Linear Systems	6.2	1, 3, 7, 19, 21
	11/29	Jacobians, the space of 2x2 linear equations	6.2	11, 14, 33, 34
	12/1	Ecological Models	6.3	1, 2, 3, 11, 12, 13
15	12/4	Ecological Models	6.3	14, 15, 16, 17
	12/6	The non-linear pendulum	6.4	9, 10, 11, 14, 15
	12/8	Review	.	Review Sheet Solutions

Math 303: Calculus IV with applications

Fall 2017 Stony Brook

Instructor:	Ben McMillan	**Time:**	MWF 12:00pm–12:53pm
Email:	bmcmillan@math.stonybrook.edu	**Place:**	Harriman Hall 137

Course Page: The primary webpage for this course is

www.math.stonybrook.edu/~bmcmillan/math303/

where you will find up to date information, homework, and announcements. Please bookmark it and check back regularly.

You will also find announcements and grades on the course blackboard page.

Office Hours: Office hours are an invaluable resource, one that you really should use!

My office hours this term are Mondays 1–2pm and Fridays 3–4pm in the math tower, office 2-116. I will also be available in the MLC (S-235) on Wednesdays 1:30–2:30pm.

You can find your TA's office hours at math.stonybrook.edu/office-hours

Textbook: The lecture will roughly follow Edwards and Penney's *Differential Equations and Boundary Value Problems: Computing and Modeling*, and I will generally assign homework problems from the book.

Exams: The midterm will be held in class on October 11. Please let me know if you will need DSS accomodations at least 2 weeks before this.

The final is scheduled for Thursday, Dec. 14, 5:30pm–8:00pm. Please ensure NOW that you won't have any scheduling conflicts with the final!

Homework: Each week I will post homework questions relevant to the week's lectures on the course webpage. You will submit it to your TA in recitation the following week.

Quizzes: There will be a 15 minute quiz at the end of recitation every other week.

Grading Policy: Homework: 15%, Quizzes 15%, Midterm: 30%, Final: 40%.

Americans with Disabilities Act: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631)632–6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. http://studentaffairs.stonybrook.edu/dss/index.html.

Academic Integrity: Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty is required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty please refer to the academic judiciary website at http://www.stonybrook.edu/commcms/academic_integrity/

Critical Incident Management: Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of University Community Standards any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures. Further information about most academic matters can be found in the Undergraduate Bulletin, the Undergraduate Class Schedule, and the Faculty-Employee Handbook.

The topics covered so far naturally fall into two groups—General first order differential equations and linear differential equations.

First order equations

- Initial value problems. If you know the general solution to a given equation, and some initial values, can you determine the specific solution that satisfies these specific values.
- Separable equations. How do you solve them? Such as:
$$\frac{dy}{dx} = 3x^2(y^2+1), \qquad y(0) = 1$$
- Change of variables. Given a suggested change of variables, how do you change a differential equation into the new variable? How do you make good guesses on what change of variable to make? For example,
$$x^2y' = xy + y^2$$
- Slope fields. What are they? What do they tell you about the behavior of solutions?
- Equilibrium solutions, stability, instability. Given an autonomous first order equation, what are the equilibrium solutions? Which ones are stable, which are unstable? What does this tell you about the behavior of the non-equilibrium solutions? For example, what does such an analysis tell you about solutions to
$$\frac{dx}{dt} = x^2 - 5x + 4?$$

Linear Equations

- Can you recognize linear equations? What is a linear operator?
- The superposition principal for homogeneous equations. Given two solutions to a homogeneous linear equation, how does the superposition principal give you more solutions?
- The existence and uniqueness theorem for solutions to linear equations. What is the right amount a initial data needed to specify a unique solution? How many derivatives at a point can you specify?
- Independence of solutions. When are solutions 'different enough' to give us all solutions (answer: when they're independent.). How do you use the Wronskian to test for independence? For example, show that $e^x \sin x$ and $e^x \cos x$ are independent.
- Constant coefficient homogeneous equations. How does the characteristic polynomial tell you solutions to such an equation? For example,
$$y'' + 2y' - 15y = 0.$$
- Constant coefficient inhomogeneous equations. How does variation of parameters give you a particular solution in good cases? How do you combine this with the associated homogeneous equation to find the general solution? What is the general solution to
$$y'' + 2y' + 5y = e^x \sin x?$$

For the following matrices, find:

- All eigenvalues and state their multiplicity.
- As many independent eigenvectors as possible.
- A complete set of generalized eigenvectors.
- Describe all of the chains of eigenvectors.

$$A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & -1 & 2 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

Finally, use the (generalized) eigenvalues that you found for D to create the 'diagonalization' matrix G and compute $G^{-1}DG$.

A note on inverses: Given an invertible matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}, \end{pmatrix}$$

we can use the adjugate matrix $Adj(A)$ to define the inverse of A, where

$$Adj(A) = \begin{pmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{pmatrix}$$

and A_{ij} is the determinant of the 2×2 matrix we get from deleting the ith row and jth column in A. For example,

$$A_{21} = a_{12}a_{33} - a_{13}a_{32},$$
$$A_{31} = a_{12}a_{23} - a_{13}a_{22},$$

and

$$A_{22} = a_{11}a_{33} - a_{13}a_{31}.$$

Note that this is slightly different from what I wrote in lecture. **This is correct, while the lecture was wrong**. (Precisely: this is the *transpose* of what I wrote in lecture.)

With this definition,

$$A^{-1} = \frac{1}{\det(A)} Adj(A).$$

The final will be focused on material we learned after the midterm, but some questions will need skills from before. For example, we've seen that you can sometimes solve a system of equations by reducing it to a single separable equation, so if you've forgotten how to solve separable equations, you won't be able to solve such a system of equations.

- (5.1) You will need to know your linear algebra basics—adding matrices, multiplying matrices, inverting 2×2 matrices, matrix multiplication is not commutative.
- (5.1) Understanding eigenvalues and eigenvectors of a square matrix is critical for solving linear equations. What are the eigenvalues/eigenvectors for

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}? \qquad B = \begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{pmatrix}? \qquad C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}?$$

- (5.5) Sometimes a matrix does not have enough independent eigenvectors, so we need generalized eigenvectors. What are the eigenvalues and generalized eigenvectors for

$$D = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}? \qquad E = \begin{pmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}?$$

- (5.2) You should be able to use these eigenvectors to find solutions to the linear, homogeneous, constant coefficient systems of equations:

$$\frac{d}{dt}\vec{x} = A\vec{x} \qquad \frac{d}{dt}\vec{x} = B\vec{x} \qquad \frac{d}{dt}\vec{x} = C\vec{x} \qquad \frac{d}{dt}\vec{x} = D\vec{x} \qquad \frac{d}{dt}\vec{x} = E\vec{x}$$

- (5.1) Make sure you understand the superposition principle for linear homogeneous systems. You use this to find solutions satisfying an initial condition. What is the solution $\vec{f}(t)$ to

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

  for which

$$\vec{f}(0) = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}?$$

- (5.1) When are solutions to systems of equations linearly independent? What is the Wronskian of solutions

$$\vec{f}_1 = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \qquad \vec{f}_2(t) = \begin{pmatrix} 7e^{5t} \\ e^{5t} \end{pmatrix}?$$

  Are they independent?
- (5.3) It's possible to classify 2×2 linear, homogeneous, constant coefficient equations

$$\frac{d}{dt}\vec{x} = F\vec{x},$$

  where F is a 2×2 matrix, by the eigenvalues of F. You should either remember the list, the diagram I drew in lecture, or (preferrably) understand it well enough that you can draw a cartoon of the solutions given just the eigenvalues.
- (5.6) Solving a linear, homogeneous, constant coefficient equation

$$\frac{d}{dt}\vec{x} = F\vec{x}$$

ultimately rests on being able to exponentiate the matrix F. Typically, if you want to do this by hand, you will diagonalize the matrix, writing

$$F = GHG^{-1}$$

where G is a matrix of eigenvectors for F and H is a diagonal matrix. Then

$$e^{Ft} = e^{GHG^{-1}t} = Ge^{Ht}G^{-1}.$$

You should know how to exponentiate a diagonal matrix.

For example, what is

$$e^{At}? \qquad e^{Ct}? \qquad e^{Dt}?$$

- (5.6) The matrix-valued function $\Phi(t) = e^{Ft}$ is the fundamental solution to

$$\frac{d}{dt}\vec{x} = F\vec{x}.$$

For any vector $\vec{c}$, the function

$$\vec{f}(t) = \Phi(t) \cdot \vec{c}$$

gives the unique solution with initial values $\vec{c}$.

- (6.1/6.2/6.3) Non-linear systems of equations can be very complicated, but we can get a good understanding of them by studying their critical points. In particular, if the linearization (AKA the Jacobian) of a system at a critical point has non-zero, non-pure-imaginary eigenvalues, then very close to the critical point, the solutions to the non-linear equation look very similar to the solutions of the linearization.

Then, to understand the behavior of our non-linear system away from the critical points, we have something of a puzzle to piece together the solutions. The various ecological models of 6.3 give good examples of this.

Try to draw a reasonable cartoon of the solutions to

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - x^2 - xy \\ -2y + xy \end{pmatrix}$$

by analyzing and piecing together the critical points.

The final will be focused on material we learned after the midterm, but some questions will need skills from before. For example, we've seen that you can sometimes solve a system of equations by reducing it to a single separable equation, so if you've forgotten how to solve separable equations, you won't be able to solve such a system of equations.

- (5.1) You will need to know your linear algebra basics—adding matrices, multiplying matrices, inverting 2×2 matrices, matrix multiplication is not commutative.
- (5.1) Understanding eigenvalues and eigenvectors of a square matrix is critical for solving linear equations. What are the eigenvalues/eigenvectors for

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}? \qquad B = \begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{pmatrix}? \qquad C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}?$$

  Answer: The matrix A has eigenvalues $9, 6, 0$ and eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

  B has eigenvectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

  C has eigenvalues $1 \pm 2i$ and eigenvectors

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- (5.5) Sometimes a matrix does not have enough independent eigenvectors, so we need generalized eigenvectors. What are the eigenvalues and generalized eigenvectors for

$$D = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}? \qquad E = \begin{pmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}?$$

  Answer: For D, we have a chain of length 2, and a chain of length 1. These are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  For E, a chain of length 3,

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \leftarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

- (5.2) You should be able to use these eigenvectors to find solutions to the linear, homogeneous, constant coefficient systems of equations:

$$\frac{d}{dt}\vec{x} = A\vec{x} \qquad \frac{d}{dt}\vec{x} = B\vec{x} \qquad \frac{d}{dt}\vec{x} = C\vec{x} \qquad \frac{d}{dt}\vec{x} = D\vec{x} \qquad \frac{d}{dt}\vec{x} = E\vec{x}$$

  Answer: For the first equation, the general solution is

$$\vec{f}(t) = \begin{pmatrix} e^{9t} & e^{6t} & 1 \\ e^{9t} & -2e^{6t} & 0 \\ e^{9t} & e^{6t} & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

For the third,

$$\vec{f}(t) = e^t \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

For the fourth,

$$\vec{f}(t) = \begin{pmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

- (5.1) Make sure you understand the superposition principle for linear homogeneous systems. You use this to find solutions satisfying an initial condition. What is the solution $\vec{f}(t)$ to

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

for which

$$\vec{f}(0) = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}?$$

Answer:

$$\vec{f}(t) = \begin{pmatrix} e^{9t} + e^{6t} - 2 \\ e^{9t} - 2e^{6t} \\ e^{9t} + e^{6t} + 2 \end{pmatrix}$$

- (5.1) When are solutions to systems of equations linearly independent? What is the Wronskian of solutions

$$\vec{f}_1 = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \qquad \vec{f}_2(t) = \begin{pmatrix} 7e^{5t} \\ e^{5t} \end{pmatrix}?$$

Are they independent?

Answer: The Wronskian is $-6e^{4t}$, which is never zero, so they are independent.

- (5.3) It's possible to classify 2×2 linear, homogeneous, constant coefficient equations

$$\frac{d}{dt}\vec{x} = F\vec{x},$$

where F is a 2×2 matrix, by the eigenvalues of F. You should either remember the list, the diagram I drew in lecture, or (preferrably) understand it well enough that you can draw a cartoon of the solutions given just the eigenvalues.

- (5.6) Solving a linear, homogeneous, constant coefficient equation

$$\frac{d}{dt}\vec{x} = F\vec{x}$$

ultimately rests on being able to exponentiate the matrix F. Typically, if you want to do this by hand, you will diagonalize the matrix, writing

$$F = GHG^{-1}$$

where G is a matrix of eigenvectors for F and H is a diagonal matrix. Then

$$e^{Ft} = e^{GHG^{-1}t} = Ge^{Ht}G^{-1}.$$

You should know how to exponentiate a diagonal matrix.

For example, what is

$$e^{At}? \qquad e^{Ct}? \qquad e^{Dt}?$$

Oh boy! (Don't worry, on the final I won't ask you to do anything this complicated by hand) e^{At} is

$$\frac{1}{6}\begin{pmatrix} 2e^{9t}+e^{6t}+3 & 2e^{9t}-2e^{6t} & 2e^{9t}+e^{6t}-3 \\ 2e^{9t}-2e^{6t} & 2e^{9t}+4e^{6t} & 2e^{9t}-2e^{6t} \\ 2e^{9t}+2e^{6t}-3 & 2e^{9t}-2e^{6t} & 2e^{9t}+e^{6t}+3 \end{pmatrix}.$$

e^{Ct} is given by

$$e^{t}\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}.$$

We compute e^{Dt} by recalling that if $AB = BA$ for two matrices, then $e^{A+B} = e^{A}e^{B}$. Then we find

$$\begin{pmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}.$$

- (5.6) The matrix-valued function $\Phi(t) = e^{Ft}$ is the fundamental solution to

$$\frac{d}{dt}\vec{x} = F\vec{x}.$$

For any vector $\vec{c}$, the function

$$\vec{f}(t) = \Phi(t)\cdot\vec{c}$$

gives the unique solution with initial values $\vec{c}$.

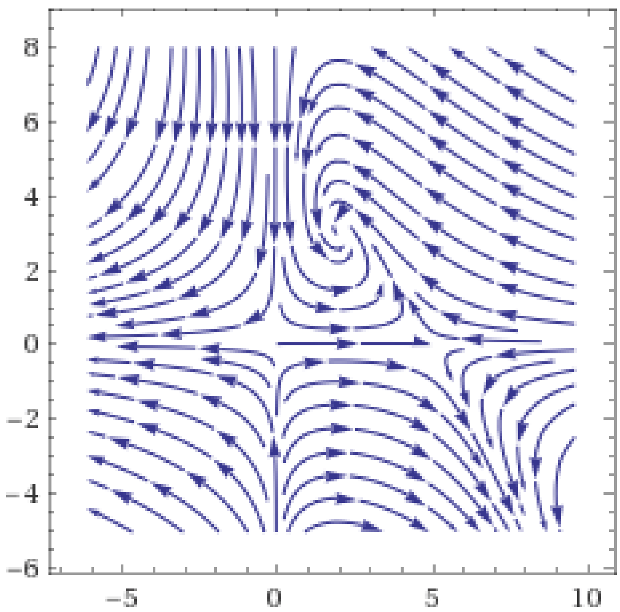
- (6.1/6.2/6.3) Non-linear systems of equations can be very complicated, but we can get a good understanding of them by studying their critical points. In particular, if the linearization (AKA the Jacobian) of a system at a critical point has non-zero, non-pure-imaginary eigenvalues, then very close to the critical point, the solutions to the non-linear equation look very similar to the solutions of the linearization.

Then, to understand the behavior of our non-linear system away from the critical points, we have something of a puzzle to piece together the solutions. The various ecological models of 6.3 give good examples of this.

Try to draw a reasonable cartoon of the solutions to

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x - x^2 - xy \\ -2y + xy \end{pmatrix}$$

by analyzing and piecing together the critical points.


8
6
4
2
0
−2
−4
−6
−5
0
5
10