

# **MAT303 - Calculus IV with Applications**

## **Spring Semester 2015**

- **Instructor:** Yaar Solomon (The best way to reach me is by Email).
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  - **Office:** Math Tower 2-119.
  - **Office hours:** Tuesdays 10:30-12:30AM, and by appointment.
  - **Class:** TuTh 8:30am-9:50AM at the Library E4320.
- **Teaching Assistant:** Zheng Zhang.
  - **Email:** zzhang at math.sunysb.edu
  - **Office:** Math Tower 5-125B.
  - **Office hours:** Thursdays 10:00-11:00AM
  - **Class:** **R03** M 11:00am-11:53AM at the Library E4320, **R04** Th 11:00am-11:53AM at Earth and Space 181.
- **Textbook:**
  - Edwards and Penney - *Differential equations and boundary value problems; 4th edition.*
- **Important Information:**
  - [Syllabus](#)
  - [Homework](#) **CHECK FOR UPDATES**
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  - [Announcements](#) **CHECK FOR UPDATES**
  - [Exams](#)

# MAT303 - CALCULUS IV WITH APPLICATIONS

## 1. INSTRUCTOR AND TA INFORMATION

Instructor: Yaar Solomon

- yaar.solomon@stonybrook.edu
- Office hours: Tu 10:30-12:30, Math Tower, room 2-119.

Teaching Assistance: Zheng Zhang

- zzhang@math.sunysb.edu
- Office hours: Th 10:00-11:00, Math Tower, room 5-125B.

## 2. COURSE DESCRIPTION

Differential equation is an equation relating an unknown function and its derivatives. Various scientific laws can be translated into differential equations. The course is dedicated to standard techniques for solving ordinary differential equations, including some numerical methods, and their applications in different branches of science.

## 3. TEXTBOOK

Edward and Penney

*Differential Equations and Boundary Value Problems; Computing and Modelling, Fourth Edition* (chapters 1-8).

## 4. COURSE PLAN

The course will cover the following topics:

- Methods for solving first order differential equation.
- Mathematical models and numerical methods.
- High order linear differential equations.
- Linear systems of differential equations.
- Non-linear systems of differential equations.

If time permits we will discuss other methods for solving differential equations such as the Laplace Transform and power series (chapters 7 and 8 in the textbook).

## 5. PREREQUISITES

Students should know integration and differentiation techniques and to be familiar with basic complex numbers and basic aspects of linear algebra.

## 6. HOMEWORK

We will have homework assignments that will be uploaded every week or two to the course website, and will be due in the recitation sessions. The due dates will be published on the assignments. In these assignments you will be required to apply the definitions, theorems, and methods that were learned in class. The main goal of the exercises is to let you practice the new concepts that were learned in class, and solving them will be essential for understanding the material. I expect you to put a lot of effort on solving the exercises.

## 7. EXAMS AND GRADING POLICY

- **Final Exam:** Tuesday, May 12, 11:15-1:45. 40% of the course grade. Failing the exam means failing the course. There will be no makeup.
- **Midterm:** Thursday, March 12, 8:30-9:50 AM - in class midterm, 30% of the course grade. There will be no makeup.
- **Homework:** 30% of the course grade.

## 8. DISABILITIES

If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at

<http://studentaffairs.stonybrook.edu/dss/>

or (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

<http://www.sunysb.edu/facilities/ehs/fire/disabilities>

## 9. ACADEMIC INTEGRITY

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another persons work as your own is always wrong. Faculty are required to report any suspected instance of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

<http://www.stonybrook.edu/uaa/academicjudiciary/>

## 10. CRITICAL INCIDENT MANAGEMENT

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, and/or inhibits students ability to learn.

## MAT303 - Homework

One can not study mathematics by watching someone else solving problems! The only way to really understand new mathematics is by sitting and solving exercises on your own. You are encouraged to work in groups and discuss the homework problems with your classmates, but the work that you hand in must be written by each student individually. In particular, in the exams I will assume that you are comfortable with all the homework assignments that I gave.

- The homework assignments and the week where they are due will be published below.
- You will have to hand in your assignments to the TA in your recitation that week.
- You are encouraged to print your homework using LaTeX. I'll be glad to help, and to supply a sample file for you, if you are interested.
- Please submit your homework on time, no late homework will be accepted.
- Homework average will take 30% of the course grade! Please put a lot of effort into them.

Homework #	Due	Assignment	Remarks
1	2/9/2015	<a href="#">Homework 1</a>	
2	2/23/2015	<a href="#">Homework 2</a>	
3	3/9/2015	<a href="#">Homework 3</a>	The typos in the due date and Q3(b) are now fixed
4	3/23/2015	<a href="#">Homework 4</a>	
5	Th 4.9.15 or M 4.13.15	<a href="#">Homework 5</a>	New exercise 5, the old one is now 6.
6	Th 4.9.15 or M 4.13.15	<a href="#">Homework 6</a>	This was exercise 5, Note the new due date.
7	Th 4.16.15 or M 4.20.15	<a href="#">Homework 7</a>	
8	Th 4.23.15 or M 4.27.15	<a href="#">Homework 8</a>	
9	M 5.4.15 or Th 5.7.15	<a href="#">Homework 9</a>	
10	Not for submission	<a href="#">Homework 10</a>	
Practice Exam		<a href="#">Practice Exam</a>	

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## MAT303 - Partial Solutions and Additional Notes

Uploaded at	Description	Notes	Remarks
3/9/2015	Exact equation	<a href="#">Exact equation</a>	
3/9/2015	Solutions for exercise 2	<a href="#">Solutions 2</a>	There will not be solutions for all the exercises
3/10/2015	A solution for a question similar to Ex4 Q2	<a href="#">Ex4 example for Q2</a>	You may also use slope fields to draw
3/10/2015	Additional practice from the textbook	<a href="#">Additional practice</a>	
3/12/2015	Solutions for the midterm	<a href="#">Midterm solutions</a>	
4/14/2015	A solutions for 1(f) of Ex7	<a href="#">Ex7, 1(f)</a>	
4/22/2015	A solutions for 4(c) of Ex8	<a href="#">Ex8, 4(c)</a>	
5/7/2015	Practice exam solutions	<a href="#">Practice exam solutions</a>	

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## **MAT303 - Announcements**

- **1.28.2015:** Welcome to MAT303 - Calculus IV with Applications. Our course will be separated from the parallel MAT303 course that is given by professor Artem Dudko, though you are encouraged to look at the exercises there for additional practice. Please read the syllabus carefully and don't hesitate to write me an email with any question or remark that you have.
  - **3.10.2015:** I have uploaded the following documents to the website (they are under the 'Partial solutions and additional notes' tab): (i) My notes from class on exact equations and finding integrating factors for them. (ii) Solutions for exercise 2. (iii) An example for a solution of a question similar to question 2 in exercise 4. (iv) Addition practice problems from the textbook.
  - **4.14.2015:** I have uploaded a solution for question 1(f) from exercise sheet 7.
  - **4.22.2015:** I have uploaded a solution for question 4(c) from exercise sheet 8.
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# MAT303 - Exams

In both the midterm and the final no electronic devices or any written material is allowed (as always).

## **Grading policy:**

- **Final exam** 40%.
- **Midterm** 30%.
- **Homework** 30%.
- There will be no makeups. Failing the exam means failing the course.

## **Final Exam:**

- **When?** Tuesday, May 12, 11:15AM-1:45PM.
- **Where?** TBA.

## **Midterm:**

- **When?** Thursday March 12, at 8:30AM-9:50AM.
  - **Where?** In class, Library E4320.
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# MAT303 - Calculus IV with Applications

## Homework 1

Due in recitation on the week of February 9, 2015

1. For each of the following sections verify that the given function is a solution for the corresponding differential equation.

$$(a) \quad y(x) = \frac{1}{1+x^2}, \quad y' = -2xy^2.$$

$$(b) \quad y(x) = \frac{1}{x} - \ln x, \quad x^2y'' + xy' - y = \ln x.$$

2. Verify that  $y(x)$  satisfies that differential equation, then find a value for the constant  $C$  for which  $y(x)$  is a solution for the initial value problem.

$$(a) \quad y(x) = Ce^{-x} + x - 1, \quad y' = x - y, \quad y(0) = 5.$$

$$(b) \quad y(x) = (x + C) \cos(x), \quad y' + y \tan(x) = \cos(x), \quad y(\pi) = 0.$$

3. Find a differential equation that is a mathematical model for the situation described (do not try to solve the equations).

(a) The acceleration of a given car is proportional to the difference between 100 km/h and its velocity.

(b) The difference between a population  $P$  and its *time rate of change*<sup>1</sup> is proportional to the third root of the population.

(c) A city has a fixed population of  $P$  persons. The time rate of change of the number  $S$  of those persons infected with a certain disease is proportional to the product of those who have the disease and those who do not.

4. Find the most general solution for the following differential equations.

(a)  $\frac{dy}{dx} = x^2e^{x^3}$ .

(b)  $y'(x) = \ln x$ .

(c)  $\frac{dx}{dt} = e^t \sin(t)$ .

(d)  $\frac{d^2y}{dx^2} = x + \sin(x)$ .

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<sup>1</sup>The rate of change with respect to time  $t$ .

# MAT303 - Calculus IV with Applications

## Homework 2

Due in recitation on the week of February 23, 2015

1. Find the most general solutions:

(a)  $y' = 2xy^3$ .

(b)  $\frac{dy}{dx} = 3x^2 \sec y$ .

(c)  $(1 + x^2) \tan(y)y' = 2x$ .

2. A cylindrical water tank with length 10 meters and radius 5 meters is situated with its axis horizontal. A circular hole of radius 10 cm is opened at the bottom. How long will it take for the water to drain completely, given that the tank was half full?

Remark: (i) you may assume that  $g = 10$  meters per sec. per sec.

(ii) Recall Torricelli's Law from page 41 in the textbook (that was done in class).

3. Find the most general solutions:

(a)  $y' - 2xy = e^{x^2}$ .

(b)  $xy' = 4y + x^6 \cos x$ .

(c)  $xy' + (2x - 2)y = 4x^4$ .

4. Find the particular solution for the following initial value problems.

(a)  $y' = (1 - y) \cos x$ ,  $y(\pi) = 2$ .

(b)  $x \frac{dy}{dx} - y = 2x^2y$ ,  $y(1) = 1$ .

(c)  $(x^2 + 4)y' + 3xy = x$ ,  $y(0) = 1$ .

(d)  $y' = x^2y^2 - x + x^2 - xy^2$ ,  $y(2) = 0$ .

5. Consider a reservoir with 6 million cubic meters of water, and an initial pollutant concentration of 25%. There is a daily inflow of 200,000 cubic meters of water with a pollutant concentration of 5%, and an equal daily outflow of the well-mixed water in the reservoir. How long will it take to reduce the pollutant concentration to 10%?

# MAT303 - Calculus IV with Applications

## Homework 3

Due in recitation on the week of March 9, 2015

1. Find the general and the particular solutions.

- (a)  $y' = e^{2x-y+1} + 2, \quad y(0) = 1.$
- (b)  $\frac{xy'}{6} + y = (1 + e^x)y^{2/3}, \quad y(1) = 8.$
- (c)  $t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0.$
- (d)  $(x + y - 2)^2y' = 1 - (x + y - 2)^2, \quad y(2) = 2.$
- (e)  $x^2y' = y^2 - 3xy + 4x^2, \quad y(1) = 3.$
- (f)  $3xy^2y' = 3x^4 + y^3, \quad y(2) = 3.$
- (g)  $xy \sin\left(\frac{y}{x}\right)y' = x^2 + y^2 \sin\left(\frac{y}{x}\right), \quad y(1) = \pi.$

2. Find the general solution (use integrating factors where needed).

- (a)  $3y^3 - y \cos(xy) + (9xy^2 - x \cos(xy))y' = 0.$
- (b)  $3t^2y + 2ty + y^3 + (t^2 + y^2)y' = 0.$
- (c)  $e^x \sin(x^2y) + 2xye^x \cos(x^2y) + x^2e^x \cos(x^2y)y' = 0.$
- (d)  $y^2(x - 3y) + (1 - 3xy^2)y' = 0.$
- (e)  $xy^2(xy' + y) = 1.$

3. Consider the non-linear equation (this is a Riccati equation)

$$(*) \quad y' + a_2(x)y^2 + a_1(x)y + a_0(x) = 0,$$

where  $a_0(x), a_1(x), a_2(x)$  are continuous on some interval  $J$ , and  $a_2(x) \neq 0$ .

- (a) Given a particular solution  $y_1(x)$  for equation (\*), show that the substitution  $y = y_1(x) + \frac{1}{u}$  transform it to a linear equation with  $P(x) = -(2y_1(x)a_2(x) + a_1(x))$  and  $Q(x) = a_2(x)$ .
- (b) Use the above method to solve

$$y' + xy^2 - 2x^2y + x^3 = x + 1,$$

where  $y_1(x) = x - 1$  (check that this is indeed a particular solution).

(c) Use the above method to solve

$$y' + y^2 - (1 + 2e^x)y + e^{2x} = 0, \quad y(\ln(2)) = \frac{5}{2}$$

where  $y_1(x) = e^x$  (check that this is indeed a particular solution).

4. Read pages 72-73 from the textbook and solve the following equations. You may assume that  $x, y, y'$  are all positive, if needed.

(a)  $yy'' = 4(y')^2$ .

(b)  $xy'' + y' = 6x$ .

(c)  $x^3y'' + 3x^2y' = 8x - 5$ .

(d)  $yy'' + (y')^2 = yy'$ .

# MAT303 - Calculus IV with Applications

## Homework 4

Due in recitation on the week of March 23, 2015

- Let  $F = \frac{N_x - M_y}{x \cdot M - y \cdot N}$ . Show that if  $F$  is a continuous function of the variable  $t = x \cdot y$  alone, then the equation  $M(x, y) + N(x, y)y' = 0$  has an integrating factor of the form  $\mu(t) = e^{\int F(t)dt}$ . [That is, the equation  $\mu(t)M(x, y) + \mu(t)N(x, y)y' = 0$  is exact].
- For each one of the equations (a), (b), (c) below do the following.
  - Find the critical points  $d_1 < d_2$ .
  - Solve the equation.
  - Sketch the graph of the solution  $y(x)$  by studying the behavior of  $y$ , as  $x \rightarrow \pm\infty$ , for different values of  $y_0 = y(0)$  (you should consider  $y_0 < d_1$ ,  $d_1 < y_0 < d_2$ , and  $d_2 < y_0$ ).
  - Finally, use (iii) to decide for each critical point if it is stable or unstable.
  - $y' = y^2 - 4$ .
  - $y' = y^2 - 7y + 6$ .
  - $y' = -y^2 + 7y - 10$ .
- Verify that the given functions are solutions for the given equation, write the general solution, and find a particular solution that satisfies the initial conditions.
  - $y^{(3)} - 9y' = 0$ ,  $y_1 = 1$ ,  $y_2 = \sin(3x)$ ,  $y_3 = \cos(3x)$ ,  
 $y(0) = 3$ ,  $y'(0) = -1$ ,  $y''(0) = 2$ .
  - $y^{(3)} + 2y'' - y' - 2y = 0$ ,  $y_1 = e^x$ ,  $y_2 = e^{-x}$ ,  $y_3 = e^{-2x}$ ,  
 $y(0) = 1$ ,  $y'(0) = 2$ ,  $y''(0) = 0$ .
- Check for each set of functions whether it is linearly dependent or linearly independent. In the former case, find the linear combination that gives 0.
  - $\{x^2 + 6, 2x^2 - 3, 4x^2 + 1\}$ .
  - $\{e^x, x^{-2}, x^{-2} \ln x\}$ .

(c)  $\{\sin(2x), \sin x, \sin x \cos x\}$ .

(d)  $\{e^{ax}, e^{bx}, e^{cx}\}$ , where  $a, b, c$  are three distinct numbers.

5. Prove that if  $f(x)$  is differentiable, and not identically zero, on the interval  $(a, b)$  then  $f(x)$  and  $xf(x)$  are linearly independent on  $(a, b)$ .

# MAT303 - Calculus IV with Applications

## Homework 5

Due in recitation on Thursday 4.9.2015 or Monday 4.13.2015

1. For the following pairs of matrices decide whether the multiplications  $AB$  and  $BA$  are defined or not, and if they are defined compute them.

(a)

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 3 & 0 \\ 2 & -1 \\ 4 & -2 \\ 3 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 0 \\ 1 & -2 \\ 4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 1 & 1 \\ 2 & -1 & 3 & 6 \end{pmatrix}.$$

(d)

$$A = \begin{pmatrix} -t & -t^2 \\ e^t & t \\ \sin t & 2 \\ -2 & \cos t \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{t} & e^{-t} & -2 & t \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 1 & 2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Compute: (a)  $AB$ . (b)  $BA$ . (c)  $A^2 + 2B + 3I$ .

3. Find the corresponding matrices for the following systems of linear equations, and solve them using Gaussian elimination.

$$\begin{array}{rcl} x & & +z = -1 \\ \text{(a)} \quad 2x & -2y & +4z = 2 \\ 3x & +3y & +3z = -3 \end{array}$$

$$\begin{array}{rcl} 2x & +y & +z = 5 \\ \text{(b)} & -x & +2y +2z = 5 \\ & -2x & +3y +z = 5 \end{array}$$

4. Compute the determinant of the following matrices.

(a)

$$\begin{pmatrix} 1 & 5 \\ -3 & 4 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 1 & 2 & -2 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(d)

$$\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & -1 & -1 \\ 2 & 0 & -2 & 0 \end{pmatrix}.$$

# MAT303 - Calculus IV with Applications

## Homework 6

Due in recitation on Thursday 4.9.2015 or Monday 4.13.2015

1. Let  $g(x)$  be a monic polynomial with integer coefficients. That is

$$g(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0,$$

where  $a_0, \dots, a_{n-1}$  are integers (monic means that the leading coefficient is equal to 1). Prove that if  $\frac{p}{q}$  is a rational root of  $g(x)$  (that is if  $g(\frac{p}{q}) = 0$ ) then

- (a)  $\frac{p}{q} = m$  is an integer.  
(b) The number  $m$  divides  $a_0$  (that is  $\frac{a_0}{m}$  is also an integer).

**Guidance:** We may assume that  $\frac{p}{q}$  is an irreducible fraction.

For (a), set  $g(\frac{p}{q}) = 0$  and multiply by  $q^{n-1}$  to deduce that  $\frac{p^n}{q}$  is an integer, and hence  $q = 1$ .

For (b), set  $g(m) = 0$  and divide by  $m$  deduce that  $\frac{a_0}{m}$  is an integer.

2. Use Question (1) to find the general solution for the following equations.

- (a)  $y^{(3)} + 2y'' - 13y' + 10y = 0$ .  
(b)  $y^{(3)} - 7y'' + 16y' - 12y = 0$ .  
(c)  $y^{(3)} - 2y'' + 4y' - 8y = 0$ .  
(d)  $y^{(3)} - 5y'' + 9y' - 5y = 0$ .  
(e)  $y^{(4)} + y^{(3)} - 7y'' - y' + 6y = 0$ .  
(f)  $y^{(3)} - y'' + 8y' + 10y = 0$ .

3. (a) Find a particular solution for (2a) that satisfies  $y(0) = 1, y'(0) = 2, y''(0) = 46$ .  
(b) Find a particular solution for (2b) that satisfies  $y(0) = 1, y'(0) = 1, y''(0) = 1$ .

4. Let  $f(x)$  be a polynomial with real coefficients. That is

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0,$$

where  $a_n, \dots, a_0$  are real numbers. Prove that if  $z = a + bi$  complex root of  $f(x)$  (that is if  $f(z) = 0$ ) then  $\bar{z} = a - bi$  is also a root of  $f(x)$ .

**Guidance:** Plug  $\bar{z}$  to  $f(x)$  and take the complex conjugate of that expression. Recall that the complex conjugate respect sums and products (namely,  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$  and  $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$ ), and that the coefficients  $a_n, \dots, a_0$  are all real.

# MAT303 - Calculus IV with Applications

## Homework 7

Due in recitation on Thursday 4.16.2015 or Monday 4.20.2015

1. Find a particular solution  $y_p$  and the complementary solution  $y_c$  and use them to represent the general solution.

(a)  $y'' + 2y' + 5y = 2e^x$ .

(b)  $y'' + 4y' + 4y = x + 2$ .

(c)  $y'' + 5y' + 6y = x^2$ .

(d)  $y'' + 3y' + y = \cos(2x)$ .

(e)  $y^3 + 2y'' + 2y' + y = \sin x + \cos x$ .

(f)  $y'' + 2y' - 3y = 2xe^x$ .

(g)  $y'' - y' + 2y = 4x \sin(2x)$ .

2. Find the general solution.

(a)  $y'' + 4y = 3 \sin(2x)$ .

(b)  $y^{(3)} + 2y'' - y' - 2y = 3e^{-x}$ .

3. Find the general solution for

$$y'' - 2y' - 8y = 3 \sin x + 2e^{3x} + 4x^2 - 1$$

4. Find the general solution for

$$y^{(3)} - 3y'' + 3y' - y = 2e^x + e^{2x}$$

# MAT303 - Calculus IV with Applications

## Homework 8

Due in recitation on Thursday 4.23.2015 or Monday 4.27.2015

1. Find the corresponding first order system of differential equations (you have to define the new variables). If it is a linear system, give the matrix representation of it as well ( $x$  and  $y$  are functions of the independent variable  $t$ ).

(a)  $\frac{1}{3}x^{(3)} - tx'' + 3x' - 2x + 5.$

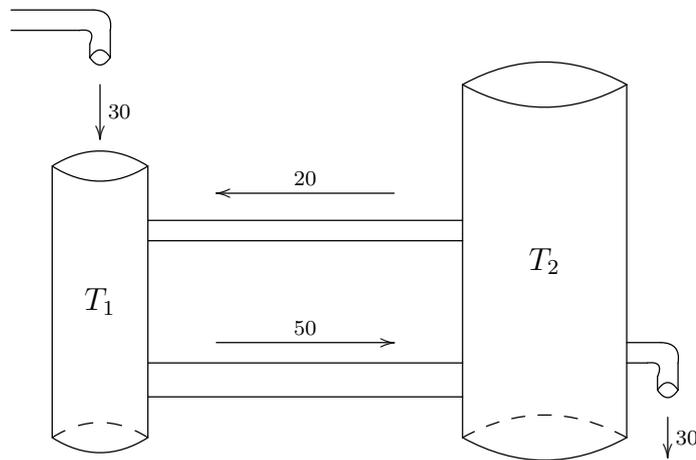
(b)  $2x^3 - x' = 2x'' + 3x' - x^2 - 2.$

(c)  $\left\{ \begin{array}{l} x'' = 5x' + 2x + 2y' \\ y'' = tx - 3y' - y + \sin(t) \end{array} \right\}$

2. In each of the following sections define the variables (in words) and build a system of differential equations that describes the situation.

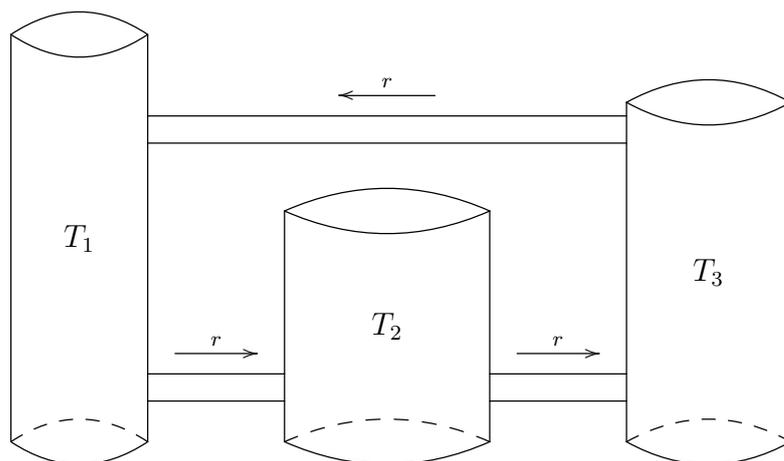
- (a) The flow between the tanks, the inflow, and the outflow are in gal/min, and in the directions of the arrows below. The inflow contains 1 kg of salt per gallon. The volumes in gallons of the brine in the tanks are:

$$V(T_1) = 90, \quad V(T_2) = 150.$$



- (b) Here the flow between the tanks is  $r = 10$  gal/min everywhere, and the volumes in gallons of the brine in the tanks are:

$$V(T_1) = 50, \quad V(T_2) = 25, \quad V(T_3) = 50.$$



3. Question 24, page 255 (section 4.1) from the textbook: Define the variables (in words), draw the diagrams of the forces that act of each of the masses, and explain how to get the system of differential equations.

4. Solve the following systems by elimination.

$$(a) \begin{aligned} x_1' &= 4x_1 + 2x_2 \\ x_2' &= -x_1 + x_2 \end{aligned}$$

$$(b) \begin{aligned} x_1' &= -4x_1 - x_2 \\ x_2' &= 5x_1 + 2x_2 \end{aligned}$$

$$(c) \begin{aligned} x_1' &= -2x_1 - 4x_2 + 2x_3 \\ x_2' &= -2x_1 + x_2 + 2x_3 \\ x_3' &= 4x_1 + 2x_2 + 5x_3 \end{aligned}$$

# MAT303 - Calculus IV with Applications

## Homework 9

Due in recitation on Monday 5.4.2015 or Thursday 5.7.2015

1. Find the general solution.

(a)  $X' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} X.$

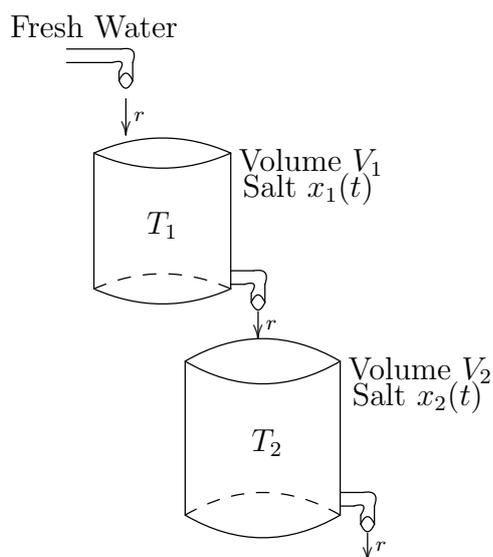
(b)  $X' = \begin{pmatrix} 7 & -5 \\ 4 & -3 \end{pmatrix} X.$

(c)  $X' = \begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{pmatrix} X.$

(d)  $X' = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix} X.$

(e)  $X' = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} X.$

2.  $x_1(t), x_2(t)$  denote the amount (in kg) of salt at time  $t$  in the two brine tanks described in the diagram below. Denote by  $V_1, V_2$  the volume (in gallons) of the brine in  $T_1, T_2$  respectively.



- (a) Set up a system of differential equations that  $x_1(t), x_2(t)$  satisfy, in terms of  $V_1, V_2$  and  $r$ .
- (b) Given that  $r = 6$  gal/min,  $V_1 = 15$  gal, and  $V_2 = 24$  gal, find the general solution.
- (c) Given additionally that  $x_1(0) = 6$  kg,  $x_2(0) = 1$  kg, find the particular solution.

3. Find the general solution.

(a)  $X' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} X.$

(b)  $X' = \begin{pmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{pmatrix} X.$

(c)  $X' = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix} X.$

4. Find the general solution.

(a)  $X' = \begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix} X + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}.$

(b)  $X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} 3t \\ -3 \end{pmatrix}.$

# MAT303 - Calculus IV with Applications

## Homework 10

Practice - not for submission

### Formulas

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Laplace Transform:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$$

$$\mathcal{L}(f^{(3)}(t)) = s^3\mathcal{L}(f(t)) - s^2f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2 + k^2}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2 + k^2}$$

---

1. Use the Laplace transform to solve the following initial value problems.

(a)  $x'' + x = \sin(2t)$ ,  $x(0) = x'(0) = 0$ .

(b)  $x'' + x = \cos(3t)$ ,  $x(0) = 1, x'(0) = 0$ .

(c)  $x'' - x' - 2x = 0$ ,  $x(0) = 0, x'(0) = 2$ .

(d)  $x'' - 6x' + 8x = 2$ ,  $x(0) = x'(0) = 0$ .

(e)  $x'' - 2x' - 8x = e^t$ ,  $x(0) = 0, x'(0) = 0$ .

(f)  $x'' - 2x' - 8x = e^{2t}$ ,  $x(0) = 0, x'(0) = 0$ .

(g)  $x^{(3)} + x'' - 6x' = 0$ ,  $x(0) = 0, x'(0) = x''(0) = 1$ .

2. Compute the Laplace transform of  $t \sin(kt)$  and  $t \cos(kt)$ .

# MAT303 - Calculus IV with Applications

## Practice Exam

### Formulas

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#### Laplace Transform:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$$

$$\mathcal{L}(f^{(3)}(t)) = s^3\mathcal{L}(f(t)) - s^2f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2 + k^2}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2 + k^2}$$

---

#### Torricelli's Law:

$$\frac{A(y) dy}{\sqrt{y}} \frac{dy}{dt} = -a\sqrt{2g},$$

where  $A(y)$  is the horizontal cross section of the tank at height  $y$ ,  $a$  is the area of the hole, and  $g$  is gravity.

---

#### Integration factors for exact equations:

$$e^{\int \frac{M_y - N_x}{N} dx} \quad \text{or} \quad e^{\int \frac{N_x - M_y}{M} dy}$$

---

## Part A

1. (14 points)

Find the general solution.

$$x(x + y)y' + y(3x + y) = 0.$$

2. (14 points)

Solve the following initial value problem.

$$\sec(x)y' = 2y + y^4, \quad y(\pi) = -2.$$

3. (14 points)

Find the general solution for the following system.

$$\begin{cases} x' = 5x + 9y \\ y' = -x - y \end{cases}$$

4. (14 points)

Find the general solution for the following system.

$$X' = \begin{pmatrix} -1 & -2 & 4 \\ 6 & 11 & -18 \\ 3 & 7 & -12 \end{pmatrix} X$$

5. (14 points)

Find the general solution of the following non-homogeneous equation.

$$y^{(3)} - 13y' + 12y = 3e^x.$$

6. (14 points)

Use Laplace Transform to solve the following initial value problem.

$$x'' + 5x' + 6x = \sin(t), x(0) = -1, x'(0) = -5.$$

## Part B - Theory

7. (4 points)

Which of the following is TRUE for a first order differential equation:

- (a) A linear equation is exact.
- (b) An exact equation is linear.
- (c) A linear equation has an integration factor that makes it exact.
- (d) Non of the above.

8. (4 points)

$v_1e^{\lambda_1 t}$ ,  $v_2e^{\lambda_2 t}$  and  $v_3te^{\lambda_3 t}$  are three linearly independent solutions for the system  $X' = AX$  on some interval  $I$ , where  $A$  is a  $3 \times 3$  matrix (note the additional  $t$  in the third solution). Which of the following is necessarily FALSE:

- (a)  $\lambda_1 \neq \lambda_2$ .
- (b)  $\lambda_1 = \lambda_2 = \lambda_3$ .
- (c) The matrix  $A$  has 3 distinct eigenvalues.
- (d)  $\lambda_1 = \lambda_3$  or  $\lambda_2 = \lambda_3$  (or both).

9. (4 points)

Which of the following is TRUE for the differential equation

$$y^{(2)} + a_1y' + a_0y = 0.$$

- (a) It always has a unique solution which is defined on  $\mathbb{R}$ .
- (b) Given real numbers  $a, b_1, b_2$ , it has a unique solution on  $\mathbb{R}$  that satisfies

$$y(a) = b_1, y'(a) = b_2.$$

- (c) Given real numbers  $a, b_1, b_2$ , it has a unique solution on  $\mathbb{R}$  that satisfies

$$y(b_1) = a, y(b_2) = a.$$

- (d) Non of the above.

10. (4 points)

It is given that

$$\det \begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix} = 0$$

On some interval  $I$ . Which of the following is TRUE:

- (a)  $\{x_1(t), x_2(t)\}$  is a linearly dependent set of functions.
- (b) If  $\{x_1(t), x_2(t)\}$  is a linearly dependent set of functions then  $x_1(t), x_2(t)$  are solutions to an equation of the form  $x'' + a_1(t)x' + a_0(t)x = 0$ .
- (c) If  $x_1(t), x_2(t)$  are solutions to an equation of the form  $x'' + a_1(t)x' + a_0(t)x = 0$  then  $\{x_1(t), x_2(t)\}$  is a linearly dependent set of functions.
- (d) All of the above.

Exact Differential Equations

• Def: An equation of the form

$$M(x,y) + N(x,y) \cdot y' = 0$$

is called exact if there exists some function

$F(x,y)$  so that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

(•  $F$  is called the potential of the equation.)

- In that case (by the chain rule):  $F(x, y(x))$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = M + N \cdot y'$$

integrate w.r.t.  $x$

$\Rightarrow$

$$F(x, y(x)) = C$$

is the general solution.

- How do we know if the equation is exact?

Recall: If the mixed - second - partial - derivative are continuous on an open set in the plane then they are equal!

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial y} \right) = \frac{\partial^2 F}{\partial x \partial y} (x,y) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial x} \right) = \frac{\partial M}{\partial y}$$

$\Rightarrow$  Necessary condition.

- ~~Then~~ If the equation  $M(x,y) + N(x,y)y' = 0$  is exact then

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

(assuming that  $M, N$  has continuous partial derivatives)

Thm: Assume that  $M, N$  are continuous with continuous partial derivatives on the rectangle  $R = (a,b) \times (c,d)$ .

Then the equation  $M(x,y) + N(x,y)y' = 0$

is exact in  $R \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$  on  $R$ .

proof.  $\Rightarrow$  we saw.

$\Leftarrow$  Need to construct a function  $F(x,y)$  on  $R$  with  $\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N$ .

Define  $F(x,y) = \int M(x,y) dx + g(y)$

where  $g(y)$  will be defined later (Note:  $\frac{\partial F}{\partial x} = M$ ).

We also want  $N = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \int M(x,y) dx + g'(y)$ .

so we must define  $g'(y) = N - \frac{\partial}{\partial y} \int M(x,y) dx$ ,

but for that to make sense  $N(x,y) - \frac{\partial}{\partial y} \int M(x,y) dx$  must be a function of  $y$  alone. Let's verify that by showing that the derivative w.r.t.  $x$  is 0:

$$\frac{\partial}{\partial x} \left[ N - \frac{\partial}{\partial y} \int M(x,y) dx \right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \int M(x,y) dx$$

$$\stackrel{\substack{\uparrow \\ \text{assumption}}}{=} \frac{\partial M}{\partial y} - \frac{\partial}{\partial y} \underbrace{\frac{\partial}{\partial x} \int M(x,y) dx}_M = \frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} = 0 !$$

So  $g'(y) = N(x,y) - \frac{\partial}{\partial y} \int M(x,y) dx$  depends only on  $y$ , hence we can find  $g(y)$  by integrating  $dy$ . Therefore

$$F(x,y) = \int M(x,y) dx + \int \left[ N(x,y) - \frac{\partial}{\partial y} \int M(x,y) dx \right] dy \quad \square$$

Method of solution: Like in the proof.

Define  $F(x,y) = \int M(x,y) dx + g(y)$  [or  $\int N(x,y) dy + h(x)$ ]

derive with respect to  $y$  and equate to  $N$  [or  $\frac{\partial}{\partial x} \int N(x,y) dy + h'(x) = M(x,y)$ ].

But first check that the equation is exact by

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Examples: (1)  $\cos y + y \cos x + (x \sin x - x \sin y) y' = 0$

Sol:  $M(x,y) = \cos y + y \cos x \Rightarrow \frac{\partial M}{\partial y}(x,y) = -\sin y + \cos x$   
 $N(x,y) = x \sin x - x \sin y \Rightarrow \frac{\partial N}{\partial x}(x,y) = \cos x - \sin y$

$M, N$  has continuous partial derivatives everywhere  $\Rightarrow$  exact! Define

$$F(x,y) = \int (\cos y + y \cos x) dx + g(y) = x \cos y + y \sin x + g(y)$$

want  $N = \frac{\partial F}{\partial y} = -x \sin y + \sin x + g'(y) \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = C. \Rightarrow \boxed{F(x,y) = x \cos y + y \sin x + C}$   
 general solution

$$\textcircled{2} e^x \sin y + x^2 + e^x \cos y \cdot y' = 0$$

$$\text{sols. } M(x,y) = e^x \sin y + x^2 \Rightarrow \frac{\partial M}{\partial y} = e^x \cos y$$

$$N(x,y) = e^x \cos y \quad \frac{\partial N}{\partial x} = e^x \cos y \quad \checkmark$$

$$F(x,y) = \int (e^x \sin y + x^2) dx + g(y) = e^x \sin y + \frac{x^3}{3} + g(y)$$

$$\Rightarrow N = \frac{\partial F}{\partial y} = e^x \cos y + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$$\Rightarrow \boxed{F(x,y) = e^x \sin y + \frac{x^3}{3} + C}$$

Alternatively:

$$F(x,y) = \int e^x \cos y dy + h(x) = e^x \sin y + h(x) \Rightarrow$$

$$M = \frac{\partial F}{\partial x} = e^x \sin y + h'(x) \Rightarrow h'(x) = x^2 \Rightarrow h(x) = \frac{x^3}{3}$$

Note: The equation  $e^x \sin y + e^x \cos y \cdot y' = 0$  is also exact. Divide by  $e^x \rightarrow \sin y + \cos y \cdot y' = 0$ , and here  $\frac{\partial \sin y}{\partial y} = \cos y \neq \frac{\partial \cos y}{\partial x} = 0$  not exact!

But these equations are "the same" and should have the same solution!

Conclusion: Exactness depends on the precise form in which the equation is written!

Finding integrating factor: Given that  $M + Ny' = 0$

is not exact, find  $\mu(x,y)$  s.t.  $\mu M + \mu N \cdot y' = 0$  is exact:

(1) IF  $\frac{My - Nx}{N}$  is a function of  $x$  alone, pick  
 $\mu = e^{\int \frac{My - Nx}{N} dx}$

(2) IF  $\frac{Nx - My}{M}$  is a function of  $y$  alone, pick  
 $\mu = e^{\int \frac{Nx - My}{M} dy}$

- Back to the example:  $\sin y + \cos y \cdot y' = 0$  even though it is separable eq.

Here  $\frac{My - Nx}{N} = \frac{\cos y - 0}{\cos y} = 1$  - function of  $x$  alone

$\Rightarrow \mu = e^{\int 1 dx} = e^x$  is an integrating factor that makes the equation exact!  $e^x \sin y + e^x \cos y \cdot y' = 0$

- Also  $\frac{Nx - My}{M} = \frac{0 - \cos y}{\sin y} = -\cot(y)$  is a function of  $y$  alone  $\Rightarrow \mu = e^{\int \cot y dy}$  is another integrating factor! } sometimes one of them is better!!

Example: Solve  $x^2 + 2x + y + (3x^2y - x)y' = 0$

$M_y = 1$   
 $N_x = 6xy - 1$  } not exact! But  $\frac{My - Nx}{N} = \frac{2 - 6xy}{3x^2y - x} = \frac{2 - 6xy}{-x(1 - 3xy)} = -\frac{2}{x}$  a function of  $x$  alone!

$$\text{Pick } \mu(x) = e^{-\int \frac{2}{x} dx} = e^{-2 \ln|x|} = \frac{1}{x^2}$$

$$\Rightarrow \mu M + \mu N y' = 0 \text{ is } 1 + \frac{2}{x} + \frac{y}{x^2} + (3y - \frac{1}{x})y' = 0$$

$$\left. \begin{array}{l} M_y = \frac{1}{x^2} \\ N_x = \frac{1}{x^2} \end{array} \right\} \text{exact!}$$

$$\Rightarrow F(x,y) = \int (3y - \frac{1}{x}) dy + h(x) = \frac{3}{2} y^2 - \frac{y}{x} + h(x)$$

$$M = \frac{\partial F}{\partial x} = +\frac{y}{x^2} + h'(x) \Rightarrow h'(x) = 1 + \frac{2}{x} \Rightarrow h(x) = \int 1 + \frac{2}{x} dx \\ = x + 2 \ln|x| + C$$

$$\Rightarrow \boxed{F(x,y) = \frac{3}{2} y^2 - \frac{y}{x} + x + 2 \ln|x| + C = 0}$$

general solution.

# HW2 - Solutions

change variables to  $y$

1. (a)  $y' = 2xy^3$

sol:  $\frac{y'}{y^3} = 2x \Rightarrow -\frac{1}{2y^2} = \int \frac{dy}{y^3} = \int \frac{1}{y^3} dx = \int 2x dx = x^2 + C$

$\Rightarrow \frac{1}{y^2} = -2x^2 - 2C \Rightarrow$

$y = \frac{1}{\sqrt{-2x^2 - 2C}}$

(b)  $\frac{dy}{dx} = 3x^2 \sec y$

sol:  $\cos(y) \cdot y' = 3x^2 \Rightarrow \int dx$

$\sin y = x^3 + C$

$y = \arcsin(x^3 + C)$

(c)  $(1+x^2) \tan(y) y' = 2x$

sol:  $\tan(y) y' = \frac{2x}{1+x^2} \Rightarrow$

$\int \tan y dy = \int \frac{2x}{1+x^2} dx = \ln(1+x^2) + C$

$\int \tan(y) dy = \int \frac{\sin y}{\cos y} dy = \left[ u = \cos y, du = -\sin y dy \right] = -\int \frac{du}{u} = -\ln u = \ln(u^{-1})$

$\Rightarrow u^{-1} = e^{\ln(1+x^2) + C} = e^C \cdot (1+x^2) \Rightarrow u = \frac{1}{e^C (1+x^2)}$

$\Rightarrow \cos(y) = \frac{1}{e^C (1+x^2)} \Rightarrow y = \arccos\left(\frac{1}{e^C (1+x^2)}\right)$

2. Torricelli's law:  $\frac{1}{\sqrt{y}} A(y) \cdot \frac{dy}{dt} = -a\sqrt{2g}$

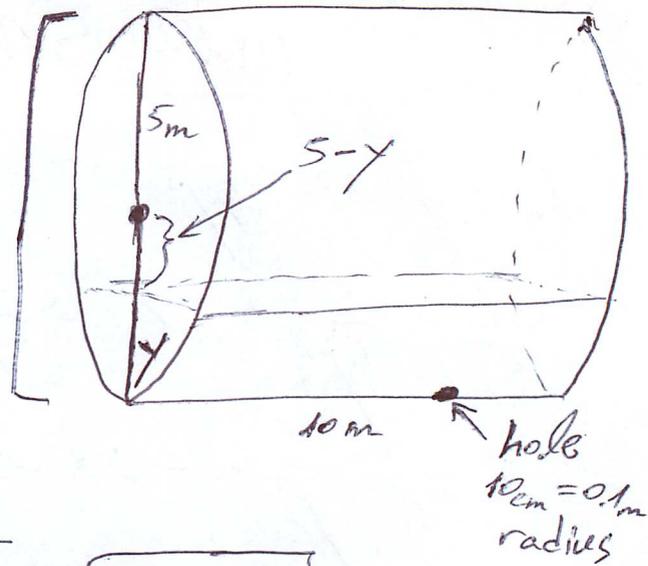
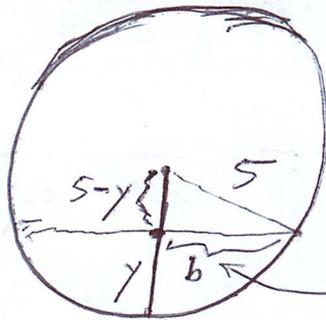
Where  $A(y)$  = the horizontal cross section of the tank at height  $y$ .

$a$  = area of the hole.

$$- a = (0.1)^2 \cdot \pi = 0.01\pi$$

$$- g = 10$$

$$- A(y) = ?$$



$$\sqrt{5^2 - (5-y)^2} = \sqrt{10y - y^2}$$

$$\Rightarrow A(y) = 2b \cdot 10 = 20\sqrt{10y - y^2}$$

$$\Rightarrow \frac{A(y)}{5y} \cdot y' = 20\sqrt{10y - y^2} \cdot \sqrt{y}^{-1} y' = 20\sqrt{10-y} \cdot y'$$

$$-a\sqrt{2g} = -0.01\pi \cdot \sqrt{20}$$

Toricelli:

$$\Rightarrow \sqrt{10-y} \cdot y' = \frac{-0.01\pi}{\sqrt{20}} \Rightarrow \int \sqrt{10-y} dy = \frac{-0.01\pi}{\sqrt{20}} t + C$$

$$\int \sqrt{10-y} dy = \left[ \begin{array}{l} u=10-y \\ du=-dy \end{array} \right] = -\int \sqrt{u} du = -u^{3/2} \cdot \frac{2}{3} = -\frac{2}{3}(10-y)^{3/2}$$

$$\Rightarrow \frac{2}{3}(10-y)^{3/2} = \frac{0.01\pi}{\sqrt{20}} t + C$$

For  $t=0$ ,  $y=5$  ( $y(0)=5$ ). So  $C = \frac{2}{3} \cdot 5^{3/2}$

Then the tank is empty when  $y=0$ , that is:

$$\frac{2}{3} \cdot 10^{3/2} = \frac{0.01\pi}{\sqrt{20}} t + \frac{2}{3} 5^{3/2} \Rightarrow t = \frac{2}{3} (10^{3/2} - 5^{3/2}) \cdot \frac{100\sqrt{20}}{\pi}$$

$$\approx 1940 \text{ sec} = 32 \text{ min and } 20 \text{ sec.}$$

$$3. (a) \underline{y}' - 2xy = e^{x^2}$$

$$\underline{\text{sol:}} \text{ Linear eq. } p(x) = e^{\int -2x dx} = e^{-x^2}$$

$$\Rightarrow \underbrace{e^{-x^2} y' - 2xe^{-x^2} y}_{\frac{d}{dx}(e^{-x^2} y)} = 1 \Rightarrow e^{-x^2} y = \int 1 dx = x + c$$

$$\Rightarrow \boxed{y = xe^{x^2} + ce^{x^2}}$$

$$(b) \underline{xy}' = 4y + x^6 \cos x$$

$$\underline{\text{sol:}} \quad y' - \frac{4}{x}y = x^5 \cos x \Rightarrow p(x) = -\frac{4}{x} \Rightarrow p(x) = e^{\int \frac{4}{x} dx} = x^{-4}$$

$$\Rightarrow \underbrace{x^{-4} y' - \frac{4}{x^{-5}} y}_{\frac{d}{dx}(x^{-4} y)} = x^5 \cos x \Rightarrow x^{-4} y = \int x^5 \cos x dx = \left[ \begin{array}{l} u=x \quad u'=1 \\ v=\cos x, \quad v'=-\sin x \end{array} \right]$$

$$= x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + c$$

$$\Rightarrow \boxed{y = x^5 \sin x + x^4 \cos x + cx^4}$$

$$(c) \underline{xy}' + (2x-2)y = 4x^4$$

$$\underline{\text{sol:}} \quad y' + (2 - \frac{2}{x})y = 4x^3 \Rightarrow p(x) = e^{\int 2 - \frac{2}{x} dx} = e^{2x - 2 \ln x} = e^{2x} / e^{\ln(x^2)}$$

$$= \frac{1}{x^2} e^{2x} \Rightarrow \underbrace{\frac{1}{x^2} e^{2x} y' + e^{2x} (\frac{2}{x^2} - \frac{2}{x^3}) y}_{\frac{d}{dx}(\frac{1}{x^2} e^{2x} y)} = 4xe^{2x}$$

$$\Rightarrow \frac{1}{x^2} e^{2x} y = \int 4xe^{2x} dx \left[ \begin{array}{l} u=x \quad u'=1 \\ v=e^{2x} \quad v'=\frac{1}{2}e^{2x} \end{array} \right] = 4 \left[ \frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} dx \right]$$

$$= 2xe^{2x} - e^{2x} + c \Rightarrow \boxed{y = x^3 - x^2 + ce^{-2x}}$$

4. (a)  $y' = (1-y)\cos x$ ,  $y(\pi) = 2$

sol.  $\frac{y'}{y-1} = -\cos x \Rightarrow \int \frac{dy}{y-1} = \int -\cos x dx \Rightarrow \ln|y-1| = -\sin x + C$

$\ln(2-1) = \frac{-\sin(\pi)}{0} + C \Rightarrow C = 0 \Rightarrow |y-1| = e^{-\sin x}$

$\Rightarrow \boxed{y = e^{-\sin x} + 1}$  (near  $y=2$  ( $y-1 > 0$ )).

(b)  $x \frac{dy}{dx} - y = 2x^2 y$ ,  $y(1) = 1$ .

sol.  $x \cdot y' = 2x^2 y + y = y(2x^2 + 1) \Rightarrow \frac{y'}{y} = \frac{2x^2 + 1}{x} = 2x + \frac{1}{x}$

$\ln|y| = \int \frac{dy}{y} = \int 2x + \frac{1}{x} dx = x^2 + \ln|x| + C$

$y(1) = 1: 0 = 1 + 0 + C \Rightarrow C = -1$

$\boxed{\ln|y| = x^2 + \ln|x| - 1}$

(c)  $(x^2+4)y' + 3xy = x$ ,  $y(0) = 1$ .

sol.  $y' + \frac{3x}{x^2+4} y = \frac{x}{x^2+4}$ ,  $P(x) = \frac{3x}{x^2+4}$ ,  $\rho(x) = e^{\int \frac{3x}{x^2+4} dx} =$

$e^{\frac{3}{2} \int \frac{2x}{x^2+4} dx} = e^{\frac{3}{2} \ln(x^2+4)} = (x^2+4)^{3/2} \Rightarrow$

$y \cdot (x^2+4)^{3/2} = \int \frac{x}{x^2+4} \cdot (x^2+4)^{3/2} dx = \int x(x^2+4)^{1/2} dx \left[ \begin{array}{l} u = x^2+4 \\ du = 2x dx \end{array} \right]$

$= \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \frac{u^{3/2}}{3/2} + C = \frac{1}{3} (x^2+4)^{3/2} + C \Rightarrow y = \frac{1}{3} + C(x^2+4)^{-3/2}$

$y(0) = 1: 1 = \frac{1}{3} + C \cdot 4^{-3/2} = \frac{1}{3} + \frac{C}{8} \Rightarrow C = 8 \cdot \frac{2}{3} = \frac{16}{3}$

$\Rightarrow \boxed{y = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}}$

(d)  $y' = x^2 y^2 - x + x^2 - x y^2$ ,  $y(2) = 0$ .

sol:  $y' = x^2(y^2+1) - x - x y^2 = x^2(y^2+1) - x(1+y^2) = (x^2-x)(1+y^2)$

$\Rightarrow \frac{y'}{1+y^2} = x^2-x \Rightarrow \arctan(y) = \int \frac{dy}{1+y^2} = \int x^2-x dx = \frac{x^3}{3} - \frac{x^2}{2} + C$

$y(2)=0$ :  $0 = \arctan(0) = \frac{8}{3} - \frac{4}{2} + C = \frac{2}{3} + C \Rightarrow C = -\frac{2}{3}$

$\Rightarrow \boxed{y = \tan\left(\frac{x^3}{3} - \frac{x^2}{2} - \frac{2}{3}\right)}$

5.  $V(0) = 6000000 \text{ m}^3$  ( $=v(t)$  for all  $t$ ).

$r_i = r_o = 200000 \text{ m}^3$

$C_i = 5\%$ ,  $C_o = \frac{x(t)}{v(t)} = \frac{x}{6000000}$

$X'(0) = 25\% \cdot 6000000 = 1500000$

The equation is:  $X' = r_i C_i - r_o C_o = 200000 \left(0.05 - \frac{x}{6000000}\right)$

$\Rightarrow X' = 10000 - \frac{1}{30} X \Rightarrow X' + \frac{1}{30} X = 10000$

$p(t) = e^{\int \frac{dt}{30}} = e^{\frac{t}{30}} \Rightarrow X \cdot e^{\frac{t}{30}} = \int 10000 e^{\frac{t}{30}} dt = \frac{10000 e^{\frac{t}{30}}}{\frac{1}{30}} + C = 300000 e^{\frac{t}{30}} + C$

$\Rightarrow X(t) = 300,000 + C e^{-\frac{t}{30}}$

$-X(0) = 1500000$ :  $1500000 = 300000 + C \Rightarrow C = 1200000$

$\Rightarrow X(t) = 300000 + 1200000 e^{-\frac{t}{30}}$

when 10%?  $10\% \cdot 6000000 = 600000 = 300000 + 1200000 e^{-\frac{t}{30}}$

$\Rightarrow e^{-\frac{t}{30}} = \frac{1}{4} \Rightarrow -\frac{t}{30} = \ln\left(\frac{1}{4}\right) \Rightarrow t = 30 \ln 4 \approx 41.59$   
days days

Question 2, Ex. sheet 4, with  $y' = y^2 - y - 2$

(i) critical points:  $y^2 - y - 2 = 0 \Rightarrow \underline{y = -1, y = 2}$

(ii)  $\frac{y'}{(y+1)(y-2)} = 1 \Rightarrow \int \frac{dy}{(y+1)(y-2)} = \int 1 dx = x + C$

$\Rightarrow \frac{1}{3} \int \left[ \frac{1}{y-2} - \frac{1}{y+1} \right] dy = \frac{1}{3} [\ln|y-2| - \ln|y+1|] = \frac{1}{3} \ln \left| \frac{y-2}{y+1} \right|$

$\Rightarrow \ln \left| \frac{y-2}{y+1} \right| = 3x + C_1 \Rightarrow \left| \frac{y-2}{y+1} \right| = C_1 e^{3x}$  ( $C_1 = e^{3C} > 0$  always)

(iii) To draw the graph we study the behavior of the solutions as  $x \rightarrow \pm\infty$  for different values of  $y_0$ :

$y_0 < -1$ :  $\frac{-y+2}{-y-1} = C_1 e^{3x} \Rightarrow \frac{y-2}{y+1-3} = (y+1) C_1 e^{3x}$

$\Rightarrow (y+1)(1 - C_1 e^{3x}) = 3 \Rightarrow \boxed{y = \frac{3}{1 - C_1 e^{3x}} - 1}$

$-1 < y_0 < 2$ :  $\frac{-y+2}{y+1} = C_1 e^{3x} \Rightarrow 3 - y - 1 = (y+1) C_1 e^{3x}$

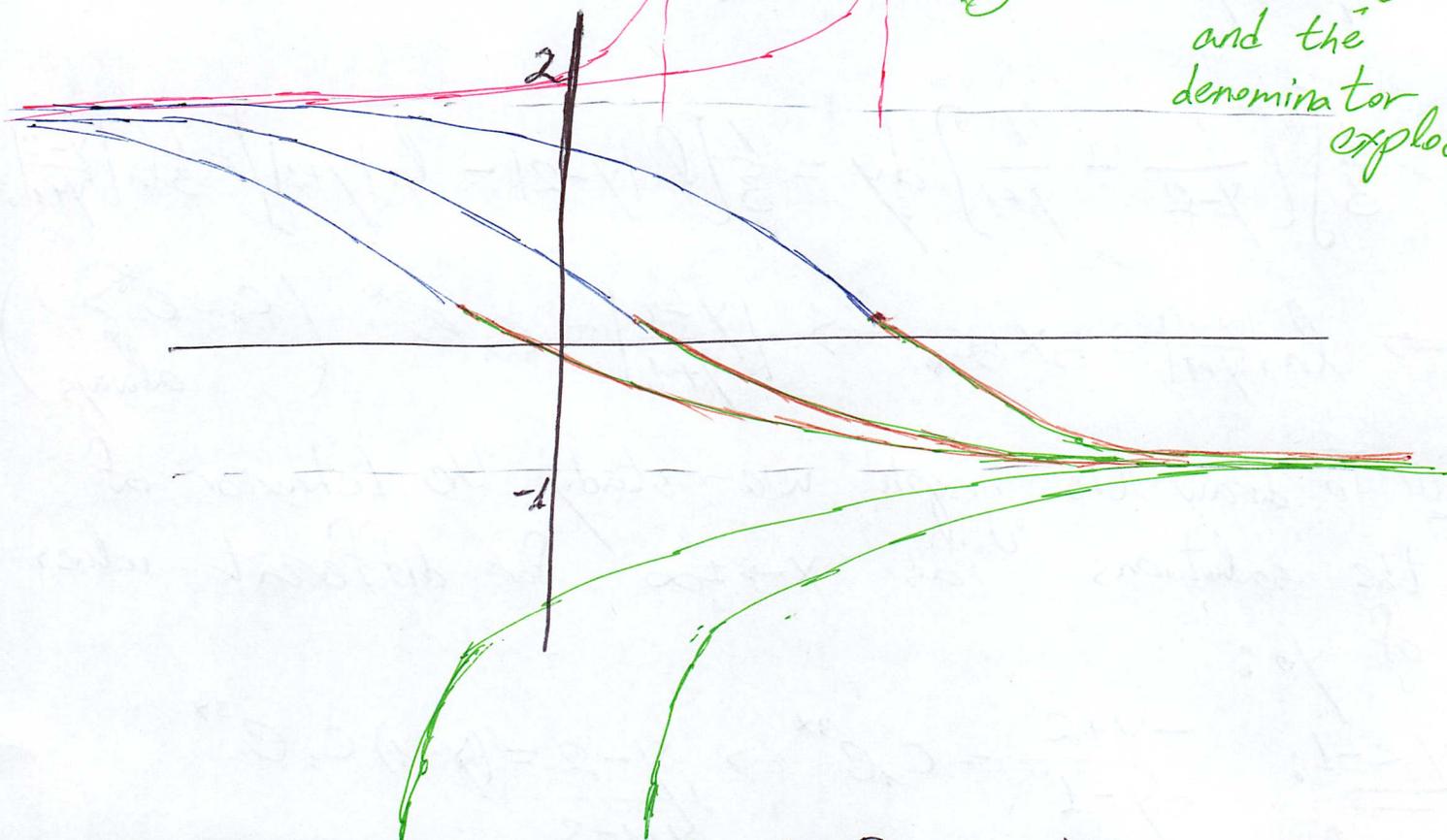
$\Rightarrow 3 = (y+1)(1 + C_1 e^{3x}) \Rightarrow \boxed{y = \frac{3}{1 + C_1 e^{3x}} - 1}$

$y_0 > 2$ :  $\frac{y-2}{y+1} = C_1 e^{3x}$  (this is the same equation as  $y_0 < -1$ )

For  $y_0 < -1$ :  $\lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} \left( \frac{3}{1 - ce^{3x}} - 1 \right) = -1^-$

$\lim_{x \rightarrow -\infty} y(x) = \lim_{x \rightarrow -\infty} \left( \frac{3}{1 - ce^{3x}} - 1 \right)$

for some  $x$   $ce^{3x} = 1$   
and the denominator explodes



For  $-1 < y_0 < 2$ :  $\lim_{x \rightarrow \infty} y(x) = \lim_{x \rightarrow \infty} \left( \frac{3}{1 + ce^{3x}} - 1 \right) = -1^+$

$\lim_{x \rightarrow -\infty} y(x) = \lim_{x \rightarrow -\infty} \left( \frac{3}{1 + ce^{3x}} - 1 \right) = 2^-$

For  $y_0 > 2$ : symmetric to  $y_0 < -1$

the denominator explodes for some point  $x$

Remarks: (1) The point  $x$  for which  $1 - c_1 e^{3x} = 0$  depends on  $y_0$  (since  $c_1$  depends on  $y_0$ ), but it is not important for our cause to find it exactly.

(2) We can see from the picture already from  $y_0 < -1$  and  $-1 < y_0 < 2$  that

(I)  $d = -1$  is a stable critical point.

(II)  $d = 2$  is an unstable critical point.

(and this answers (iv)).

# MAT303 - Calculus IV with Applications

## Additional Practice

- Section 1.3 - Questions 1-10, 18, 21, 22.
- Section 1.4 - Questions 1-29, 54-57, 59-63.
- Section 1.5 - Questions 1-28, 31-37.
- Section 1.6 - Questions 1-58, 63-65.
- Also:
  - $2\frac{y}{x} + 6y^{3/2} + (1 + 3x\sqrt{y})y' = 0.$
  - $y^2 \cos(xy) + (xy \cos(xy) + 2 \sin(xy))y' = 0.$
  - $e^{x^2y}(y + \frac{1}{2}xy') = 0.$
- Section 2.2 - Questions 1-10.

# MAT303 - Calculus IV with Applications

## Midterm

Instructor: Yaar Solomon

Thursday, March 12, 2015

Name (First - Last): \_\_\_\_\_

Stony Brook ID: \_\_\_\_\_

Signature: \_\_\_\_\_

### Instructions

1. Start when told to; stop when told to.
2. No notes, books, calculators, etc...
3. Turn OFF your cell phone and all other unauthorized electronic devices.
4. Write coherent mathematical statements and show your work on all problems.
5. Please write clearly.

(1) (25pts)	(2) (25pts)	(3) (25pts)	(4) (25pts)	TOTAL

1. (25 points)

Find the general solution for the equation, for  $y > 0$

Sol:

$$y + (2x - e^y)y' = 0.$$

$$M(x,y) = y \Rightarrow M_y = 1 \quad \text{not exact.}$$

$$N(x,y) = 2x - e^y \Rightarrow N_x = 2$$

$$\frac{N_x - M_y}{M} = \frac{2 - 1}{y} = \frac{1}{y} \quad (\text{depends only on } y) \Rightarrow \rho(x) = e^{\int \frac{1}{y} dx} = y$$

$$\Rightarrow y^2 + (2xy - ye^y)y' = 0 \quad \text{is exact.}$$

Check:  $M_y = 2y$ ,  $N_x = 2y$  ✓

$$F(x,y) = \int M(x,y) dx + g(y) = xy^2 + g(y)$$

$$\frac{\partial F}{\partial y} = N: \quad 2xy + g'(y) = N = 2xy - ye^y \Rightarrow g'(y) = -ye^y$$

$$\Rightarrow g(y) = -\int ye^y dy = \left[ \begin{array}{l} u=y \\ v=e^y \end{array} \right] = -ye^y + e^y + c$$

$$\Rightarrow F(x,y) = xy^2 - ye^y + e^y + c$$

general solution:

$$\boxed{xy^2 - ye^y + e^y + c = 0}$$

2. (25 points)

Find the general and the particular solution for the following initial value problem

$$x^2 y' + 2xy = y^3, \quad y(1) = 2 \quad (y \neq 0).$$

Sol: Bernoulli: equation with  $n=3$

$$u = y^{1-3} = y^{-2} \Rightarrow u' = -2y^{-3} \cdot y' \quad (y = u^{-1/2})$$

Divide the equation by  $y^3$  gives:

$$x^2 y^{-3} y' + 2xy^{-2} = 1 \Leftrightarrow -\frac{1}{2}x^2 \cdot u' + 2xu = 1$$

$$\Rightarrow u' - \frac{4}{x}u = -\frac{2}{x^2} \quad P(x) = -\frac{4}{x} \quad (\text{linear eq.})$$

$$\Rightarrow \rho(x) = e^{\int -\frac{4}{x} dx} = x^{-4} \quad \text{integrating factor.}$$

$$\Rightarrow \underbrace{u'x^{-4} - 4x^{-5}u}_{\frac{d}{dx}(u \cdot x^{-4})} = -2x^{-6} \Rightarrow u x^{-4} = -2 \int x^{-6} dx = \frac{2}{5}x^{-5} + C$$

$$\Rightarrow u = \frac{2}{5} \cdot \frac{1}{x} + Cx^4 \Rightarrow y = u^{-1/2} = \frac{1}{\sqrt{\frac{2}{5x} + Cx^4}}$$

$$\underline{y(1)=2}: \quad 2 = \frac{1}{\sqrt{\frac{2}{5} + C}} \Rightarrow 4 = \frac{1}{\frac{2}{5} + C} \Rightarrow C = \frac{1}{4} - \frac{2}{5}$$

$$\Rightarrow C = -\frac{3}{20}$$

$$\Rightarrow \boxed{y = \frac{1}{\sqrt{\frac{2}{5x} - \frac{3}{20}x^4}}}$$

3. (25 points)

A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at a rate of 2 gal/min, and the (perfectly mixed) solution leaves the tank at a rate of 3 gal/min; thus the tank is empty after exactly 1 hour. Find the amount of salt in the tank after  $t$  minutes.

Sol:  $V(0) = 60 \text{ gal}$      $r_i = 2$  ,  $r_o = 3 \text{ gal/min} \Rightarrow V(t) = 60 - t$

$C_i = 1$  ,  $C_o = \frac{X(t)}{V(t)} = \frac{X(t)}{60-t}$  ,  $X(0) = 0$ .

$\Rightarrow X' = 2 - \frac{3X}{60-t}$     linear eq.     $X' + \frac{3X}{60-t} = 2$

$P(t) = \frac{3}{60-t} \Rightarrow p(t) = e^{\int \frac{3}{60-t} dt} = e^{-3 \ln(60-t)} = \frac{1}{(60-t)^3}$

$\frac{X'}{(60-t)^3} + \frac{3X}{(60-t)^4} = \frac{2}{(60-t)^3} \Rightarrow X \cdot \frac{1}{(60-t)^3} = \int \frac{2}{(60-t)^3} dt$

$\frac{d}{dx} \left( X \cdot \frac{1}{(60-t)^3} \right)$

$= \frac{2}{-2} (60-t)^{-2} / (-1) + C$

inner derivative

$\Rightarrow \boxed{X(t) = (60-t) + C(60-t)^3}$

$X(0) = 0: 0 = 60 + C \cdot 60^3 \Rightarrow C = -\frac{60}{60^3} = -\frac{1}{3600}$

$\Rightarrow \boxed{X(t) = 60 - t - \frac{1}{3600} (60-t)^3}$

4. (25 points)

- (6 points) Define the notion of a critical point for the equation  $\frac{dx}{dt} = F(x)$ .
- (6 points) Define the notion of a stable critical point.
- (7 points) Solve the equation  $y' = y^2 - 6y + 8$ .
- (6 points) Find the critical points of the equation  $y' = y^2 - 6y + 8$ , decide for each one of them if it is stable or not, and carefully explain your decision in words (you do not have to use the  $\epsilon, \delta$ -definition from (b)).

Sol: (a) A critical point of  $\frac{dx}{dt} = F(x)$  is a number  $d$  such that  $F(d) = 0$  (zeros of  $F(x)$ ).

(b) A critical point  $d$  is called stable if for every  $\epsilon > 0$  there is  $\delta > 0$  such that any solution  $x(t)$  that satisfies  $|x(0) - d| < \delta$ , has  $|x(t) - d| < \epsilon$  for all  $t > 0$ .

(c)  $y' = y^2 - 6y + 8 = (y-2)(y-4) \Rightarrow \frac{y'}{(y-2)(y-4)} = 1 \Rightarrow$

$$\int \frac{dy}{(y-2)(y-4)} = \int 1 dx = x + C$$

$$\rightarrow \frac{1}{2} \int \left[ \frac{1}{y-4} - \frac{1}{y-2} \right] dy = \frac{1}{2} [\ln|y-4| - \ln|y-2|] = \frac{1}{2} \ln \left| \frac{y-4}{y-2} \right|$$

$$\Rightarrow \ln \left| \frac{y-4}{y-2} \right| = 2x + 2C \Rightarrow \boxed{\left| \frac{y-4}{y-2} \right| = C_1 e^{2x}} \quad (C_1 = e^{2C})$$

(d) The critical points are 2 and 4.

If  $2 < y(0) < 4$ :  $\frac{4-y}{y-2} = C e^{2x} \Rightarrow 2 - (y-2) = (y-2) C e^{2x}$   
 $\Rightarrow 2 = (y-2)(1 + C e^{2x}) \Rightarrow y = \frac{2}{1 + C_1 e^{2x}} + 2 \xrightarrow{x \rightarrow \infty} 2^+$

So The critical point  $d=4$  is unstable.

If  $y(0)$  is very close to 4 from below  $y(x) \xrightarrow{x \rightarrow \infty} 2$ , and in particular does not stay close to 4.

If  $y(0) < 2$ :  $\frac{4-y}{2-y} = c_1 e^{2x} \Rightarrow \frac{y-4}{y-2} = c_1 e^{2x} \Rightarrow y-2-2 = (y-2)c_1 e^{2x}$   
 $\Rightarrow (y-2)(1-c_1 e^{2x}) = 2 \rightarrow y = \frac{2}{1-c_1 e^{2x}} + 2 \xrightarrow{x \rightarrow \infty} 2$

We deduce that  $d=2$  is a stable critical point since if  $y(0)$  is close to 2 (either from above or from below) we have  $y(x) \xrightarrow{x \rightarrow \infty} 2$ . In particular, the solution curve stays close to 2.

## Exercise 7: Question 1 (F)

$$y'' + 2y' - 3y = 2xe^x$$

Solution: Homogeneous part  $y_c$ :

$$r^2 + 2r - 3 = 0 \rightsquigarrow (r-1)(r+3) = 0 \rightarrow r=1, r=-3$$

$$y_c(x) = C_1 e^x + C_2 e^{-3x}$$

Non-homogeneous part  $y_p$ :

Here  $f(x) = 2xe^x$ , and  $xe^x$  does not appear in  $y_c$ , but one of the terms of its derivative does. So we should follow RULE 2 here [read the rules carefully in the textbook].

Our usual trial solution would be here

$$\tilde{y}_p = Ax e^x + B e^x$$

and by RULE 2 we should multiply this by  $x^n$ , where  $n$  is the minimal integer so that none of the terms appears in  $y_c$ . So here  $n=1$ , and

$$y_p = Ax^2 e^x + Bx e^x$$

Then

$$\begin{aligned}y' &= \underline{2Ax}e^x + Ax^2e^x + Be^x + \underline{Bxe^x} \\ &= Be^x + (2A+B)xe^x + Ax^2e^x\end{aligned}$$

$$\begin{aligned}y'' &= \underline{Be^x} + \underline{(2A+B)e^x} + \underline{(2A+B)xe^x} + \underline{2Ax}e^x + Ax^2e^x \\ &= (2A+2B)e^x + (4A+B)xe^x + Ax^2e^x\end{aligned}$$

Then

$$2e^x = y'' + 2y' - 3y.$$

$$\begin{aligned}&= \underline{(2A+2B)e^x} + \underline{(4A+B)xe^x} + \underline{Ax^2e^x} + 2(\underline{Be^x} + \underline{(2A+B)xe^x} + \underline{Ax^2e^x}) \\ &\quad - 3(\underline{Ax^2e^x} + \underline{Bxe^x})\end{aligned}$$

$$= e^x(2A+4B) + xe^x(8A) + x^2e^x(0).$$

Compare coefficients: 
$$\begin{cases} 2A+4B = 0 \\ 8A = 2 \end{cases}$$

$$\Rightarrow A = \frac{1}{4}, B = -\frac{1}{8} \Rightarrow$$

$$y_p = \frac{1}{4}x^2e^x - \frac{1}{8}xe^x$$

General solution:

$$y(x) = c_1e^x + c_2e^{-3x} + \frac{1}{4}x^2e^x - \frac{1}{8}xe^x$$

# Homework 8 - Question 4(c)

$$\left. \begin{aligned} \text{(i)} \quad X_1' &= -2X_1 - 4X_2 + 2X_3 \\ \text{(ii)} \quad X_2' &= -2X_1 + X_2 + 2X_3 \\ \text{(iii)} \quad X_3' &= 4X_1 + 2X_2 + 5X_3 \end{aligned} \right\} \text{ using elimination (only)}$$

Sol:

$$\text{(i)} \Rightarrow X_3 = \frac{1}{2}X_2'' + X_1' - \frac{1}{2}X_2' \quad \textcircled{1}$$

$$\textcircled{1} \& \text{(ii)} \Rightarrow \frac{1}{2}X_2'' + X_1' - \frac{1}{2}X_2' = 4X_1 + 2X_2 + \frac{5}{2}X_2' + 5X_1 - \frac{5}{2}X_2 = 9X_1 - \frac{1}{2}X_2 + \frac{5}{2}X_2'$$

$$\cdot 2 \Rightarrow X_2'' + 2X_1' - X_2' = 18X_1 - X_2 + 5X_2'$$

$$\Rightarrow X_1 = \frac{1}{18} [X_2'' - 6X_2' + 2X_1' + X_2] \quad \textcircled{2}$$

$$\text{(i)} - \text{(ii)} \Rightarrow X_1' - X_2' = -5X_2 \Rightarrow X_1' = X_2' - 5X_2 \quad \textcircled{3}$$

$$\textcircled{2} \& \textcircled{3} \Rightarrow X_1 = \frac{1}{18} [X_2'' - 6X_2' + 2X_2' - 10X_2 + X_2] = \frac{1}{18} [X_2'' - 4X_2' - 9X_2]$$

$$\Rightarrow X_1' = \frac{1}{18} [X_2^{(3)} - 4X_2'' - 9X_2'] \quad \textcircled{4}$$

$$\textcircled{3} \& \textcircled{4} \Rightarrow X_2^{(3)} - 4X_2'' - 9X_2' = 18(X_2' - 5X_2) = 18X_2' - 90X_2$$

$$\Rightarrow \boxed{X_2^{(3)} - 4X_2'' - 27X_2' + 90X_2 = 0}$$

Char. eq.  $f(r) = r^3 - 4r^2 - 27r + 90 = 0$  rational roots  $\{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 9, \pm 10, \dots\}$

$$f(1) = 1 - 4 - 27 + 90 \neq 0$$

$$f(2) = 8 - 16 - 54 + 90 \neq 0$$

$$f(3) = 27 - 36 - 81 + 90 = 0 \Rightarrow \text{Divide by } (r-3)$$

$$\begin{array}{r}
 r^2 - r - 30 \\
 \hline
 r^3 - 4r^2 - 27r + 90 \quad | \quad r-3 \\
 \hline
 r^3 - 3r^2 \\
 \hline
 -r^2 - 27r + 90 \\
 -r^2 + 3r \\
 \hline
 -30r + 90 \\
 -30 + 90 \\
 \hline
 0
 \end{array}
 \Rightarrow
 \begin{cases}
 r^3 - 4r^2 - 27r + 90 = \\
 (r-3)(r^2 - r - 30) = \\
 (r-3)(r-6)(r+5) \Rightarrow \underline{\underline{\text{roots}}} \\
 \underline{\underline{3, 6, -5}}
 \end{cases}$$

$$\Rightarrow X_2(t) = c_1 e^{3t} + c_2 e^{6t} + c_3 e^{-5t}$$

$$\textcircled{4} \Rightarrow X_1(t) = \frac{1}{18} [X_2'' - 4X_2' - 9X_2] =$$

$$= \frac{1}{18} \left[ (9c_1 e^{3t} + 36c_2 e^{6t} + 25c_3 e^{-5t}) - 4(3c_1 e^{3t} + 6c_2 e^{6t} - 5c_3 e^{-5t}) + 9(c_1 e^{3t} + c_2 e^{6t} + c_3 e^{-5t}) \right]$$

$$= \frac{1}{18} [-12c_1 e^{3t} + 3c_2 e^{6t} + 36c_3 e^{-5t}] = \boxed{-\frac{2}{3}c_1 e^{3t} + \frac{1}{6}c_2 e^{6t} + 2c_3 e^{-5t}} \quad X_1(t)$$

$$\textcircled{1} \Rightarrow X_3(t) = \frac{1}{2} X_2'(t) + X_1(t) - \frac{1}{2} X_2(t) =$$

$$= \frac{1}{2} [3c_1 e^{3t} + 6c_2 e^{6t} - 5c_3 e^{-5t}] + \left[ -\frac{2}{3}c_1 e^{3t} + \frac{1}{6}c_2 e^{6t} + 2c_3 e^{-5t} \right] - \frac{1}{2} [c_1 e^{3t} + c_2 e^{6t} + c_3 e^{-5t}]$$

$$= \boxed{\frac{1}{3}c_1 e^{3t} + \frac{2}{3}c_2 e^{6t} - c_3 e^{-5t}} \quad X_3(t)$$

$$\text{So } \vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = C_1 \begin{pmatrix} -2/3 \\ 1 \\ 1/3 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1/6 \\ 1 \\ 2/3 \end{pmatrix} e^{6t} + C_3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} e^{-5t}$$

(we may multiply by constants to make the vector "nicer", since the  $C_1, C_2, C_3$  are arbitrary).

$$\Rightarrow \vec{X}(t) = C_1 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix} e^{6t} + C_3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} e^{-5t}$$

# MAT303 - Calculus IV with Applications

## Practice Exam

### Formulas

---

#### Laplace Transform:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$$

$$\mathcal{L}(f^{(3)}(t)) = s^3\mathcal{L}(f(t)) - s^2f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2 + k^2}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2 + k^2}$$

---

#### Torricelli's Law:

$$\frac{A(y) dy}{\sqrt{y}} \frac{dy}{dt} = -a\sqrt{2g},$$

where  $A(y)$  is the horizontal cross section of the tank at height  $y$ ,  $a$  is the area of the hole, and  $g$  is gravity.

---

#### Integration factors for exact equations:

$$e^{\int \frac{My - Nx}{N} dx} \quad \text{or} \quad e^{\int \frac{Nx - My}{M} dy}$$

---

## Part A

1. (14 points)

Find the general solution.

$$x(x+y)y' + y(3x+y) = 0.$$

Sol:  $M = y(3x+y) = 3xy + y^2 \Rightarrow M_y = 3x + 2y$  not exact.  
 $N = x(x+y) = x^2 + xy \Rightarrow N_x = 2x + y$

But  $\frac{M_y - N_x}{N} = \frac{x+y}{x(x+y)} = \frac{1}{x}$  (depends only on  $x$ !)

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\ln x} = x$$

$$\Rightarrow \underbrace{3x^2y + xy^2}_{\tilde{M}} + \underbrace{(x^3 + x^2y)}_{\tilde{N}} y' = 0 \quad \left\{ \begin{array}{l} \text{check} \\ \tilde{M}_y = 3x^2 + 2xy \\ \tilde{N}_x = 3x^2 + 2xy \end{array} \right. \checkmark$$

$$F(x,y) = \int \tilde{N} dy = x^3y + \frac{1}{2}x^2y^2 + h(x)$$

$$\tilde{M} = \frac{\partial F}{\partial x} = 3x^2y + xy^2 + h'(x) \Rightarrow h'(x) = 0 \Rightarrow h(x) = C$$

Solution:  $x^3y + \frac{1}{2}x^2y^2 = C$

2. (14 points)

Solve the following initial value problem.

$$\sec(x)y' = 2y + y^4, \quad y(\pi) = -2.$$

Sol:  $y' - 2\cos x y = \cos x \cdot y^4$  Bernoulli  $n=4$   
[  $u = y^{1-n} = y^{-3}$ ,  $u' = -3y^{-4} \cdot y'$  ]

$\frac{1}{y^4}$ :  $y^{-4} y' - 2\cos x y^{-3} = \cos x$   
 $-\frac{1}{3}u' - 2\cos x \cdot u = \cos x \Rightarrow u' + 6\cos x \cdot u = -3\cos x$

$$\mu(x) = e^{\int 6\cos x dx} = e^{6\sin x}$$

$$e^{6\sin x} u' + 6\cos x e^{6\sin x} \cdot u = -3e^{6\sin x} \cos x$$
$$\frac{d}{dx} (e^{6\sin x} \cdot u) \Rightarrow u \cdot e^{6\sin x} = -3 \int e^{6\sin x} \cos x dx \quad \left[ \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right]$$
$$= -3 \int e^{6t} dt = -\frac{1}{2} e^{6t} + C$$
$$= -\frac{1}{2} e^{6\sin x} + C$$

$$\Rightarrow u = -\frac{1}{2} + C e^{-6\sin x}$$

$$\Rightarrow y = u^{-1/3} = \frac{1}{\sqrt[3]{-\frac{1}{2} + C e^{-6\sin x}}}$$

$y(\pi) = -2$ :  $-2 = \frac{1}{\sqrt[3]{-\frac{1}{2} + C}} \Rightarrow -8 = \frac{1}{-\frac{1}{2} + C} \Rightarrow \frac{1}{2} + C = -\frac{1}{8} \Rightarrow C = -\frac{3}{8}$

$$y = \frac{1}{\sqrt[3]{-\frac{1}{2} + \frac{3}{8} e^{-6\sin x}}}$$

3. (14 points)

Find the general solution for the following system.

$$\begin{cases} x' = 5x + 9y \\ y' = -x - y \end{cases}$$

Sol:  $X = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow X' = \begin{pmatrix} 5 & 9 \\ -1 & -1 \end{pmatrix} X$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 5 & -9 \\ 1 & \lambda + 1 \end{vmatrix} = \lambda^2 - 4\lambda - 5 + 9 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$\lambda = 2$ :  $\begin{pmatrix} -3 & -9 \\ 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{V_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$V_2$ :  $(A - \lambda I)V_2 = V_1 \Rightarrow \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \underline{V_2} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Sol:  $X(t) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{2t} + c_2 e^{2t} \left( \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right)$   
 $= e^{2t} \left( c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right)$

$$x(t) = e^{2t} (-3c_1 - c_2 + 3c_2 t)$$

$$y(t) = e^{2t} (c_1 + c_2 t)$$

4. (14 points)

Find the general solution for the following system.

$$X' = \begin{pmatrix} -1 & -2 & 4 \\ 6 & 11 & -18 \\ 3 & 7 & -12 \end{pmatrix} X$$

Sol: 
$$\begin{vmatrix} \lambda+1 & 2 & -4 \\ -6 & \lambda-11 & 18 \\ -3 & -7 & \lambda+12 \end{vmatrix} = (\lambda+1)(\lambda^2 + \lambda - 132 + 126) - 2(-6\lambda + 72 + 54)$$

$$-4(42 + 3\lambda - 33) = (\lambda+1)(\lambda^2 + \lambda - 6) + 12(\lambda+3) - 12(\lambda+3)$$

$$= (\lambda+1)(\lambda+3)(\lambda-2) \Rightarrow \lambda = -1, -3, 2.$$

$\lambda = -1$ : 
$$\begin{pmatrix} 0 & 2 & -4 \\ -6 & -12 & 18 \\ -3 & -7 & 11 \end{pmatrix} \xrightarrow{1/6} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda = -3$ : 
$$\begin{pmatrix} -2 & 2 & -4 \\ -6 & -14 & 18 \\ -3 & -7 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 3 & 7 & -9 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 10 & -15 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

$\lambda = 2$ : 
$$\begin{pmatrix} 3 & 2 & -4 \\ -6 & -9 & 18 \\ -3 & -7 & 14 \end{pmatrix} \xrightarrow{1/6} \begin{pmatrix} 1 & 1/2 & -2 \\ 3 & 2 & -4 \\ -3 & -7 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & -2 \\ 0 & -5/2 & 5 \\ 0 & -5/2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$X(t) = C_1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} e^{-3t} + C_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{2t}$$

5. (14 points)

Find the general solution of the following non-homogeneous equation.

$$y''' - 13y' + 12y = 3e^x.$$

Sol: Char. poly.  $f(r) = r^3 - 13r + 12 = 0$  ( ~~$y_c$~~  first)

$$f(1) = 1 - 13 + 12 = 0 \Rightarrow r^3 - 13r + 12 = (r-1)(r^2 + r - 12) \\ = (r-1)(r+4)(r-3)$$

$r = 1, -4, 3$  are the roots.

$$\Rightarrow y_c = c_1 e^x + c_2 e^{-4x} + c_3 e^{3x}$$

$y_p$ : Rule 2, pick  $y_p(x) = Ax e^x \Rightarrow y_p'(x) = Ae^x + Ax e^x \\ \Rightarrow y_p''(x) = 2Ae^x + Ax e^x \Rightarrow y_p^{(3)}(x) = 3Ae^x + Ax e^x$

$$\Rightarrow 3Ae^x + Ax e^x - 13Ae^x - 13Ax e^x + 12Ax e^x = 3e^x$$

$$\Rightarrow -10Ae^x = 3e^x \Rightarrow A = -\frac{3}{10}$$

$$y_p(x) = -\frac{3}{10} x e^x$$

$$y(x) = c_1 e^x + c_2 e^{-4x} + c_3 e^{3x} - \frac{3}{10} x e^x$$

6. (14 points)

Use Laplace Transform to solve the following initial value problem.

$$x'' - 7x' + 6x = \sin(3t), x(0) = 3, x'(0) = 2.$$

Sol:  $\mathcal{L}(x'' + 5x' + 6x) = s^2 \mathcal{L}(x) - sx(0) - x'(0) + 5(s\mathcal{L}(x) - x(0))$

$$+ 6\mathcal{L}(x) = \mathcal{L}(x)(s^2 + 5s + 6) + s \cdot 5 + 5$$

$$\bullet \mathcal{L}(\sin(t)) = \frac{1}{s^2 + 1}$$

$$\Rightarrow \mathcal{L}(x) = \frac{1}{(s^2 + 1)(s^2 + 5s + 6)} + \frac{-s}{s^2 + 5s + 6} = \frac{-s^3 - s + 1}{(s^2 + 1)(s + 2)(s + 3)}$$

$$= \frac{As + B}{s^2 + 1} + \frac{C}{s + 2} + \frac{D}{s + 3} = \frac{As(s+2)(s+3) + B(s+2)(s+3) + C(s+2)(s^2+1) + D(s+2)(s^2+1)}{(s^2+1)(s+2)(s+3)}$$

$s=2$ :  $C \cdot (1) \cdot 5 = 8 + 2 + 1 \Rightarrow C = \frac{11}{5} = \frac{22}{10}$

$s=-3$ :  $D(-1) \cdot 10 = 27 + 3 + 1 \Rightarrow D = -\frac{31}{10}$

$s=0$ :  $1 = 6B + 3C + 2D \Rightarrow B = -\frac{1}{6} \left[ \frac{3 \cdot 22}{10} - \frac{2 \cdot 31}{10} \right] = -\frac{1}{6} \cdot \frac{4}{10} = -\frac{2}{30}$

coefficient of  $s^3$ :  $-1 = A + C + D \Rightarrow A = -1 - C - D = -1 - \frac{22}{10} + \frac{31}{10} = -\frac{1}{10}$

$$\Rightarrow \mathcal{L}(x) = -\frac{1}{10} \cdot \frac{s}{s^2 + 1} - \frac{2}{30} \cdot \frac{1}{s^2 + 1} + \frac{22}{10} \cdot \frac{1}{s + 2} - \frac{31}{10} \cdot \frac{1}{s + 3}$$

$$\stackrel{\mathcal{L}^{-1}}{\Rightarrow} \boxed{x(t) = -\frac{1}{10} \cos(t) - \frac{2}{30} \sin t + \frac{22}{10} e^{-2t} - \frac{31}{10} e^{-3t}}$$

## Part B - Theory

7. (4 points)

Which of the following is TRUE for a first order differential equation:

- (a) A linear equation is exact.
- (b) An exact equation is linear.
- (c) A linear equation has an integration factor that makes it exact.
- (d) Non of the above.

$$y' + p(x)y = q(x) \quad \begin{cases} M = p(x)y - q(x) \\ N = 1 \end{cases} \Rightarrow \begin{matrix} M_y = p(x) \\ N_x = 0 \end{matrix}$$
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int p(x) dx}$$

← (this integration factor that we use to solve it, makes it exact.)

8. (4 points)

$v_1 e^{\lambda_1 t}$ ,  $v_2 e^{\lambda_2 t}$  and  $v_3 t e^{\lambda_3 t}$  are three linearly independent solutions for the system  $X' = AX$  on some interval  $I$ , where  $A$  is a  $3 \times 3$  matrix (note the additional  $t$  in the third solution). Which of the following is necessarily FALSE:

- (a)  $\lambda_1 \neq \lambda_2$ .
- (b)  $\lambda_1 = \lambda_2 = \lambda_3$ .
- (c) The matrix  $A$  has 3 distinct eigenvalues.
- (d)  $\lambda_1 = \lambda_3$  or  $\lambda_2 = \lambda_3$  (or both).

9. (4 points)

Which of the following is TRUE for the differential equation

$$y^{(2)} + a_1 y' + a_0 y = 0.$$

(a) It always has a unique solution which is defined on  $\mathbb{R}$ .

(b) Given real numbers  $a, b_1, b_2$ , it has a unique solution on  $\mathbb{R}$  that satisfies

$$y(a) = b_1, y'(a) = b_2.$$

(c) Given real numbers  $a, b_1, b_2$ , it has a unique solution on  $\mathbb{R}$  that satisfies

$$y(b_1) = a, y(b_2) = a.$$

(d) Non of the above.

The existence & uniqueness Thm tells you that (b) is correct. The solution may be complex, but still it is defined on  $\mathbb{R}$ . It will be unique on the domain where  $a_1(t), a_0(t)$  are continuous  $\rightarrow$  everywhere (here)

10. (4 points)

It is given that

$$\det \begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix} = 0$$

On some interval  $I$ . Which of the following is TRUE:

(a)  $\{x_1(t), x_2(t)\}$  is a linearly dependent set of functions.

(b) If  $\{x_1(t), x_2(t)\}$  is a linearly dependent set of functions then  $x_1(t), x_2(t)$  are solutions to an equation of the form  $x'' + a_1(t)x' + a_0(t)x = 0$ .

(c) If  $x_1(t), x_2(t)$  are solutions to an equation of the form  $x'' + a_1(t)x' + a_0(t)x = 0$  then  $\{x_1(t), x_2(t)\}$  is a linearly dependent set of functions.

(d) All of the above.

↑  
using (b)  
of the Thm  
on Wronskian