MAT 303: Calculus IV with Applications
Fall 2012

Welcome to MAT 303

MAT 303: Calculus IV with Applications
Fall 2012

General Information

Differential equation is an equation relating an unknown function and its derivatives. Various scientific laws can be translated into differential equations. The course is dedicated to standard techniques for solving ordinary differential equations, including numerical methods, and their applications in different branches of science such as physics, biology, chemistry, economics and social sciences.

Instructor:
Artem Dudko, artem.dudko@stonybrook.edu
Lectures: MWF 12:00-12:53
Office hours: MW 2:00-2:53 (Math Tower 3114) and F 11:00-11:53 (Math Learning Center, Math Tower S-240A)

Teacing Assistant:
Fadi Elkhatib
Tutorials: W 10:00-10:53 (Physics P113) and F 1:00-1:53 (Library E4330)

Textbook: Edwards & Penney, Differential Equations with Boundary Value Problems: Computing and Modelling, Fourth Edition, Prentice Hall (Chapters 1-6)

Topics: an introduction to first order differential equations; phase plane analysis; numerical methods; higher order linear equations and systems; nonlinear phenomena.

Prerequisite is completion of one of the standard calculus sequences (either MAT 125-127 or MAT 131-132 or MAT 141-142) with a grade C or higher in MAT 127, 132 or 142 or AMS 161. Also, MAT 203/205 (Calculus III) and AMS 261/MAT 211 (Linear Algebra) are recommended. Informally, students should know integration and differentiation techniques and, desirably, be familiar with complex numbers and basic aspects of linear algebra.

Tests, quizzes, assignments:
Every week there will be either a homework assignment or a quiz (alternating). The quizzes will be written on Fridays during the last 20 minutes of the class. You should hand in your assignments to the instructor during Friday class. For instance, the first quiz is on Friday,

August 31. The first homework assignment will be due on Friday, September 7. No late assignments will be accepted.
Midterm Test I: Monday, October 1.
Midterm Test II: Monday, November 5.
Final Exam: Thursday, December 13, 5:30PM-8:00PM
Last day of classes: Friday, December 7.

Course grade is computed by the following scheme:
Homework and Quizzes: 20%
Midterm Test I: 20%
Midterm Test II: 20%
Final Exam: 40%

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

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Midterm 1 solutions

Midterm 2 solutions

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Recommended problems

Problems from the book to exercise

MAT 303 Assignment 1.

Hand in to the instructor in class on Friday, September 7.

Problem 1. Verify by substitution that the functions

$$y_1 = e^{-x} \text{ and } y_2 = xe^{-x}$$

are solutions of the differential equation

$$y'' + 2y' + y = 0.$$

Problem 2. Find by inspection at least one solution of the differential equation

$$xy' - y = 1/x^3.$$

Problem 3. Find the general solution of the differential equation

$$\tfrac{dx}{dt} = te^{t^2} - t.$$

Problem 4. Solve the initial value problem

$$\tfrac{dx}{dt} = \tfrac{1}{\sqrt{1-t^2}},$$
$$x(0) = 1.$$

Problem 5. A function g satisfies the following condition. Every straight line normal to the graph of the function g passes through the point $(0, 0)$. Write a differential equation having g as its solution. Give an example of a function g satisfying this condition.

Problem 6. A car travelling at $60 mi/h$ $(88 ft/s)$ skids $176 ft$ after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

Problem 7. Show by substitution that the formula

$$y(x) = \frac{2}{1+Ce^x} - 1, \qquad (1)$$

where C is a constant, gives *a general* solution of the differential equation

$$2y' = y^2 - 1.$$

Show that formula (1) is not *the general* solution of the given equation by finding a solution which is not described by (1).

MAT 303 Assignment 2.

Hand in to the instructor in class on Friday, September 22.

Problem 1. Consider the differential equation

$$y' = (x+y)^{\frac{1}{3}} - 1. \qquad (1)$$

Describe all pairs of numbers (x_0, y_0) for which Theorem of Existence and Uniqueness guaranties that the initial value problem $y(x_0) = y_0$ has a unique solution.

Problem 2. Solve the differential equation (1). Describe all pairs (x_0, y_0) for which the initial value problem $y(x_0) = y_0$:

a) has a unique solution,
b) do not have any solutions,
c) has more than one solution.

Problem 3. Separate variables and use partial fractions to solve the initial value problem

$$\tfrac{dx}{dt} = 3x(5-x), \quad x(0) = 8.$$

Problem 4. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of $5L/s$, and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

Problem 5. Verify that the given differential equation is exact; then solve it.

$$\tfrac{1}{x}\sin y\, dx + (\ln x \cos y + y)dy = 0.$$

Problem 6. Show that the following differential equation is homogeneous:

$$x(\ln x - \ln t + 1)dt = t dx.$$

Solve the initial value problem $x(1) = 1$.

Problem 7. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

MAT 303 Assignment 3.

Hand in to the instructor in class on Monday, October 15.

Problem 1. Let $P(t)$ be a rabbit population satisfying the extinction-explosion equation

$$\frac{dP}{dt} = aP^2 - bP.$$

The threshold population is $M = 300$ rabbits. In 1975 the population is 150 rabbits. In 1985 the population is 100 rabbits. Find the size of the population in 1995.

Problem 2. Find all critical points of the autonomous differential equation

$$\frac{dx}{dt} = (x-1)^2(x-2)(x-3).$$

Determine their types (stable, unstable, or semistable).

Problem 3. Apply Euler's method with step size $h = 0.25$ to find approximate value of the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y}{x^2+1}, \quad y(2) = 1$$

at point $x = 3$.

Problem 4. Verify that functions $y_1 = e^{-x}\sin x$, $y_2 = e^{-x}\cos x$ are solutions of the differential equation

$$y'' + 2y' + 2y = 0.$$

Find a solution of this equation of the form $y = c_1 y_1 + c_2 y_2$ such that

$$y(\pi) = 0, \quad y'(\pi) = 1.$$

Problem 5. Show that the functions $y_1 = x$, $y_2 = x^{-2}$ are solutions of $x^2y'' + 2xy' - 2y = 0$. Find a solution y of this differential equation such that $y(-1) = -3, y'(-1) = 0$.

Problem 6. Show that the functions $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

Problem 7. Find constants b, c such that the quadratic function $y(x) = x^2+bx+c$ is a solution of the second order differential equation

$$y'' - 2y' + y = x^2.$$

Problem 8. Find the general solution of the second order differential equation

$$y'' - 5y' + 4y = 0.$$

MAT 303 Assignment 4.

Hand in to the instructor in class on Monday, October 29.

Problem 1. Show directly that the functions

$$f(x) = 3x - 2,\ g(x) = 2x^2 - x,\ h(x) = 3x^2 - 1$$

are linearly dependent on the real line. That is, find constants c_1, c_2, c_3 (not all equal to zero) such that $c_1 f(x) + c_2 g(x) + c_3 h(x)$ is identically equal to zero.

Problem 2. Using Wronskian, show that the functions

$$f(x) = e^x,\ g(x) = \sin x,\ h(x) = x \sin x$$

are linearly independent on the real line.

Problem 3. 1) Verify that the functions $y_1 = 1,\ y_2 = x^3,\ y_3 = \ln x$ are solutions of the differential equation

$$x^2 y^{(3)} - 2y' = 0.$$

2) Show that y_1, y_2, y_3 are linearly independent.
3) Solve the initial value problem

$$y(1) = 2,\ y'(1) = 2,\ y''(1) = 7.$$

Problem 4. Find the general solution of the differential equation

$$y^{(4)} - \tfrac{3}{2}y'' + \tfrac{1}{2}y = 0.$$

Problem 5. Assume that a homogeneous differential equation with constant coefficients has the characteristic equation of the form

$$(2r - 3)(r - 1)^2(r + 2)^2 = 0.$$

Using polynomial differential operators show that $y = xe^{-2x}$ is a particular solution of this differential equation.

Problem 6. Solve the initial value problem

$$y'' + 2y' + 5y = 0, \;\; y(\pi) = 0, \; y'(\pi) = 1.$$

Problem 7. Assume that the roots of a characteristic polynomial of a homogeneous differential equation with constant coefficients are:

$$0, 0, 0, 1, 1 + 3i, 1 - 3i, 5.$$

Write the general solution of this differential equation.

MAT 303 Assignment 5.

Hand in to the instructor in class on Monday, November 26.

Problem 1. Find the general solution of

$$y'' + 2y = x \sin x.$$

Problem 2. Solve the initial value problem

$$y'' - 2y' + y = e^x, \;\; y(1) = 2e, \; y'(1) = 4e.$$

Problem 3. Solve the initial value problem

$$x'' + 16x = 7\cos 3t, \;\; x(\tfrac{\pi}{2}) = -1, \; x'(\tfrac{\pi}{2}) = 3.$$

Write the solution in the form $x(t) = C\sin(\omega t + \alpha)\sin(\omega_0 t + \alpha)$. Sketch the graph of $x(t)$.

Problem 4. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \;\; B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Calculate the product AB. Verify that $\det(AB) = \det(A)\det(B)$.

Problem 5. Solve the following system of differential equations by reducing it to a single second order differential equation:

$$x_1' = x_1 + 2x_2, \; x_2' = -2x_1 + x_2, \;\; x_1(0) = 1, \; x_2(0) = 2.$$

Problem 6. Using Wronskian show that the following vector-functions are linearly independent:

$$\begin{bmatrix} 1 \\ x \\ x^2+1 \end{bmatrix}, \;\; \begin{bmatrix} x^2-1 \\ 2x \\ 1 \end{bmatrix}, \;\; \begin{bmatrix} x^2 \\ 3x \\ x^2-1 \end{bmatrix}.$$

Problem 7. Show that there is no continuous 2×2 matrix-function $P(t)$ on $\mathbb{R}$ such that the vector-functions

$$X_1 = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \; X_2 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

are solutions of the system $X' = PX$ on $\mathbb{R}$.

MAT 303 Assignment 6.

Hand in to the instructor in class on Friday, December 7.

Problem 1. Solve the initial value problem

$$x' = \frac{2}{t}x + \frac{y}{t},\ y' = (t - \frac{2}{t})x - \frac{y}{t},\ x(\pi) = 0,\ y(\pi) = -2\pi$$

using the method of elimination.

Problem 2. Solve the nonhomogeneous system

$$x_1' = -3x_1 + 4x_2 - 4e^t,\quad x_2' = 6x_1 - 5x_2 + 6e^t.$$

Problem 3. Prove that the trajectories of the system

$$x' = y,\ y' = x$$

are hyperbolas.

In Problems 4–5, given a matrix A and a vector X_0, solve the initial value problem

$$X' = AX,\quad X(0) = X_0$$

using the eigenvalue method.

Problem 4.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 0 \\ -1 & 5 & 2 \end{bmatrix},\quad X_0 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}.$$

Problem 5.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix},\quad X_0 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$

In problems 6, 7 solve the system $X' = AX$, determine the type of the critical point 0 and sketch the phase portrait of the system.

Problem 6.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

Problem 7.

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}.$$

MAT 303 Computational project (for extra credit).
Print your project and hand in to the instructor in class on Friday, November 29.

For your own personal computational project, let a, b, c, d be the four last digits of your student ID number, set

$$p = \tfrac{10a+b+1}{10},\ q = \tfrac{10c+d+1}{10}.$$

Problem 1. Consider the differential equation

$$\tfrac{dy}{dx} = \tfrac{1}{p}\cos(x - yp). \tag{1}$$

(a) Find a symbolic general solution using a computer algebra system (like Maple or Mathematica).

(b) Determine the symbolic particular solution corresponding to several typical initial conditions of the form $y(x_0) = y_0$.

(c) Determine the possible values m, n such that the straight line $y = mx + n$ is a solution curve of the equation (1).

(d) Plot a direction field and some typical solution curves. Can you make a connection between the symbolic solution and your (linear and nonlinear) solution curves?

Problem 2. Consider the homogeneous differential equation with constant coefficients

$$y^{(3)} + py' + qy = 0. \tag{2}$$

(a) Find the roots of the corresponding characteristic equation using a computer algebra system. Define the corresponding particular solutions y_1, y_2, y_3 of the equation (2).

(b) Consider the general solution $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ of (2). Take sample nonzero values c_1, c_2, c_3 of your choice and let $Y(x)$ be the particular solution of (2), corresponding to these values. Plot $Y(x)$.

(c) Calculate the numbers $s_0 = Y(1), s_1 = Y'(1), s_2 = Y''(1)$. Using a computer algebra system, solve the initial value problem

$$y^{(3)} + py' + qy = 0, \quad y(1) = s_0, y'(1) = s_1, y''(1) = s_2.$$

Plot the solution and compare it with the function $Y(x)$.

Problem 3. Using a computer algebra system, solve the system of linear differential equations

$$x' = x + py, y' = qx + y.$$

Draw the direction field corresponding to this system and a few solution curves, illustrating the behavior of the general solution.

Quiz 1 with solutions

Problem 1. Show that the function $y(x) = \sqrt{x}e^x$ is a solution of the equation

$$2xy' = (2x+1)y$$

on the infinite segment $(0, +\infty)$.

Solution. Using the product rule for differentiation, we obtain

$$y' = (\sqrt{x}e^x)' = \tfrac{1}{2\sqrt{x}}e^x + \sqrt{x}e^x.$$

Thus,

$$2xy' = \sqrt{x}e^x + 2x\sqrt{x}e^x = (1+2x)\sqrt{x}e^x = (2x+1)y.$$

Problem 2. Solve the initial value problem

$$\begin{aligned} \tfrac{dx}{dt} &= 2t^2 + \cos t - 1, \\ x(0) &= 1. \end{aligned}$$

Solution. We have:

$$dx = (2t^2 + \cos t - 1)dt.$$

Integrating, we obtain:

$$x = \int (2t^2 + \cos t - 1)dt = \tfrac{2}{3}t^3 + \sin t - t + C.$$

Substituting $t = 0$, we find: $1 = x(0) = C$. Thus,

$$x(t) = \tfrac{2}{3}t^3 + \sin t - t + 1.$$

Quiz 2 with solutions

Problem 1. Solve the initial value problem:

$$y' = 2xy^2 + y^2 + 2x + 1, \;\; y(0) = 1.$$

Solution. The right-hand side of the equation can be rewritten as $(2x+1)(y^2+1)$. Thus, the equation is separable. Separating the variables and integrating, we obtain:

$$\begin{aligned} \tfrac{dy}{y^2+1} &= (2x+1)dx, \\ \int \tfrac{dy}{y^2+1} &= \int (2x+1)dx, \\ \arctan y &= x^2 + x + C. \end{aligned}$$

Substituting $x = 0$ we find that

$$C = \arctan(y(0)) = \arctan(1) = \tfrac{\pi}{4}.$$

Thus, the answer is

$$y(x) = \tan(x^2 + x + \tfrac{\pi}{4}).$$

Problem 2. Find the general solution of the first-order differential equation:

$$t^2 \tfrac{dx}{dt} + x = 1.$$

Describe all pairs (a, b) for which the initial value problem $x(a) = b$ does not have a solution.

This equation can be solved either as linear or as separable. In the quiz, to present one solution is enough.

Solution 1 (linear). The equation can be rewritten as linear first-order differential equation:

$$\tfrac{dx}{dt} + \tfrac{x}{t^2} = \tfrac{1}{t^2}.$$

The integrating factor is:

$$\rho(t) = \exp(\textstyle\int \tfrac{1}{t^2} dt) = \exp(-\tfrac{1}{t}).$$

Multiplying both sides of the original equation by $\rho(t)$, we obtain

$$\tfrac{d}{dt}(\exp(-\tfrac{1}{t})x) = \exp(-\tfrac{1}{t})\tfrac{1}{t^2},$$

$$\exp(-\tfrac{1}{t})x = \int \exp(-\tfrac{1}{t})\tfrac{1}{t^2}dt =$$

$$\int \exp(-\tfrac{1}{t})d(-\tfrac{1}{t}) = \exp(-\tfrac{1}{t}) + C.$$

Thus, $x(t) = 1 + C\exp(\frac{1}{t})$.

Solution 2 (separable). The equation can be rewritten as a linear first-order differential equation

$$\tfrac{dx}{x-1} = -\tfrac{dt}{t^2}$$

(assuming $x \neq 1$). Integrating, we obtain:

$$\ln|x-1| = \tfrac{1}{t}, \ \ (x-1) = \pm A\exp(-\tfrac{1}{t}).$$

Now, since we divided by $x-1$, we need to check $x = 1$. Substituting $x = 1$ into the original equation we see that $x = 1$ is a solution. Therefore, the general solution is

$$x(t) = 1 + C\exp(\tfrac{1}{t}),$$

where C is any number.

Pairs (a, b)**.**For each pair (a, b) such that $b \neq 0$ the constant C can be found uniquely:

$$b = 1 + C\exp(\tfrac{1}{a}) \Rightarrow C = (b-1)\exp(-\tfrac{1}{a}).$$

Thus, for such pair (a, b) the solution of the initial value problem exists and is unique. However, point $t = 0$ belongs to the domain of the definition of $x(t)$ only for $C = 0$ (when $x(t) = 1$ for all t). This solution satisfies the initial condition $x(0) = 1$. Therefore, for each $b \neq 1$ the initial condition $x(0) = b$ can not be satisfied. Thus, the initial value problem does not have a solution for the pairs of the form $(0, b)$, where $b \neq 1$.

Quiz 3 solutions

Problem 1. Suppose that a body moves through a resisting medium with resistance proportional to the square of its velocity v, so that

$$\frac{dv}{dt} = -kv^2.$$

The body starts moving with the velocity $10m/s$. After 10 seconds its velocity decreases to $5m/s$. Find the velocity of the body in $1.5min$ ($90s$) after it started moving.

Solution. To find a formula for the velocity let us solve the given differential equation. Separating the variables, we obtain:

$$\frac{dv}{v^2} = -kdt, \quad -\frac{1}{v} = -kt + C, \quad v(t) = \frac{1}{kt - C}.$$

Substituting $t = 0$ and using the initial condition $v(0) = 10$ we get:

$$\frac{1}{-C} = 10 \ \Rightarrow \ C = -0.1, \ v(t) = \frac{1}{kt + 0.1}.$$

Substituting $t = 10$ and using the condition $v(10) = 5$ we get:

$$\frac{1}{10k + 0.1} = 5 \ \Rightarrow \ k = 0.01, \ v(t) = \frac{1}{0.01t + 0.1}.$$

Thus, the velocity after 90 seconds is $v(90) = \frac{1}{0.9+0.1} = 1$.

Problem 2. Given the initial value problem

$$\frac{dx}{dt} = \frac{t}{x}, \quad x(0) = 1$$

apply Euler's method with step size $h = 0.1$ to find an approximation of the value $x(0.2)$.

Solution. Since the initial condition is given at $t = 0$ we set $t_0 = 0$. Construct according to the Euler's method points $t_k = t_0 + kh$: $t_1 = 0.1, \ t_2 = 0.2$. Given $x_0 = x(0) = 1$ we define inductively approximations x_k of $x(t_k)$ by the formula

$$x_{k+1} = x_k + hf(t_k, x_k) = x_k + \frac{0.1t_k}{x_k}.$$

We have:

$$x_1 = x_0 + \frac{0.1t_0}{x_0} = 1, \quad x_2 = x_1 + \frac{0.1t_1}{x_1} = 1 + \frac{0.1 \cdot 0.1}{1} = 1.01.$$

Thus, $x(0.2)$ is approximately equal to $x_2 = 1.01$.

Quiz 4 solutions

Problem 1. Solve the initial value problem:

$$2y'' + y' - y = 0, \;\; y(0) = 3, \; y'(0) = 0.$$

Solution. To solve the homogeneous differential equation with constant coefficients we first solve the corresponding characteristic equation:

$$2r^2 + r - 1 = 0, \;\; r_{1,2} = \tfrac{-1\pm\sqrt{1+4\cdot 2}}{2\cdot 2}, \; r_1 = -1, r_2 = \tfrac{1}{2}.$$

Thus, we have particular solutions

$$y_1 = e^{r_1 x} = e^{-x}, \; y_2 = e^{r_2 x} = e^{x/2}$$

and the general solution is of the form

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{x/2}.$$

To solve the initial value problem plug $x = 0$ and use the initial conditions:

$$3 = y(0) = c_1 + c_2, \; 0 = y'(0) = -c_1 + \tfrac{1}{2}c_2 \;\; \Rightarrow \;\; c_2 = 2c_1, 3c_1 = 3, c_1 = 1, c_2 = 2.$$

Thus, $y(x) = e^{-x} + 2e^{x/2}$.

Problem 2. Using Wronskian, prove that the functions $y_1 = x, \; y_2 = \ln x, \; y_3 = x \ln x$ are linearly independent on $(0, +\infty)$.

Solution. By definition, Wronskian of y_1, y_2, y_3 is the following determinant:

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & \ln x & x \ln x \\ 1 & \frac{1}{x} & 1 + \ln x \\ 0 & -\frac{1}{x^2} & \frac{1}{x} \end{vmatrix}.$$

According to Theorem on Wronskian, if Wronskian is not identically equal to zero (that is, there exists a point x for which $W(x) \neq 0$) then the functions are linearly independent. Thus, we need to show that $W(x)$ is not identically equal to zero.

One of the ways, we can first calculate the determinant. Then we obtain

$$W(x) = \tfrac{2}{x} - \tfrac{\ln x}{x}.$$

One has: $W(x) = 0$ if and only if $\ln x = 2$, that is $x = e^2$. Therefore, $W(x)$ is not identically equal to 0.

Another way, instead of calculating the determinant for arbitrary x (which can be messy), we can guess a point x at which the determinant is not equal to zero and for which calculations are simple. For instance, take $x = 1$. Then we have:

$$W(1) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}.$$

This determinant is much simpler to calculate. We have $W(1) = 2$. Therefore, $W(x)$ is not identically equal to zero and functions are linearly independent.

MAT 303 Assignment 2 solutions.

Problem 1. Consider the differential equation

$$y' = (x+y)^{\frac{1}{3}} - 1. \tag{1}$$

Describe all pairs of numbers (x_0, y_0) for which Theorem of Existence and Uniqueness guaranties that the initial value problem $y(x_0) = y_0$ has a unique solution.

Solution. Theorem of Existence states the following. If $f(x,y)$ is continuous in some rectangle containing the point (x_0, y_0), then the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

has a solution (not necessarily unique) on some interval containing x_0. Since $f(x,y) = (x+y)^{\frac{1}{3}} - 1$ is continuous everywhere, Theorem of Existence guaranties existence of a solution of (1) for all (x_0, y_0).

Theorem of Uniqueness states the following. If $f(x,y)$ and $\frac{d}{dy}f(x,y)$ both are continuous in some rectangle containing point (x_0, y_0), then the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0$$

has a *unique* solution on some interval containing x_0. The functions

$$f(x,y) = (x+y)^{\frac{1}{3}} - 1, \text{ and } \frac{d}{dy}f(x,y) = \frac{1}{3}(x+y)^{-\frac{2}{3}}$$

are continuous at all points except where $x = -y$. Therefore, Theorem of Uniqueness guaranties uniqueness of solution for all

(x_0, y_0) except the points for which $x_0 = -y_0$.

Problem 2. Solve the differential equation (1). Describe all pairs (x_0, y_0) for which the initial value problem $y(x_0) = y_0$:

$a)$ has a unique solution,

$b)$ do not have any solutions,

$c)$ has more than one solution.

Solution. Substitute $v = x + y$. We have: $\frac{dv}{dx} = \frac{dy}{dx} + 1$. Therefore,

$$\tfrac{dv}{dx} = v^{\frac{1}{3}} - 1 + 1 = v^{\frac{1}{3}}.$$

This is a separable equation. Assuming $v \neq 0$, we obtain:

$$v^{-\frac{1}{3}} dv = dx, \quad \int v^{-\frac{1}{3}} dv = \int dx,$$

$$\tfrac{3}{2} v^{\frac{2}{3}} = x + C \Rightarrow v = \pm(\tfrac{2}{3}x + \tfrac{2}{3}C)^{\frac{3}{2}}.$$

Thus,

$$y = \pm(\tfrac{2}{3}x + \tfrac{2}{3}C)^{\frac{3}{2}} - x.$$

Since we assumed $v \neq 0$, we need to check whether $v = 0$ is a solution. Clearly, it is a solution. Therefore, $y = -x$ is a particular solution of (1). Thus, the solutions of (1) are

$$y = \pm(\tfrac{2}{3}x + C)^{\frac{3}{2}} - x, \text{ where } C \text{ is a constant; } y = -x. \quad (2)$$

Let (x_0, y_0) be a pair of numbers. Let us find all solutions of (1) such that $y(x_0) = y_0$. Observe that the solution $y = -x$ satisfies the initial condition $y(x_0) = y_0$ if and only if $y_0 = -x_0$. The solution $y = \pm(\frac{2}{3}x + C)^{\frac{3}{2}} - x$ satisfies the initial condition $y(x_0) = y_0$ if and only if

$$y_0 = \pm(\tfrac{2}{3}x_0 + C)^{\frac{3}{2}} - x_0.$$

This condition defines C uniquely, namely:

$$C = (y_0 + x_0)^{\frac{2}{3}} - \tfrac{2}{3}x_0.$$

Thus, if $y_0 \neq -x_0$, then the initial value problem has a unique solution and it is of the form

$$y = (\tfrac{2}{3}x + C)^{\frac{3}{2}} - x \qquad (3)$$

for some C. If $y_0 = -x_0$ then the initial value problem has 2 solutions: $y = -x$ and a solution of the form (3). The answer for the second part of the problem is:
a) all pairs such that $y_0 \neq -x_0$,
b) no such pairs;
c) all pairs of the form $(x_0, -x_0)$.

Problem 3. Separate variables and use partial fractions to solve the initial value problem

$$\tfrac{dx}{dt} = 3x(5-x), \quad x(0) = 8.$$

Solution. Observe that $x = 0$ and $x = 5$ are particular solutions of the equation $\frac{dx}{dt} = 3x(5-x)$. If $x \neq 0$ and $x \neq 5$, we have:

$$\tfrac{dx}{3x(5-x)} = dt.$$

Use partial fractions:

$$\tfrac{1}{x(5-x)} = \tfrac{A}{x} + \tfrac{B}{5-x} = \tfrac{A(5-x)+Bx}{x(5-x)}.$$

Then $5A + (B-A)x = 1$. Therefore, $A = \frac{1}{5}$ and $B = A$. Thus,

$$\int \tfrac{dx}{3x(5-x)} = \tfrac{1}{15}\int (\tfrac{1}{x} + \tfrac{1}{5-x})dx = \tfrac{1}{15}(\ln|x| - \ln|5-x|) = \tfrac{1}{15}\ln|\tfrac{x}{5-x}| = t + C.$$

Exponentiating, we obtain:

$$\tfrac{x}{5-x} = Ke^{15t}, \ \text{ where } \ K = const.$$

It is convenient to plug the initial condidion $x(0) = 8$ in this formula. We obtain:

$$-\tfrac{8}{3} = K.$$

Thus,

$$x(t) = \tfrac{5Ke^{15t}}{1+Ke^{15t}} = \tfrac{40e^{15t}}{8e^{15t}-3}.$$

Problem 4. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of $5L/s$, and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

Solution. Observe that no new salt is coming into the tank, but some salt is leaving the tank with the solution. Thus, the amount of salt in the tank changes with time. Let $x(t)$ be the amount of salt in the tank at time t. Then the ratio of the salt in the solution is $x/1000$ (kg/L). Therefore, salt leaves the tank with the speed

$$5 \cdot x/1000 = x/200 \ (kg/s).$$

We obtain that

$$\tfrac{dx}{dt} = -x/200.$$

Solving this separable equation we find

$$\tfrac{dx}{x} = -\tfrac{dt}{200}, \ \ \ln|x| = C - \tfrac{t}{200},$$
$$x = Ae^{-\tfrac{t}{200}}, \ \text{ where } \ A = \text{const}.$$

Observe that $x(0) = 100\ kg$ (the amount of salt at the beginning). Therefore, $A = 100$. Thus, $x(t) = 100e^{-\frac{t}{200}}$. Solving the equation

$$100e^{-\frac{t}{200}} = 10$$

we find that 10 kg of salt remains after $t = -200 \ln \frac{1}{10} \approx 461$ seconds.

Problem 5. Verify that the given differential equation is exact; then solve it.

$$\tfrac{1}{x} \sin y \, dx + (\ln x \cos y + y)dy = 0.$$

Solution. By Criteria of Exactness, a differential equation $M(x, y)dx + N(x, y)dy$ is exact if and only if $\frac{dM}{dy} = \frac{dN}{dx}$. We have:

$$\tfrac{d}{dy}(\tfrac{1}{x} \sin y) = \tfrac{1}{x} \cos y, \quad \tfrac{d}{dx}(\ln x \cos y + y) = \tfrac{1}{x} \cos y.$$

Therefore, the equation is exact. To solve it, we need to find $F(x, y)$ such that

$$\tfrac{dF}{dx} = \tfrac{1}{x} \sin y, \quad \tfrac{dF}{dy} = \ln x \cos y + y.$$

Then the solution would be $F(x, y) = C$, where C is arbitrary constant.

From the formula for $\frac{dF}{dx}$ we obtain:

$$F = \int \tfrac{1}{x} \sin y dx = \ln x \sin y + g(y).$$

Plugging this formula into the formula for $\frac{dF}{dy}$ we obtain:

$$\ln x \cos y + y = \tfrac{dF}{dy} = \tfrac{d}{dy}(\ln x \sin y + g(y)) = \ln x \cos y + g'(y).$$

Thus, $g'(y) = y, \;\; g(y) = \frac{y^2}{2} + \text{const}$. For simplicity, take const $=$ 0. Then

$$F(x, y) = \ln x \sin y + \tfrac{y^2}{2}.$$

Thus, the solution of the exact equation is:

$$\ln x \sin y + \tfrac{y^2}{2} = C.$$

Problem 6. Show that the following differential equation is homogeneous:

$$x(\ln x - \ln t + 1)dt = tdx.$$

Solve the initial value problem $x(1) = 1$.

Solution. A first-order differential equation is called homogeneous if it can be written in the form

$$\tfrac{dx}{dt} = F(x/t).$$

We have:

$$\tfrac{dx}{dt} = \tfrac{x}{t}(\ln x - \ln t + 1) = \tfrac{x}{t}(\ln \tfrac{x}{t} + 1).$$

Therefore, this differential equation is homogeneous. To solve it, use the substitution $v = \frac{x}{t}$. One has:

$$x = vt \Rightarrow \tfrac{dx}{dt} = v + t\tfrac{dv}{dt}.$$

We obtain:

$$v + t\tfrac{dv}{dt} = v(\ln v + 1), \quad \tfrac{dv}{v \ln v} = \tfrac{dt}{t},$$

$$\int \tfrac{dv}{v \ln v} = \int \tfrac{dt}{t}, \quad \ln |\ln v| = \ln |t| + C.$$

Thus,

$$\ln v = Kt, \quad \text{or} \quad v = e^{Kt},$$

where K is some constant. Finally,

$$x(t) = vt = te^{Kt}.$$

Substituting the initial condition $x(1) = 1$ we obtain: $1 = 1e^K \Rightarrow K = 0$. Thus, the solution of the initial value problem is $x(t) = t$.

Problem 7. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

Solution. This rabbit population $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = k\sqrt{P}, \tag{4}$$

where k is some constant. According to the conditions of the problem, we have: $P(0) = 100$, $\frac{dP}{dt} = 20$. Thus, plugging $t = 0$ into the equation (4) we obtain:

$$20 = k\sqrt{100} \Rightarrow k = 2.$$

Solving the differential equation $\frac{dP}{dt} = 2\sqrt{P}$ we find:

$$\frac{dP}{2\sqrt{P}} = dt, \quad \sqrt{P} = t + C.$$

For $t = 0$ we obtain: $\sqrt{100} = 0 + C \Rightarrow C = 10$. Thus,

$$P(t) = (10 + t)^2.$$

So, after 1 year (12 months) there will be $P(12) = (10 + 12)^2 = 484$ rabbits.

Assignment 3 solutions.

Problem 1. Let $P(t)$ be a rabbit population satisfying the extinction-explosion equation

$$\frac{dP}{dt} = aP^2 - bP.$$

The threshold population is $M = 300$ rabbits. In 1975 the population is 150 rabbits. In 1985 the population is 100 rabbits. Find the size of the population in 1995.

Solution. We know that the solution of the extinction-explosion equation is of the form

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}.$$

Let time $t = 0$ measured in years correspond to year 1975. Then $P_0 = P(0) = 150$. Thus,

$$P(t) = \frac{300 \cdot 150}{150 + 150e^{300kt}} = \frac{300}{1 + e^{300kt}}.$$

Year 1985 correspond to $t = 10$. Thus,

$$100 = P(10) = \frac{300}{1 + e^{3000k}}, \quad e^{3000k} = 2, \quad k = \frac{\ln 2}{3000}.$$

Year 1995 correspond to $t = 20$. We obtain:

$$P(20) = \frac{300}{1 + e^{6000k}}.$$

Observe that $e^{6000k} = (e^{3000k})^2 = 2^2 = 4$. Thus, $P(20) = \frac{300}{5} = 60$. There are 60 rabbits in 1995.

Problem 2. Find all critical points of the autonomous differential equation

$$\frac{dx}{dt} = (x-1)^2(x-2)(x-3). \qquad (1)$$

Determine their types (stable, unstable, or semistable).

Solution. Critical points of an autonomous differential equation $\frac{dx}{dt} = f(x)$ are solutions of $f(x) = 0$. Thus, there are 3 critical points $x_1 = 1, x_2 = 3, x_3 = 3$. Using (1) we obtain:

$$x < 1 \Rightarrow \frac{dx}{dt} > 0, x(t) \nearrow, \quad 1 < x < 2 \Rightarrow \frac{dx}{dt} > 0, x(t) \nearrow,$$
$$2 < x < 3 \Rightarrow \frac{dx}{dt} < 0, x(t) \searrow, \; 3 < x \Rightarrow \frac{dx}{dt} > 0, x(t) \nearrow .$$

Thus, near 1 solutions from one side approach 1, from other side escape from 1, therefore $x_1 = 1$ is a semistable point. Near 2 solutions approach 2 from both sides, $x_2 = 2$ is a stable point. Near 3 solutions escape from 3, $x_3 = 3$ is an unstable point.

Problem 3. Apply Euler's method with step size $h = 0.25$ to find approximate value of the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y}{x^2+1}, \quad y(2) = 1$$

at point $x = 3$.

Solution. Since the initial condition is given at $x_0 = a = 2$ and we are asked to find a value at $b = 3$, we will use the Euler's method for the segment $[a, b] = [2, 3]$. We have: $x_k = x_0 + kh = 2+0.25k$. Thus, $3 = x_4$. Inductively, define approximations y_k of $y(x_k)$ by the formulas $y_0 = 1, \; y_{k+1} = y_k + hf(x_k, y_k) = y_k + \frac{0.25 y_k}{x_k^2+1}$. We obtain the following values:

$$y_1 = 1.05, \; y_2 \approx 1.0933, \; y_3 \approx 1.1310, \; y_4 \approx 1.1640.$$

Thus, $y(3) \approx 1.1640$.

Problem 4. Verify that functions $y_1 = e^{-x}\sin x,\ y_2 = e^{-x}\cos x$ are solutions of the differential equation

$$y'' + 2y' + 2 = 0.$$

Find a solution of this equation of the form $y = c_1y_1 + c_2y_2$ such that

$$y(\pi) = 0, \ \ y'(\pi) = 1.$$

Solution. We have:

$$\begin{aligned} y_1' = -e^{-x}\sin x + e^{-x}\cos x,\ y_1'' &= -2e^{-x}\cos x, \\ y_1'' + 2y_1' + 2y_1 &= -2e^{-x}\cos x + \\ 2(-e^{-x}\sin x + e^{-x}\cos x) + 2e^{-x}\sin x &= 0, \end{aligned}$$

therefor, y_1 is a solution. Similarly, we obtain that y_2 is a solution. Let $y = c_1y_1 + c_2y_2$ satisfy the initial condition. Then

$$\begin{aligned} 0 = y(\pi) = c_1y_1(\pi) + c_2y_2(\pi) = -c_2e^{-\pi} \ &\Rightarrow \ c_2 = 0, \\ 1 = y'(\pi) = c_1y_1'(\pi) = -c_1e^{-\pi} \ &\Rightarrow \ c_1 = -e^{\pi}. \end{aligned}$$

Thus, $y(x) = -e^{\pi - x}\sin x$.

Problem 5. Show that the functions $y_1 = x,\ y_2 = x^{-2}$ are solutions of $x^2y'' + 2xy' - 2y = 0$. Find a solution y of this differential equation such that $y(-1) = -3, y'(-1) = 0$.

Solution. We have:

$$\begin{aligned} x^2y_1'' + 2xy_1' - 2y_1 &= 2x - 2x = 0, \\ x^2y_2'' + 2xy_2' - 2y_2 &= 6x^{-2} - 4x^{-2} - 2x^{-2} = 0, \end{aligned}$$

therefor, y_1 and y_2 are solutions. By Principle of superposition, the general solution is of the form: $y = c_1y_1 + c_2y_2 = c_1x + c_2x^{-2}$.

Plugging $x = -1$ and using the initial conditions, we obtain:

$$-c_1 + c_2 = -3,\ c_1 + 2c_2 = 0 \quad \Rightarrow \quad c_1 = 2, c_2 = -1.$$

Thus, $y(x) = 2x - x^{-2}$.

Problem 6. Show that the functions $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

Solution. Since $y_1' = 0, y_1'' = 0$, y_1, obviously, is a solution. One has:

$$y_2 y_2'' + (y_2')^2 = x^{1/2}(-\tfrac{1}{4})x^{-3/2} + (\tfrac{1}{2}x^{-1/2})^2 = 0,$$

therefor y_2 is a solution. Further, for $y = y_1 + y_2$ we have

$$yy'' + (y')^2 = (1 + x^{1/2})(-\tfrac{1}{4})x^{-3/2} + (\tfrac{1}{2}x^{-1/2})^2 = -\tfrac{1}{4}x^{-3/2}.$$

Thus, $y = y_1 + y_2$ is not a solution.

Problem 7. Find constants b, c such that the quadratic function $y(x) = x^2 + bx + c$ is a solution of the second order differential equation

$$y'' - 2y' + y = x^2.$$

Solution. We have:

$$\begin{aligned} y' = 2x + b,\ y'' = 2,\ x^2 = y'' - 2y' + y = 2 - 2(2x + b) + \\ (x^2 + bx + c) = x^2 + (b - 4)x + (c - 2b + 2), \end{aligned}$$

therefor, $b = 4, c = 2b - 2 = 6$. Thus, $y(x) = x^2 + 4x + 6$.

Problem 8. Find the general solution of the second order differential equation

$$y'' - 5y' + 4y = 0.$$

Solution. Solving the characteristic equation $r^2 - 5r + 4 = 0$, we obtain the roots

$$r_{1,2} = \frac{5 \pm \sqrt{5^2 - 4 \cdot 4}}{2} = \frac{5 \pm 3}{2}, \quad r_1 = 1, r_2 = 4.$$

Therefore, we have particular solutions $y_1 = e^x, y_2 = e^{4x}$ and the general solution is of the form:

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{4x}.$$

MAT 303 Assignment 4 solutions.

Problem 1. Show directly that the functions

$$f(x) = 3x - 2,\ g(x) = 2x^2 - x,\ h(x) = 3x^2 - 1$$

are linearly dependent on the real line. That is, find constants c_1, c_2, c_3 (not all equal to zero) such that $c_1 f(x) + c_2 g(x) + c_3 h(x)$ is identically equal to zero.

Solution. We have

$$c_1(3x - 2) + c_2(2x^2 - x) + c_3(3x^2 - 1) =$$
$$(2c_2 + 3c_3)x^2 + (3c_1 - c_2)x + (-2c_1 - c_3).$$

In order for this expression to be identically equal to zero we need that each of the coefficients is zero: $2c_2 + 3c_3 = 0,\ 3c_1 - c_2 = 0,\ 2c_1 + c_3 = 0$ (otherwise the expression would be a polynomial which is not identically zero). From the second and the third equations we get $c_2 = 3c_1, c_3 = -2c_1$. Plugging this into the first equation we obtain: $2 \cdot 3c_1 - 3 \cdot 2c_2 = 0$, which is always true. Thus, we can take any $c_1 \neq 0$, say $c_1 = 1$, and find the corresponding values $c_2 = 3, c_3 = -2$ for which $c_1 f(x) + c_2 g(x) + c_3 h(x)$ is identically equal to zero.

Problem 2. Using Wronskian, show that the functions

$$f(x) = e^x,\ g(x) = \sin x,\ h(x) = x \sin x$$

are linearly independent on the real line.

Solution. By Theorem on Wronskian, we need to show that

$W(x)$ is not identically equal to 0, that is, there exists x_0 for which $W(x_0) \neq 0$. We have:

$$W(x) = \begin{bmatrix} e^x & \sin x & x \sin x \\ e^x & \cos x & x \cos x + \sin x \\ e^x & -\sin x & -x \sin x + 2 \cos x. \end{bmatrix}$$

Guess a point x_0 for which $W(x_0)$ is easy to calculate and $W(x_0) \neq 0$. Take $x_0 = 0$. We have:

$$W(0) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = 2 \neq 0.$$

Therefor, the functions are linearly independent.

Problem 3. 1) Verify that the functions $y_1 = 1,\ y_2 = x^3,\ y_3 = \ln x$ are solutions of the differential equation

$$x^2 y^{(3)} - 2y' = 0.$$

2) Show that y_1, y_2, y_3 are linearly independent.
3) Solve the initial value problem

$$y(1) = 2,\ y'(1) = 2,\ y''(1) = 7.$$

Solution. 1) One has: $y_1' = y_1^{(3)} = 0,\ \ y_2' = 3x^2, y_2^{(3)} = 6,\ y_3' = \frac{1}{x}, y_3^{(3)} = \frac{2}{x^3}$. Plugging each of the functions y_1, y_2, y_3 into the equation we obtain 0. Therefor, they are solutions.

2) To verify linear independence let us calculate the Wronskian:

$$W(x) = \begin{bmatrix} 1 & x^3 & \ln x \\ 0 & 3x^2 & \frac{1}{x} \\ 0 & 6x & -\frac{1}{x^2} \end{bmatrix},\ \ W(1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & -1 \end{bmatrix} = -9 \neq 0.$$

By Theorem on Wronskian, the functions are linearly independent.

3) By Principle of Superposition, the general solution is of the form:

$$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 + c_2 x^3 + c_3 \ln x.$$

Plugging the initial conditions we obtain:

$$c_1 + c_2 = 2,\ 3c_2 + c_3 = 2,\ 6c_2 - c_3 = 7.$$

Solving this system, we find: $c_1 = 1, c_2 = 1, c_3 = -1$. Thus, $y(x) = 1 + x^3 - \ln x$.

Problem 4. Find the general solution of the differential equation

$$y^{(4)} - \tfrac{3}{2} y'' + \tfrac{1}{2} y = 0.$$

Solution. First, let us solve the characteristic equation: $r^4 - \frac{3}{2} r + \frac{1}{2} = 0$. If we set $s = r^2$ we obtain a quadratic equation: $s^2 - \frac{3}{2} s + \frac{1}{2} = 0$. Its solutions are: $s_1 = -1, s_2 = \frac{1}{2}$. Now we can find r by solving: $r^2 = -1 \ \Rightarrow\ r_{1,2} = \pm i$ and $r^2 = \frac{1}{2} \ \Rightarrow\ r_{3,4} = \pm \frac{1}{\sqrt{2}}$. The complex roots $\pm i$ give solutions $y_1 = \cos x, y_2 = \sin x$. The real roots $\pm \frac{1}{\sqrt{2}}$ give solutions $y_{3,4} = e^{\pm \frac{x}{\sqrt{2}}}$. Thus, the general solution is:

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 e^{-\frac{x}{\sqrt{2}}} + c_4 e^{\frac{x}{\sqrt{2}}}.$$

Problem 5. Assume that a homogeneous differential equation with constant coefficients has the characteristic equation of the form

$$(2r - 3)(r - 1)^2 (r + 2)^2 = 0.$$

Using polynomial differential operators show that $y = xe^{-2x}$ is a particular solution of this differential equation.

Solution. A homogeneous differential equation with constant coefficient with the characteristic polynomial $p(r)$ can be written as $p(D)y = 0$. Thus, the equation described in the problem can be written in terms of the polynomial differential operator as follows:

$$(2D-3)(D-1)^2(D+2)^2 y = 0.$$

For $y = xe^{-2x}$ we have:

$$(D+2)y = y' + 2y = e^{-2x}, \ \ (D+2)^2 y = (D+2)(e^{-2x}) = (e^{-2x})' + 2e^{-2x} = 0.$$

Therefor, $(2D-3)(D-1)^2(D+2)^2 y = (2D-3)(D-1)^2 0 = 0$, which shows that y is a solution.

Problem 6. Solve the initial value problem

$$y'' + 2y' + 5y = 0, \ \ y(\pi) = 0, \ y'(\pi) = 1.$$

Solution. First, solve the characteristic equation: $r^2 + 2r + 5 = 0$. We have: $r_{1,2} = -1 \pm 2i$. Therefor, the general solution is: $y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$. To find c_1, c_2 plug the initial conditions. We have,

$$y'(x) = e^{-x}((-c_1 + 2c_2)\cos 2x - (2c_1 + c_2)\sin 2x).$$

Thus,

$$0 = y(\pi) = e^{-\pi} c_1 \ \Rightarrow \ c_1 = 0, \ \ 1 = e^{-\pi}(-c_1 + 2c_2) \ \Rightarrow \ c_2 = \tfrac{e^{\pi}}{2}.$$

The answer is $y(x) = \frac{e^{\pi}}{2} e^{-x} \sin 2x$.

Problem 7. Assume that the roots of a characteristic polynomial of a homogeneous differential equation with constant coefficients are:

$$0, 0, 0, 1, 1 + 3i, 1 - 3i, 5.$$

Write the general solution of this differential equation.

Answer: $y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^x + e^x (c_5 \cos 3x + c_6 \sin 3x) + c_7 e^{5x}$.

Assignment 5 solutions.

Problem 1. First, find the general solution y_c of the associated homogeneous equation $y'' + 2y = 0$. The characteristic equation is $r^2 + 2 = 0$, its solutions are $r_{1,2} = \pm i\sqrt{2}$. Thus, the general solution of the homogeneous equation is $y_c(x) = c_1\sqrt{2}\sin x + c_2\sqrt{2}\cos x$.

Now, let us find a particular solution of the given non-homogeneous equation. Let $f(x) = x\sin x$. Then

$$f'(x) = \sin x + x\cos x, f''(x) = 2\cos x - x\sin x,$$
$$f^{(3)}(x) = -3\sin x - x\cos x.$$

Thus, $f^{(3)}(x)$ does not have any new terms ($\sin x$ and $x\cos x$ appeared already in $f'(x)$). Therefor, the function $f(x)$ and its derivatives contain only the following terms:

$$\sin x, \cos x, x\sin x, x\cos x.$$

Neither one of these terms solves the associated homogeneous equation. Therefor, as a particular solution we should try

$$y_p(x) = a\sin x + b\cos x + cx\sin x + dx\cos x.$$

We have:

$$x\sin x = y_p'' + 2y_p = -a\sin x - b\cos x + c(2\cos x - x\sin x) +$$
$$d(-2\sin x - x\cos x) + 2(a\sin x + b\cos x + cx\sin x + dx\cos x).$$

Comparing coefficient in front of each term, we get: $a - 2d = 0, b + 2c = 0, c = 1, d = 0$. Thus, $a = 2d = 0, b = -2c = -2$.

Answer: $y_p(x) = x\sin x - 2\cos x$.

Problem 2. First, let us find the general solution of the given non-homogeneous equation. The characteristic equation $r^2 - 2r + 1 = 0$, $r_{1,2} = 1$. Therefor, the solution of the associated homogeneous equation is: $y_c(x) = (c_1 + c_2 x)e^x$. For the particular solution of the non-homogeneous equation, our first guess is ae^x. However, it is a solution of the homogeneous equation. Therefor, we need to multiply our guess by x^s so that $ax^s e^x$ would not be a solution of the homogeneous equation. Clearly, $s = 1$ does not work (xe^x is also a solution of the homogeneous equation), but $s = 2$ works. Set $y_p(x) = ax^2 e^x$. Then we have:

$$e^x = y_p'' - 2y_p + y_p = a((2+4x+x^2)e^x - 2(2x+x^2)e^x + x^2 e^x) = 2ae^x.$$

Therefor, $a = \frac{1}{2}$ and $y_p(x) = \frac{x^2}{2}e^x$.

Now, to solve the initial value problem plug the conditions into the general solution $y(x) = (c_1 + c_2 x)e^x + \frac{x^2}{2}$ of the non-homogeneous equation:

$$2e = y(1) = (c_1 + c_2)e + \tfrac{e}{2}, \quad 4e = y'(1) = (c_1 + 2c_2)e + \tfrac{3e}{2},$$
$$c_2 e + e = 2e, c_2 = 1, \quad c_1 = 2 - 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Answer: $y(x) = (\frac{1}{2} + x + \frac{x^2}{2})e^x$.

Problem 3. The solution of the associated homogeneous equation $x'' + 16x = 0$ is $x_c(t) = c_1 \sin 4t + c_2 \cos 4t$. Since $\cos 3t$ and $\sin 3t$ are not solutions of the homogeneous equation, as a particular solution of the non-homogeneous equation we may try $x_p(t) = a \sin 3t + b \cos 3t$. We have:

$$7\cos 3t = x_p'' + 16x_p = (-9a \sin 3t - 9b \cos 3t) + 16(a \sin 3t + b \cos 3t).$$

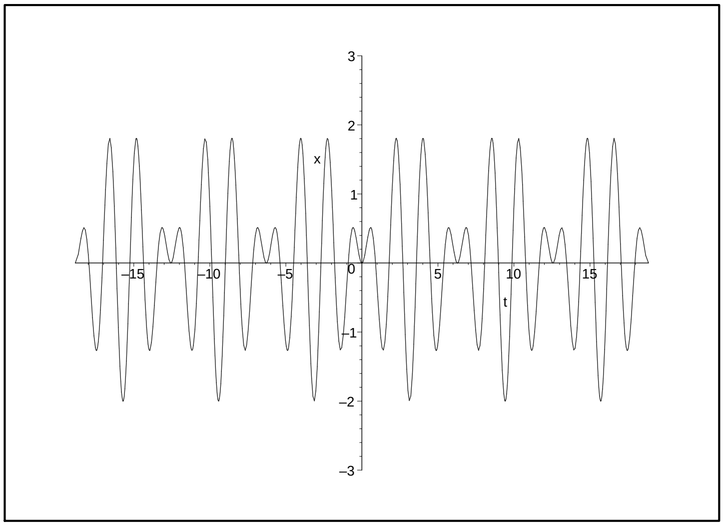

Figure 1: Graph of the solution.

Thus, $a = 0, b = 1$. The general solution of the non-homogeneous equation is:

$$x(t) = x_c + x_p = c_1 \sin 4t + c_2 \cos 4t + \cos 3t.$$

For the solution of the initial value problem we have:

$$-1 = x(\tfrac{\pi}{2}) = c_2, \quad 3 = x'(\tfrac{\pi}{2}) = 4c_1 + 3, \quad c_1 = 0.$$

Thus, $x(t) = -\cos 4t + \cos 3t$.

To write the solution in the required form we use the formula:

$$\cos u - \cos v = 2 \sin \tfrac{v-u}{2} \sin \tfrac{v+u}{2}.$$

We obtain:

$$x(t) = \cos 3t - \cos 4t = 2 \sin \tfrac{t}{2} \sin \tfrac{7t}{2}.$$

Problem 4. One has:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

So,

$$det(A) = -2, det(B) = 0, det(AB) = 0 = det(A)det(B).$$

Problem 5. From the first equation we get: $x_2 = \frac{x_1' - x_1}{2}$. Therefor, $x_2' = \frac{x_1'' - x_1'}{2}$. Plugging these formulas into the second equation we obtain:

$$\frac{x_1'' - x_1'}{2} = -2x_1 + \frac{x_1' - x_1}{2}.$$

Thus, $x_1'' - 2x_1' + 5x_1 = 0$. We have:

$$r^2 - 2r + 5 = 0, \;\; r_{1,2} = 1 \pm 2i, \; x_1(t) = e^t(c_1 \cos 2t + c_2 \sin 2t).$$

Further, $x_2 = \frac{x_1' - x_1}{2} = e^t(c_2 \cos 2t - c_1 \sin 2t)$. Plugging the initial conditions we get:

$$1 = x_1(0) = c_1, \; 2 = x_2(0) = c_2.$$

Answer: $x_1(t) = e^t(\cos 2t + 2 \sin 2t), \; x_2(t) = e^t(2 \cos 2t - \sin 2t)$.

Problem 6. The Wronskian of the given three vector functions is the following determinant:

$$W(x) = \begin{vmatrix} 1 & x^2 - 1 & x^2 \\ x & 2x & 3x \\ x^2 + 1 & 1 & x^2 - 1 \end{vmatrix}.$$

By Theorem on Wronskian, we need to show that $W(x)$ is not identically equal to 0, that is there exist at least one point x_0 for which $W(x_0) \neq 0$. Try to guess such a value x_0. Let $x_0 = 0$. It is easy to see that $W(0) = 0$, so 0 does not work. Let $x_0 = 1$. Then we have:

$$W(1) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{vmatrix} = -6.$$

Thus, Wronskian is not identically zero, therefor, the vector functions are linearly independent.

Problem 7. Assume that X_1 and X_2 are solutions of $X' = PX$ on the real line for some continuous function $P(t)$. Then by Theorem on Wronskian, one of two options hold:
a) X_1 and X_2 are linearly independent, then their Wronskian is never 0;
b) X_1 and X_2 are linearly dependent, then their Wronskian is everywhere 0.
However, we have $W(t) = \cos^2 t - \sin^2 t$. Thus, $W(\pi/4) = 0$, but $W(0) = 1$, so neither a) nor b) is true. This contradiction shows that X_1 and X_2 cannot be solutions of such a system.

MAT 303 Assignment 6 solutions.

Problem 1. Solve the initial value problem

$$x' = \tfrac{2}{t}x + \tfrac{y}{t},\ y' = (t - \tfrac{2}{t})x - \tfrac{y}{t},\ x(\pi) = 0,\ y(\pi) = -2\pi$$

using the method of elimination.

Solution. From the first equation we obtain: $y = tx' - 2x$. Therefor, $y' = tx'' - x'$. Plugging these formulas into the second equation we get:

$$tx'' - x' = (t - \tfrac{2}{t})x - \tfrac{tx'-2x}{t}, \ \text{ or } \ t(x'' - x) = 0.$$

Dividing by t we get $x'' - x = 0$. The solution of this differential equation is $x(t) = c_1e^t + c_2e^{-t}$. Thus,

$$y = tx' - 2x = t(c_1e^t - c_2e^{-t}) - 2(c_1e^t + c_2e^{-t}).$$

Plugging the initial conditions, we obtain:

$$0 = x(\pi) = c_1e^{\pi} + c_2e^{-\pi}, \ \text{ therefor } \ c_2 = -c_1e^{2\pi},$$
$$-2\pi = y(\pi) = c_1(\pi - 2)e^{\pi} - c_2(\pi + 2)e^{-\pi} = 2c_1\pi e^{2\pi}.$$

Thus, $c_1 = -e^{-2\pi},\ c_2 = 1$. Finally, $x(t) = -e^{t-2\pi} + e^{-t},\ y(t) = -t(e^{t-2\pi} + e^{-t}) + 2(e^{t-2\pi} - e^{-t})$.

Problem 2. Solve the nonhomogeneous system

$$x_1' = -3x_1 + 4x_2 - 4e^t, \ \ x_2' = 6x_1 - 5x_2 + 6e^t.$$

Solution. In the matrix form this equation can be written as

$$X' = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} X + \begin{bmatrix} -4e^t \\ 6e^t \end{bmatrix}.$$

The general solution of the non-homogeneous system is of the form $X = X_c + X_p$, where X_c is the general solution of the associated homogeneous equation

$$X' = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} X$$

and X_p is a particular solution of the nonhomogeneous equation. First, find X_c. The characteristic equation is $(-3-\lambda)(-5-\lambda)-24 = \lambda^2 + 8\lambda - 9 = 0$. Its solutions are $\lambda_1 = -9, \lambda_2 = 1$. Find the corresponding eigenvectors:

$$\lambda_1 = -9 : \begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,\ 6v_1 + 4v_2 = 0,\ v_2 = -\tfrac{3}{2}v_1,\ V_1 = \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix};$$

$$\lambda_2 = 1 : \begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,\ v_1 = v_2,\ V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, $X_c = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 = c_1 e^{-9t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let's look for a particular solution of the form $X_p(t) = e^t V$, where $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a constant vector. Plugging X_p into the nonhomogeneous equation and dividing by e^t we obtain:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$

Thus, $v_1 = -3v_1 + 4v_2 - 4,\ v_2 = 6v_1 - 5v_2 + 6$. We see that the two equations are equivalent to one equation $v_2 = v_1 + 1$. Setting $v_1 = 0, v_2 = 1$ we obtain a particular solution $X_p(t) = e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus,

$$X(t) = X_c + X_p = c_1 e^{-9t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Problem 3. Prove that the trajectories of the system

$$x' = y, \; y' = x$$

are hyperbolas.

Solution. Solving this system we obtain:

$$x(t) = c_1 e^t + c_2 e^{-t}, \; y(t) = c_1 e^t - c_2 e^{-t}.$$

To show that trajectories $(x(t), y(t))$ are hyperbolas we need to find a quadratic relation between $x(t)$ and $y(t)$. We have:

$$x^2(t) = c_1^2 e^{2t} + 2c_1 c_2 + c_2^2 e^{-2t}, \; y^2(t) = c_1^2 e^{2t} - 2c_1 c_2 + c_2^2 e^{-2t}.$$

Thus, for every solution we have $x^2(t) - y^(t) = 4c_1 c_2 = C$, where C is a constant. For $C \neq 0$ this is an equation of a hyperbola. For $C = 0$ we have $x^2 = y^2, x = \pm y$, which describes two lines. Notice that lines can be considered as degenerate hyperbolas.

In Problems 4–5, given a matrix A and a vector X_0, solve the initial value problem

$$X' = AX, \quad X(0) = X_0$$

using the eigenvalue method.

Problem 4.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 0 \\ -1 & 5 & 2 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}.$$

Solution. The characteristic equation is:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & -3-\lambda & 0 \\ -1 & 5 & 2-\lambda \end{vmatrix} = (-\lambda - 3)((2-\lambda)^2 + 1) = 0.$$

Thus, $\lambda_1 = -3, \lambda_{2,3} = 2 \pm i$.
1) $\lambda_1 = -3$.

$$\begin{bmatrix} -5 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 5 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0, \text{ we obtain } v_1 = v_3 = 0, v_2 \text{ any;}$$

$$\text{take } v_2 = 1, \text{ then } V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The corresponding solution is $X_1(t) = e^{-3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

2) $\lambda_2 = 2 + i$.

$$\begin{bmatrix} -i & 1 & 1 \\ 0 & -5-i & 0 \\ -1 & 5 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0, \text{ we get } v_2 = 0, v_3 = iv_1,$$

$$\text{let } v_1 = 1, \text{ then } V = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}.$$

The corresponding complex solution is

$$e^{\lambda_2 t} V = e^{2t}(\cos t + i \sin t) \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} = e^{2t} \begin{bmatrix} \cos t + i \sin t \\ 0 \\ -\sin t + i \cos t \end{bmatrix}.$$

To obtain real solutions take real and imaginary parts of the complex solution:

$$X_2(t) = e^{2t} \begin{bmatrix} \cos t \\ 0 \\ -\sin t \end{bmatrix}, \; X_3(t) = e^{2t} \begin{bmatrix} \sin t \\ 0 \\ \cos t \end{bmatrix}.$$

Thus, the general solution is:

$$X(t) = c_1X_1 + c_2X_2 + c_3X_3 = c_1e^{-3t}\begin{bmatrix}0\\1\\0\end{bmatrix} + c_2e^{2t}\begin{bmatrix}\cos t\\0\\-\sin t\end{bmatrix} + c_3e^{2t}\begin{bmatrix}\sin t\\0\\\cos t\end{bmatrix}.$$

Plugging the initial condition, we obtain:

$$\begin{bmatrix}c_2\\c_1\\c_3\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix}.$$

Therefor, the solution of the initial value problem is

$$X(t) = 2e^{2t}\begin{bmatrix}\cos t\\0\\-\sin t\end{bmatrix} - 2e^{2t}\begin{bmatrix}\sin t\\0\\\cos t\end{bmatrix} = 2e^{2t}\begin{bmatrix}\cos t - \sin t\\0\\-\sin t - \cos t\end{bmatrix}.$$

Problem 5.

$$A = \begin{bmatrix}1 & 1 & 1\\0 & 0 & -2\\0 & 1 & 3\end{bmatrix}, \quad X_0 = \begin{bmatrix}1\\3\\-2\end{bmatrix}.$$

Solution. The characteristic equation is:

$$\begin{vmatrix}1-\lambda & 1 & 1\\0 & -\lambda & -2\\0 & 1 & 3-\lambda\end{vmatrix} = (1-\lambda)(-\lambda(3-\lambda)+2) = (1-\lambda)(\lambda^2-3\lambda+2) = 0.$$

The roots are: $\lambda_1 = 2, \lambda_{2,3} = 1$. Let's find the corresponding eigenvectors and solutions.
1) $\lambda_1 = 2$. For the eigenvector V_1 we have:

$$\begin{bmatrix}-1 & 1 & 1\\0 & -2 & -2\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = 0, \; v_1 = v_2 + v_3 = 0.$$

Setting $v_3 = 1$ we get $V_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. The corresponding solution is $X_1(t) = e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

2) $\lambda_2 = 1$. We have:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0, \; v_2 = v_3 = 0, \; v_1 \text{ any}.$$

Thus, we can take $V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The corresponding solution is $X_2(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Observe that 1 is a repeated eigenvalue of order 2 but it has only one eigenvector. Therefor, it is a defective eigenvalue and we need to find a generalized eigenvector.

3) Generalized eigenvector V_3 can be found from the equation $(A - \lambda I)V_3 = V_2$. We have:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; v_2 + v_3 = 1, \; v_2 + 2v_3 = 0, \; v_1 \text{ any}.$$

Thus, $v_2 = 2, v_3 = -1$. Setting $v_1 = 0$ we gen $V_3 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. The solution corresponding to the generalized eigenvector is

$$X_3(t) = e^{\lambda t}(V_2 t + V_3) = e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right).$$

Thus, the general solution is

$$X(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right).$$

Plugging the initial condition we get:

$$\begin{bmatrix} c_2 \\ -c_1 + 2c_3 \\ c_1 - c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$

Therefor, $c_1 = -1, c_2 = 1, c_3 = 1$. The solution of the initial value problem is:

$$X(t) = -e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} (t+1)e^t \\ e^{2t} + 2e^t \\ -e^{2t} - e^t \end{bmatrix}.$$

In problems 6, 7 solve the system $X' = AX$, determine the type of the critical point 0 and sketch the phase portrait of the system.

Problem 6.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

Solution. The characteristic equation is $(1 - \lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6 = 0$. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 3$.

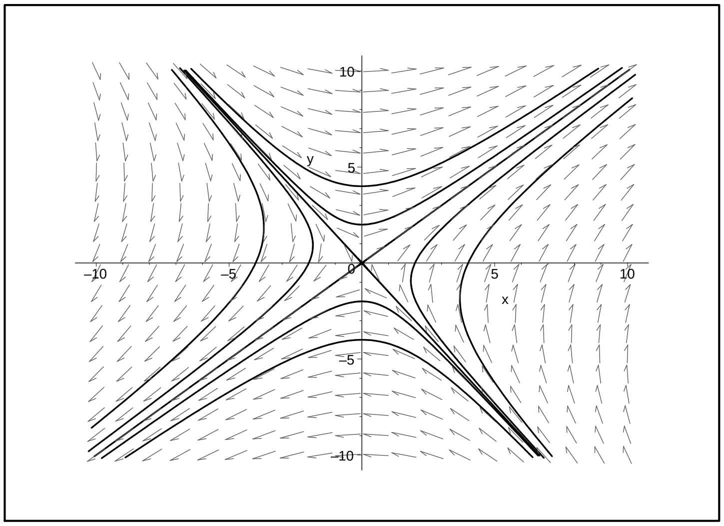

Figure 1: Phase portrait for problem 6.

1) $\lambda_1 = -2$. For the eigenvector V_1 we have:

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0, v_2 = -\tfrac{3}{2}v_1, \text{ setting } v_1 = 1 \text{ we get } V_1 = \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}.$$

The corresponding solution is $X_1(t) = e^{-2t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}$.

2) $\lambda_2 = 3$. Then

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0, \ v_1 = v_2, \text{ we can set } V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The corresponding solution is $X_2(t) = e^{-2t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}$. Thus, the general solution is $X(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix}$. According to the classification of the critical points, since $\lambda_1 < 0 < \lambda_2$, zero is an unstable saddle point. There are straight line trajectories in the directions of V_1 (converging to 0 when $t \to +\infty$) and V_2

(converging to ∞ when $t \to \infty$). Other trajectories have these lines as asymptotes.

Problem 7.

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}.$$

Solution. The characteristic equation is $(3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 5 = 0$. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 5$.
1) $\lambda_1 = 1$. Let's find the corresponding eigenvector V_1:

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,\ v_1 = -2v_2,\ \text{set } v_2 = 1,\ \text{then } V_1 = \begin{bmatrix} -2 \\ 1. \end{bmatrix}.$$

Thus, $X_1(t) = e^t \begin{bmatrix} -2 \\ 1. \end{bmatrix}$.
2) $\lambda_2 = 5$. We have:

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0,\ v_1 = 2v_2,\ set\ v_2 = 1,\ \text{then } V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The second solution is $X_2(t) = e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The general solution is

$$X(t) = c_1 e^t \begin{bmatrix} -2 \\ 1. \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since $0 < \lambda_1 < \lambda_2$, zero is an unstable improper node. There are straight line trajectories in the directions of V_1 and V_2. All other trajectories touch the direction of V_2 at zero.

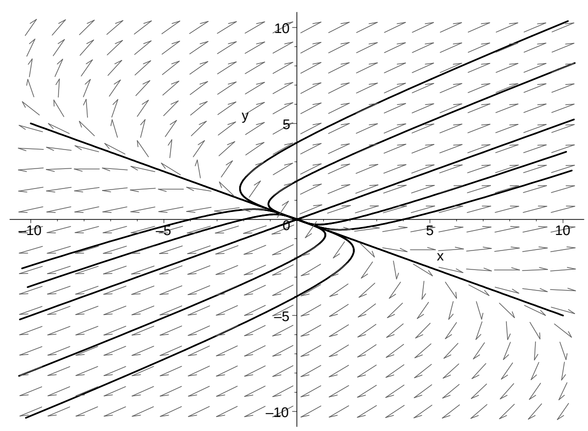

Figure 2: Phase portrait for problem 7.

MAT 303 FALL 2012 MIDTERM I

Problem 1. A car traveling at 30 mi/h (44 ft/s) gradually speeds up during 10 seconds with the acceleration given by

$$a(t) = 0.06t^2 + 2.4 \ (ft/s^2).$$

Find the distance it has traveled in these 10 seconds and its velocity at the end.

Solution. Since acceleration is the derivative of the speed (for movement by a straight line) , we obtain the following differential equation:

$$\tfrac{dv}{dt} = 0.06t^2 + 2.4.$$

Solving it by integration, we find:

$$v(t) = \int (0.06t^2 + 2.4)dt = 0.02t^3 + 2.4t + v_0 = 0.02t^3 + 2.4t + 44 \ (ft/s).$$

In particular, the velocity after $10s$ is

$$v(10) = 0.02 \cdot 10^3 + 2.4 \cdot 10 + 44 = 88 \ (ft/s).$$

Since velocity is the derivative of the distance (for movement by a straight line), we have:

$$\tfrac{dx}{dt} = v(t) = 0.02t^3 + 2.4t + 44.$$

Thus,

$$x(t) = \int (0.02t^3 + 2.4t + 44)dt = 0.005t^4 + 1.2t^2 + 44t.$$

In particular, after $10s$ the car traveled

$$0.005 \cdot 10^4 + 1.2 \cdot 10^2 + 44 \cdot 10 = 610 \ (ft).$$

Answer: The velocity at the end is 88 ft/s, the distance traveled is 610 ft.

Problem 2. Solve the initial value problem

$$x\frac{dy}{dx} = y + x^2, \;\; y(1) = 0.$$

Solution. This is a linear differential equation. First, rewrite it in the standard form:

$$\frac{dy}{dx} - \frac{y}{x} = x.$$

Calculate the multiplier:

$$\rho(x) = \exp(\int (-1/x)dx) = \exp(-\ln|x|) = \frac{1}{|x|}.$$

Since the initial condition is given at the point $x_0 = 1 > 0$, we can restrict our attention to $x > 0$, so that $\rho(x) = \frac{1}{x}$. Multiplying the differential equation by $\rho(x)$, we obtain:

$$\frac{d}{dx}(\frac{y}{x}) = \frac{1}{x}(\frac{dy}{dx} - \frac{1}{x}y) = 1.$$

Therefore,

$$\frac{y}{x} = \int 1dx = x + C, \;\; y = x^2 + Cx.$$

Plugging the initial condition $y(1) = 0$, we get:

$$0 = y(1) = 1 + C.$$

Thus, $C = -1$.

Answer: $y(x) = x^2 - x$.

Problem 3. Solve the differential equation

$$y' = (2x - y)^2 + 3.$$

Solution. On the right hand side of the equation we see a noticeable expression $2x - y$, therefore it is natural to try the substitution $v = 2x - y$. We have: $v' = 2 - y'$. Thus,

$$v' = 2 - ((2x - y)^2 + 3) = -1 - v^2.$$

This is a separable equation. Separating variables and integrating, we get

$$\tfrac{dv}{v^2+1} = -dx, \quad \tan^{-1}(v) = -x + C, \quad v = \tan(C - x).$$

Thus, we obtain:

$$y = 2x - v = 2x - \tan(C - t) = 2x + \tan(x - C).$$

Since C is arbitrary constant, we can replace C with $-C$.

Answer: $y(x) = 2x + \tan(x + C)$, where C is an arbitrary constant.

Problem 4. Show that the following differential equation is exact; then solve it.

$$\frac{1}{yx}dy + (2x - \frac{\ln y}{x^2})dx = 0.$$

Solution. Let $M(x,y) = \frac{1}{yx}$, $N(x,y) = 2x - \frac{\ln y}{x^2}$. By Criteria of Exactness, the differential equation $Mdy + Ndx = 0$ is exact if and only if $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$. We have:

$$\frac{\partial M}{\partial x} = -\frac{1}{yx^2} = \frac{\partial N}{\partial y},$$

therefore the equation is exact.

To solve the equation we need to find a function $F(x,y)$ such that $\frac{\partial F}{\partial y} = M$, $\frac{\partial F}{\partial x} = N$. Integrating the first equation, we obtain:

$$F = \int Mdy = \int \frac{1}{yx}dy = \frac{\ln y}{x} + g,$$

where $g = g(x)$ is a function of x. Substituting this formula into the second equation, we find:

$$2x - \frac{\ln y}{x^2} = \frac{\partial F}{\partial x} = -\frac{\ln y}{x^2} + g'(x).$$

Therefore, $g'(x) = 2x$, $g(x) = x^2 + C$. Since we need to find a function $F(x,y)$, we can fix any C. Let $C = 0$. Then $F(x,y) = \frac{\ln y}{x} + x^2$.

The general solution of the exact equation is $F(x,y) = C$, where C is any constant. Thus,

$$\frac{\ln y}{x} + x^2 = C, \ \ln y = Cx - x^3, \ \ y = \exp(Cx - x^3).$$

Answer: $y(x) = \exp(Cx - x^3)$, where C is a constant.

Problem 5. Assume that a population of size $P(t)$ has a constant birth rate $\beta = 0.1$ and the death rate is given by $\delta = 0.01P$.

1) Write the logistic equation which is satisfied by this population.

2) Write the general solution of this logistic equation.

3) Indicate the critical points and their types (stable or unstable).

Solution.

1) The differential equation of the population with given birth and death rates is:

$$\tfrac{dP}{dt} = (\beta - \delta)P = (0.1 - 0.01P)P = 0.01P(10 - P).$$

2) The general solution of a logistic equation $\frac{dP}{dt} = kP(M - P)$ is:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)\exp(-kMt)}.$$

In our case, $k = 0.01, \quad M = 10$. Thus,

$$P(t) = \frac{10P_0}{P_0 + (1 - P_0)\exp(-0.1t)}.$$

3) For a logistic equation there are two critical points: 0 and M. The critical point 0 is unstable, the critical point $M = 10$ is stable.

MAT 303 FALL 2012 MIDTERM II

1. Suppose that a motorboat is moving at $5m/s$ when its motor suddenly quits, and that 10 seconds later the velocity of the boat is $1m/s$. Assume that the resistance motorboat encounters is proportional to the cube of its velocity. Find the velocity of the boat in 20 seconds after the motor has quit.

Solution. Let $v(t)$ be the speed of the motorboat. From the conditions of the problem we get: $v(0) = 5, v(10) = 1$. The motorboat movement is influenced only by the water resistance, which is of the form $F = kv^3$ for some constant k. By the Newton's law, we get:

$$m\frac{dv}{dt} = ma = F = kv^3.$$

Thus, $\frac{dv}{dt} = cv^3$ for some constant $c = \frac{k}{m}$. Solving this separable equation, we get:

$$\frac{dv}{v^3} = cdt, \quad -\frac{1}{2v^2} = ct + b$$

for some constant b. Plugging $t = 0$ and $t = 10$, we get:

$$-\frac{1}{50} = b, \ -\frac{1}{2} = 10c + b, \ c = \frac{-1-2b}{20} = -\frac{24}{20\cdot 25} = -\frac{6}{125}.$$

It follows that

$$v(t) = \frac{1}{\sqrt{-2ct-2b}} = \frac{1}{\sqrt{\frac{12}{125}t + \frac{1}{25}}}.$$

In particular,

$$v(20) = \frac{1}{\sqrt{\frac{12}{125}\cdot 20 + \frac{1}{25}}} = \frac{1}{\sqrt{\frac{49}{25}}} = \frac{5}{7}.$$

Answer: $\frac{5}{7}m/s$.

2.

Verify that the functions $y_1 = x^2, y_2 = \frac{1}{x}$ are solutions of the differential equation

$$x^2y'' - 2y = 0.$$

Solve the initial value problem

$$y(1) = 5, y'(1) = -3.$$

Solution. We have: $y_1'' = 2,\ x^2y_1'' - 2y_1 = 2x^2 - 2x^2 = 0,\quad y_2'' = \frac{2}{x^3},\ x^2y_2'' - 2y_2 = \frac{2}{x} - \frac{2}{x} = 0$, therefor, y_1 and y_2 are solutions. By Theorem on General Solution for linear equations, $y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^2 + \frac{c_2}{x}$. We have:

$$y_2'(x) = 2c_1x - \frac{c_2}{x^2}.$$

Plugging the initial conditions, we obtain:

$$c_1 + c_2 = 5,\ 2c_1 - c_2 = -3,\ 3c_1 = 2,\ c_1 = \frac{2}{3},\ c_2 = 5 - c_1 = \frac{13}{3}.$$

Answer: $y(x) = \frac{2}{3}x^2 + \frac{13}{3x}$.

3.

Using Wronskian show that the functions $y_1 = x^2, y_2 = \sin x, y_3 = \cos x$ are linearly independent on $\mathbf{R}$.

Solution. The Wronskian of these function is

$$W(x) = \begin{vmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x. \end{vmatrix}.$$

We need show that Wronskian is not identically zero. For $x = 0$ we have:

$$W(0) = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{vmatrix} = 2 \neq 0.$$

By Theorem on Wronskian, the functions y_1, y_2, y_3 are linearly independent on the real line.

4.

Find the general solution of the differential equation

$$y^{(4)} - y = 2x.$$

Solution. The general solution of the non-homogeneous equation is of the form: $y = y_c + y_p$, where y_c is the general solution of the associated homogeneous equation $y^{(4)} - y = 0$ and y_p is a particular solution of the non-homogeneous equation.

1) For y_c, the characteristic equation is $r^4 - 1 = 0$. We have: $r^2 = \pm 1$. For $r^2 = 1$ we get $r_1 = -1, r_2 = 1$. For $r^2 = -1$ we get $r_3 = i, r_4 = -i$. Thus, $y_c(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$.

2) For y_p, consider the right hand side $f(x) = 2x$ of the non-homogeneous equation. The function $f(x)$ and its derivatives $f'(x) = 2, f''(x) = 0$ has only two different terms (up to scalar factors): x and 1. These terms are not solutions of the associated homogeneous equation. Therefor, as a particular solution we can take $y_p(x) = ax + b$. Plugging it into the equation, we obtain: $-(ax + b) = 2x$. Thus, $y_p(x) = ax + b = -2x$.

Answer: $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x - 2x$.

5.

In a mass-spring-dashpot system a body of mass $m = 1kg$ is attached to an ordinary spring from one side and a dashpot from the other side. The spring constant is $k = 1N/m$. The damping constant is $c = 2.5Ns/m$. The initial position of the body is at $x_0 = 0.1m$ from the equilibrium position. The initial velocity is $v_0 = 0.3m/s$. Find the position function of the body.

Solution. The position function of the body in the mass-spring-dashpot system without external force satisfies the equation $mx'' + cx' + kx = 0$. Thus, $x'' + 2.5x' + x = 0$. Let's solve this equation. The characteristic equation is $r^2 + 2.5r + 1 = 0$. Its solutions are $r_1 = -\frac{1}{2}, r_2 = -2$. Thus,

$$x(t) = c_1 e^{-t/2} + c_2 e^{-2t}.$$

The formula for the velocity is

$$v(t) = x'(t) = -\tfrac{c_1}{2} e^{-t/2} - 2c_2 e^{-2t}.$$

Plugging the conditions $x(0) = 0.1$, $v(0) = 0.3$, we obtain:

$$c_1 + c_2 = 0.1, \ -\tfrac{c_1}{2} - 2c_2 = 0.3.$$

Therefor, $2(c_1 + c_2) - \frac{c_1}{2} - 2c_2 = 0.2 + 0.3, \ \frac{3c_1}{2} = 0.5, \ c_1 = \frac{1}{3}, \ c_2 = 0.1 - c_1 = \frac{7}{30}$.

Answer: $x(t) = \frac{1}{3} e^{-t/2} - \frac{7}{30} e^{-2t}$.

Recommended problems from the course book.

Section 1.1: 5, 7, 9, 27, 40, 45.
Section 1.2: 6, 7, 8, 24, 32, 38.
Section 1.3: 15, 16, 17.
Section 1.4: 6, 9, 17, 23, 25, 27, 31, 34, 44, 61.
Section 1.5: 7, 13, 16, 20, 36.
Section 1.6: 10, 15, 16, 30, 32, 37, 40.
Section 2.1: 4, 9, 17, 26.
Section 2.2: 4, 7.
Section 2.3: 1, 4, 10, 17, 20.
Section 2.4: 2, 4, 10.
Section 3.1: 4, 7, 14, 18, 20, 22, 25, 30, 31, 34, 41, 47, 48.
Section 3.2: 1, 3, 5, 8, 10, 13, 17, 27, 30, 33.
Section 3.3: 4,10, 16, 19, 22, 23, 24, 27, 31, 39, 42.

Nonhomogeneus equations: in each of the following question find by inspection a particular solution of the given differential equation, then find the general solution of the associated homogeneous equation and compose the general solution of the nonhomogeneous differential equation. Solve the initial value problem, if the initial conditions are given.

1. $y'' - 9y = \sin(2x),\ y(0) = 2,\ y'(0) = -2/13.$
2. $y'' + 4y = 3x - 1,\ y(0) = -1/4,\ y'(0) = 3/4.$
3. $y''' + 3y'' + 3y' + y = \exp(x).$

Section 3.4: 1, 2, 15, 17, 20.
Section 3.5: 1, 3, 9, 17, 25, 30.
Section 3.6: 1, 3, 5.
Section 4.1: 5, 7, 10, 16, 19, 24.
Section 4.2: 3, 5, 7, 9, 12, 14.
Section 5.1: 1, 2, 5, 12, 20, 22, 26.
Section 5.2: 4, 8, 11, 19, 22, 26.
Section 5.4: 2, 6, 7.
Section 6.2: solve the system, determine the type of the critical point, sketch the phase portrait for the questions 5, 7, 8, 9.