MAT 303: Calculus IV

Welcome to MAT 303

Differential equations are the language in which the laws of physics are expressed, and have numerous applications in the physical, biological, and social sciences.

This course will introduce you to basic methods of solving differential equations. It is a prerequisite for all subsequent courses in the area. We will mainly focus on on linear differential equations with constant coefficients and systems of differential equations. We will also discuss many standard applications and some numerical methods.

These pages will be updated regularly throughout the semester. For specific course-related information, go to the Course info page. Other pages here contain the (tentative) course schedule, home assignments, and exam reviews.

NEW: The applets page with interactive demonstrations of some concepts we've covered in this class.

MAT 303: Calculus IV

Course Information

LECTURE

MWF 11:45am-12:40pm, Harriman Hall 116

RECITATIONS

Section R01: Tu 9:50-10:45am, Earth&Space 183
Section R02: Th 2:20-3:15pm, Earth&Space 181

INSTRUCTOR

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TA

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REQUIRED TEXT

Differential Equations and Boundary Value Problems: Computing and Modeling by Edwards & Penney, 3rd edition
(On reserve in Math/Physics library)

EXAMS

Midterm 1
Monday, October 9, in-class

Midterm 2
Mid-November, date TBA; in-class

Final Exam
Monday, December 18, 2:00-4:30pm

Make-up policy: The university policy is that makeup examinations are given only for work missed due to **unforseeable circumstances** beyond the student's control.

HOMEWORK

Homework is a fundamental part of this course, and you will have to work hard on the assigned problems in order to succeed. Assignments will be announced in class, posted on the web and will be **collected in class on Friday** of the following week. **Late homework will not be accepted**.

GRADING

Your course grade will be computed as follows: homework 20%, two in-class midterms 20% each, and the final exam 40%. The lowest homework grade will be dropped before calculating the average. In borderline cases, class participation (both lectures and recitations) will be taken into account.

HELP OUTSIDE CLASS

The Math Learning Center is located in Math Tower S-240A and offers free help to any student requesting it. It also provides a locale for students wishing to form study groups.

AMERICANS WITH DISABILITIES ACT

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. (Note that we cannot make special arrangements for students with disabilities except for those determined by DSS.) All information on and documentation of a disability condition should be supplied to me in writing at the earliest possible time.

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Course Schedule (*italic=tentative*)

WEEK OF	SECTIONS	NOTES
9/6-9/8	1.1-2	
9/11-9/15	1.3-4	
9/18-9/22	1.5-6	
9/25-9/29	1.6, 2.1-2	
10/2-10/6	2.3-4	
10/9-10/13	3.1-2	midterm 1 on 10/9
10/16-10/20	3.2-3	
10/23-10/27	3.4-5	
10/30-11/3	3.5-6, 4.1	
11/6-11/10	5.1	
11/13-11/17	5.2	midterm 2 on 11/13
11/20-11/22	5.4	
11/27-12/1	5.3-5	
12/4-12/8	*6.1-2*	
12/11-12/15	*6.3-4*	final review on 12/15
12/18		final

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Home Assignments

WEEK OF	PROBLEMS	DUE
9/6-9/8	1.1: 3, 9, 19, 23, 33, 35 1.2: 5, 7, 9, 15, 35, 43	9/13
9/11-9/16	1.3: 12, 15, 20, 21, 27 1.4: 5, 13, 17, 21, 23, 27 1.4: 33, 41 1.5: 3, 5, 11	9/20
9/18-9/22	1.5: 8, 13, 17, 27, 36 1.6: 5, 8, 16, 20, 21 1.6: 17, 31, 37	9/27
9/25-9/29	1.6: 43, 45 2.1: 5, 10, 15 2.2: 9, 11, 20, 27 2.3: 2, 10, 19, 21	10/4
10/4-10/6	2.4: 3, 9	10/11
10/9-10/13	3.1: 3, 5, 14, 19, 20, 25 3.1: 24, 28, 33, 37, 40, 44	10/18
10/16-10/20	3.2: 3, 7, 9, 21, 24 3.3: 15, 19, 23, 31, 36 3.3: 20, 37, 40	10/25
10/23-10/27	3.4: 3, 9, 15, 18 3.5: 3, 9, 13, 22, 25 3.5: 33, 39, 52	11/1
10/30-11/3	3.5: 16, 28 3.6: 1, 6, 9, 15, 19 4.1: 1, 5, 13, 19 (no graphing) 4.2: 5	11/8
11/6-11/11	5.1: 3, 4, 9, 13, 15 5.1: 23, 29, 37	11/15
11/15-11/17	5.2: 3, 5, 7, 17, 21 5.2: 8, 11, 24, 25, 26	11/27(Monday)
11/20-11/22	5.4: 2, 5, 7, 15, 23 5.4: 11, 20, 21	11/29
11/27-12/1	5.3: 3, 5, 9, 11 5.4: 35, 36 5.5: 2, 3, 7	12/6

	12/4-12/8	5.5: 17, 22, 27 6.1: 4, 6, 8 6.1: 13, 14, 16, 19, 20 6.2: 7, 12, 25, 31	12/13

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Exam Information

FINAL: December 18, 2-4pm

Reviews: December 13 and 15, in-class
Practice problems (for Midterms 1 and 2 and newer material):

SECTION	PROBLEMS	SECTION	PROBLEMS	SECTION	PROBLEMS
1.3	17, 18	3.1	26, 38	5.1	18, 26
1.4	13, 25	3.2	10, 15, 23	5.2	6, 9, 23
1.5	13, 23	3.3	12, 16	5.4	6, 8, 17
1.6	7, 28, 39	3.4	16, 20	5.5	5, 25, 26
2.2	10, 21	3.5	14, 17, 38	6.1	5, 15, 17
2.3	9, 12	4.1	6, 17	6.2	5, 16, 23, 26

Solutions: part 1, part 2, part 3

Go over all homework problems assigned so far
For more practice on Section 1 problems, you may look at Review problems on p. 76

If you wish to see old exams, you can look for them at old MAT303 web-pages. However, bear in mind that in every semester *coverage* in this course *differs*, i.e. old exams do not reflect the material you need to study. It is much better to use practice and homework problems.

MIDTERM 2 was on November 13

Practice problems:

SECTION	PROBLEMS
3.1	26, 38
3.2	10, 15, 23
3.3	12, 16
3.4	16, 20
3.5	14, 17, 38
4.1	6, 17
4.2	4

Solutions are posted.

Also, go over all homework problems assigned so far

If you wish to see old midterms, you can look for them at old MAT303 web-pages. However, bear in mind that in every semester *coverage* in this course *differs*, i.e. old midterms do not reflect the material you need to study as well as the practice and homework problems.

MIDTERM 1 was on October 9

Practice problems:

SECTION	PROBLEMS
1.3	17, 18
1.4	13, 25
1.5	13, 23
1.6	7, 28, 39, 48, 49
2.2	10, 21
2.3	9, 12

Solutions are available here.

Extra review problems:
Chapter 1 Review (p. 76), problems 1-5, 9, 15-17

Also, go over all homework problems assigned so far
You may also check out remaining problems in Chapter 1 Review

If you wish to see old midterms, you can look for them at old MAT303 web-pages. However, bear in mind that in every semester *coverage* in this course *differs*, i.e. old midterms do not reflect the material you need to study as well as the practice and homework problems.

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Applets

Here are several applets demonstrating concepts discussed in MAT 303 (will open in new windows):

- Mechanical Vibrations:

  Free damped vibrations
  : This is the graph of the solution of the equation x''+bx'+kx=0. You may vary parameters b and k and the initial values x(0) and x'(0) (just click on the b or k slides in the lower left corner or the x(0)-x'(0) plane above them to change the values). Clicking on the Roots button will show the roots of the corresponding characteristic equation.

  Non-free damped vibrations
  : This is the graph of the solution of the equation x''+bx'+kx=cos(ωt). Here you may also change the values of b, k, x(0) and x'(0) as well as the external frequency ω. The blue graph on the right corresponds to the solution of the associated homogeneous equation, x_c, the green graph to the particular solution x_p. The yellow graph is the solution $x=x_c+x_p$.

- Systems of equations:

  Phase Portraits
  : To work with this applet, turn off the "Companion Matrix" feature (click on the blue square to the left of the words "Companion Matrix").
  In the lower right corner, you have the slides for parameters a,b,c,d.
  Above is the phase portrait for the system
  x'=ax+by
  y'=cx+dy
  Changing values of these parameters changes the phase portrait.

All applets are courtesy of the d'Arbeloff Interactive Math Project at MIT.

Calculus IV with Applications MAT303
Solutions to Practice Problems for Midterm 1

1.3, 17. $f(x,y) = \frac{x-1}{y}$ and $D_y f(x,y) = -\frac{x-1}{y^2}$ are continuous in the neighborhood of $(0,1)$, thus the solution near $x = 0$ exists and is unique.

1.3, 18. $f(x,y) = \frac{x-1}{y}$ is not continuous at $(1,0)$, thus neither existence nor uniqueness are guaranteed.

1.4, 13. Separable equation: $\dfrac{y^3}{y^4+1}dy = \cos x dx$. $\displaystyle\int \frac{y^3}{y^4+1}dy = \frac{1}{4}\ln(y^4+1)$ (after substitution $u = y^4+1$), thus $\dfrac{1}{4}\ln(y^4+1) = \sin x + C_1$ or $\ln(y^4+1) = 4\sin x + C_2$ (replace $4C_1$ with C_2). Then $y = (Ce^{\sin x} - 1)^{1/4}$, $C > 0$.

1.4, 25. Separable equation: $xy' = (2x^2+1)y$, so $\dfrac{dy}{y} = \dfrac{2x^2+1}{x}dx$. $\displaystyle\int 2x + \frac{1}{x}dx = x^2 + \ln|x| + C_1$, hence after integrating both sides $\ln|y| = x^2 + \ln|x| + C_1$. Thus $y = Ce^{x^2}|x|$.

The initial condition $y(1) = 1$ gives $C = e^{-1}$, hence $y = xe^{x^2-1}$.

1.5, 13. Integrating factor $\rho(x) = e^{\int 1dx} = e^x$. Thus $(e^x y)' = e^x e^x$ and $e^x y = \displaystyle\int e^{2x}dx = \frac{1}{2}e^{2x} + C$. Dividing both sides by e^x, we get $y = \dfrac{1}{2}e^x + Ce^{-x}$.

The initial conditions $y(0) = 1$ gives $C = \dfrac{1}{2}$, hence $y = \dfrac{1}{2}(e^x + e^{-x})$.

1.5, 23. $y' + \dfrac{2x-3}{x}y = 4x^3$. Integrating factor: $\exp\left(\displaystyle\int 2 - \frac{3}{x}dx\right) = \exp(2x - 3\ln x) = e^{2x}/x^3$. Thus $\left(\dfrac{e^{2x}}{x^3}y\right)' = 4e^{2x}$ and $\dfrac{e^{2x}}{x^3}y = 2e^{2x} + C$. Finally, $y = x^3(2 + Ce^{-2x})$.

1.6, 7. $y' = \dfrac{x^2}{y^2} + \dfrac{y}{x}$. This is a homogeneous equation and substitution $v = \dfrac{y}{x}$ gives a separable equation $xv' = v^{-2}$. Thus $v^2 dv = \dfrac{dx}{x}$ and $\dfrac{v^3}{3} = \ln|x| + C$. Substituting back, we find that $y^3 = 3x^3(\ln|x| + C)$ or $y = x(3(\ln|x| + C))^{1/3}$.

1.6, 28. In this equation y is present only in the expression e^y, so we make the substitution $v = e^y$. Then $v' = e^y y'$ and the equation becomes $xv' = 2(v + x^3 e^{2x})$. Rewrite as: $v' - \frac{2}{x}v = 2x^2 e^{2x}$. Integrating factor: $e^{\int -2/x dx} = x^{-2}$. Thus $(x^{-2}v)' = 2e^{2x}$, i.e. $x^{-2}v = \int e^{2x}dx = e^{2x} + C$. Then $v = (e^{2x} + C)x^2$ or $e^y = (e^{2x} + C)x^2$. Finally, $y = \ln(e^{2x}x^2 + Cx^2)$.

1.6, 39. Checking that the equation is exact: $\frac{\partial}{\partial y}(3x^2y^3 + y^4) = 9x^2y^2 + 4y^3$, $\frac{\partial}{\partial x}(3x^3y^2 + y^4 + 4xy^3) = 9x^2y^2 + 4y^3$.

Solution: $F(x,y) = \displaystyle\int 3x^2y^3 + y^4 dx = x^3y^3 + xy^4 + C(y)$. To determine $C(y)$, $3x^3y^2 + y^4 + 4xy^3 = \dfrac{\partial}{\partial y}F(x,y) = 3x^3y^2 + 4xy^3 + C'(y)$. Hence $C'(y) = y^4$ and $C(y) = \dfrac{y^5}{5} + C_1$. Answer: $F(x,y) = x^3y^3 + xy^4 + \dfrac{y^5}{5} + C_1 = 0$ or $5x^3y^3 + 5xy^4 + y^5 + C = 0$.

1.6, 48. Substitution: $y' = p(x)$. Then $x^2p' + 3xp = 2$ or $p' + \dfrac{3}{x}p = 2x^{-2}$. Integrating factor: $e^{\int 3/x dx} = x^3$. The equation becomes $(x^3p)' = 2x$, i.e. $x^3p =$

$x^2 + C$ or $p = x^{-1} + Cx^{-3}$. Integrating p to obtain y, we get $y = \ln x + Ax^{-2} + B$ $(A = -C/2)$.

1.6, 49. Substitution: $y' = p(y)$. Then $y'' = \frac{dp}{dy}y' = \frac{dp}{dy}p$ and the equation becomes $yp'p + p^2 = yp$. Then $p' + \dfrac{p}{y} = 1$. Integrating factor: $e^{\int 1/y dy} = y$. Then $(yp)' = y$. Hence $yp = \dfrac{y^2}{2} + C_1$ or $p = \dfrac{y^2 + C}{2y}$. (Here $C = 2C_1$.) To obtain y, we solve the differential equation $y' = \dfrac{y+C}{2y}$: $\dfrac{2ydy}{y^2+C} = dx$, hence $\ln|y^2 + C| = x + A$ and $y = (\pm(Be^x - C))^{1/2}$.

2.2, 10. Critical points: $7x - x^2 - 10 = 0$ has solutions $x = 2, 5$. $x' < 0$ on $(-\infty; 2)$ and $(5; \infty)$; $x' > 0$ on $(2; 5)$. Therefore 2 is an unstable critical point and 5 is stable.

Solution: $\dfrac{dx}{7x - x^2 - 10} = dt$. From $7x - x^2 - 10 = -(x-2)(x-5)$, $\dfrac{1}{7x - x^2 - 10} = \dfrac{1}{3(x-2)} - \dfrac{1}{3(x-5)}$. Thus $\displaystyle\int \frac{dx}{7x - x^2 - 10} = \int \frac{1}{3(x-2)} - \frac{1}{3(x-5)} dx = \frac{1}{3}(\ln|x - 2| - \ln|x-5|) = \frac{1}{3}\ln\left|\frac{x-2}{x-5}\right|$. Therefore, the solution is $\dfrac{1}{3}\ln\left|\dfrac{x-2}{x-5}\right| = t + C_1$ or $\dfrac{x-2}{x-5} = Ce^{3t}$. Solving for x, we obtain $x = \dfrac{2 - 5Ce^{3t}}{1 - Ce^{3t}}$.

2.2, 21. $kx - x^3 = x(k - x^2)$, thus the critical points are $x = 0$ and $x = \pm\sqrt{k}$ (only when $k \geq 0$).

(a) If $k < 0$, the only critical point is $x = 0$. $\frac{dx}{dt} > 0$ for $x < 0$ and $\frac{dx}{dt} < 0$ for $x > 0$, thus the critical point is stable.

(b) $\frac{dx}{dt} > 0$ on $(-\infty, -\sqrt{k}$ and $(0, \sqrt{k})$; $\frac{dx}{dt} < 0$ on $(-\sqrt{k}, 0)$ and $(\sqrt{k}, \infty)$.

2.3, 9. The velocity equation is $\frac{dv}{dt} = 5 - 0.1v$. The initial acceleration is positive but as velocity increases, acceleration declines. When acceleration is zero, velocity cannot increase further because such an increase will immediately make acceleration negative and lead to a decline in velocity. Thus the maximum is achieved when $5 - 0.1v = 0$, i.e. when $v = 50$.

Note that $v = 50$ is the stable critical point of the velocity equation. Therefore we know immediately that $v = 50$ is the terminal (limiting) velocity of the boat but this is not enough to conclude that it is also the maximal velocity.

2.3, 12. $W = 640$, $B = 62.5 \cdot 8 = 500$, and $m = 640/32 = 20$. The velocity equations is $m\dfrac{dv}{dt} = -W + B - v$ ($F_R = -1 \cdot v$ because as vectors they have opposite directions). Thus $\dfrac{mdv}{-W + B - v} = dt$. Integrating both sides, we obtain $-m\ln|W - B + v| = t + C$. Since $v(0) = 0$, $C = -m\ln(W - B)$. Therefore, $v = e^{-(t+C)/m} + W - B = e^{-t/m}(W - B) + W - B$, so finally $v = (W - B)(e^{-t/m} - 1)$. The equation of motion is $\dfrac{dy}{dt} = v$, hence $y = (W - B)(-me^{-t/m} - t) + C_1$. Drums are dropped from zero depth, i.e. $y(0) = 0$ which makes the equation $y = (W - B)(-me^{-t/m} - t + m)$.

Now we compute at what time $v = -75$ and plug the answer into the equation of motion. Answer: 648 ft.

Calculus IV with Applications MAT303
Solutions to Practice Problems for Midterm II

3.1, 26. If dependent, then $f = cg$ for a constant c, i.e. $2\cos x + 3\sin x = c(3\cos x - 2\sin x)$. Then comparing coefficients at $\cos x$ and $\sin x$, we get $2 = 3c$ and $3 = -2c$ at the same time, which is impossible. Therefore, f and g are linearly independent. (Another solution: compute the Wronskian.)

3.1, 38. Char. eq-n: $4r^2 + 8r + 3 = 0$. Solutions to char eq-n: $r = -3/2, -1/2$. General sol-n: $c_1 e^{-\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}$.

3.2, 10. $W(f, g, h) = \begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & -2x^{-3}\ln x + x^{-3} \\ e^x & 6x^{-4} & 6x^{-4}\ln x - 5x^{-4} \end{vmatrix} =$
$e^x(-2x^{-3}(6x^{-4}\ln x - 5x^{-4}) - (-2x^{-3}\ln x + x^{-3})6x^{-4}) - e^x(x^{-2}(6x^{-4}\ln x - 5x^{-4}) - x^{-2}\ln x 6x^{-4}) + e^x(x^{-2}(-2x^{-3}\ln x + x^{-3}) - x^{-2}\ln x(-2x^{-3})) = e^{-x}(4x^{-7} + 5x^{-6} + x^{-5}) \neq 0$.

3.2, 15. General solution: $y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$. Then $y'(x) = (c_1 + c_2)e^x + (c_2 + 2c_3)xe^x + c_3 x^2 e^x$ and $y''(x) = (c_1 + 2c_2 + 2c_3)e^x + (c_2 + 4c_3)xe^x + c_3 x^2 e^x$. Then we have

$$\begin{cases} 2 = y(0) = c_1 \\ 0 = y'(0) = c_1 + c_2 \\ 0 = y''(0) = c_1 + 2c_2 + 2c_3 \end{cases}$$

Hence, $c_1 = 2$, $c_2 = -2$, $c_3 = 1$. Solution: $2e^x - 2xe^x + x^2 e^x$.

3.2, 23. General solution: $y(x) = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} - 2$. Then $y'(x) = -c_1 e^{-x} + 3c_2 e^{3x}$. Then we get $3 = y(0) = c_1 + c_2 - 2$ and $11 = y'(0) = -c_1 + 3c_2$. Then $c_1 = 1$, $c_2 = 4$. Solution: $e^{-x} + 4e^{3x} - 2$.

3.3, 12. Char eq-n: $r^4 - 3r^3 + 3r^2 - r = 0$ or, equivalently, $r(r^3 - 3r^2 + 3r - 1) = 0$ or, equivalently, $r(r-1)^3 = 0$. Solutions to char eq-n: r=0, r=1 (with multiplicity 3). General sol-n: $c_1 e^{0x} + c_2 e^x + c_3 x e^x + c_4 x^2 e^x = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$.

3.3, 16. Char eq-n: $r^4 + 18r^2 + 81 = 0$. Put $r^2 = s$, then $s^2 + 18s + 81 = 0$ and $s = -9$, thus $r = \pm\sqrt{9} = \pm 3i$ (each root with multiplicity 2). Alternative approach: rewrite equation as $(r^2 + 9)^2 = 0$, then as $(r + 3i)^2(r - 3i)^2 = 0$. General sol-n: $c_1 \cos 3x + c_2 x \cos 3x + c_3 \sin 3x + c_4 x \sin 3x$.

3.4, 16. $3x'' + 30x' + 63x = 0$. The characteristic equation is $3r^2 + 30r + 63 = 0$, thus $r = \dfrac{-30 \pm \sqrt{30^2 - 4 \cdot 63 \cdot 3}}{2 \cdot 3} = -7, -3$. The roots are real and distinct, therefore the system is overdamped. General solution: $x(t) = c_1 e^{-3t} + c_2 e^{-7t}$. Then $v(t) = x'(t) = -3c_1 e^{-3t} - 7c_2 e^{-7t}$. From $x(0) = 2, v(0) = 2$, we have $c_1 + c_2 = 2, -3c_1 - 7c_2 = 2$. Thus $c_1 = 4, c_2 = -2$. Position function: $4e^{-3t} - 2e^{-7t}$.

In the undamped case, the equation is $3x'' + 63x = 0$. Then $\omega_0 = \sqrt{63/3} = \sqrt{21}$. General solution: $x(t) = C\cos(\omega_0 t - \alpha) = C\cos(\sqrt{21}t - \alpha)$. Then $v(t) = x'(t) = -\sqrt{21}C\sin(\sqrt{21}t - \alpha)$. From $x(0) = 2, v(0) = 2$, we have $2 = C\cos(-\alpha)$ and $2 = -\sqrt{21}C\sin(-\alpha)$. $C = 2/\cos(-\alpha)$, thus $2 = -2\sqrt{21}\sin(-\alpha)/\cos(-\alpha) = -2\sqrt{21}\tan(-\alpha)$. It follows that $\alpha = \arctan(1/\sqrt{21})$. If $\tan\alpha = 1/\sqrt{21}$, then $\cos\alpha = \sqrt{21/22}$. Hence $C = 2\sqrt{22/21}$.

3.4, 20. $2x'' + 16x' + 40x = 0$. The characteristic equation is $2r^2 + 16r + 40 = 0$, thus $r = \dfrac{-16 \pm \sqrt{16^2 - 4 \cdot 40 \cdot 2}}{2 \cdot 2} = -4 \pm 2i$. The roots are complex, therefore the system is underdamped. General solution $x(t) = Ce^{-4t}\cos(2t - \alpha)$ (i.e. $p = 4, \omega_1 = 2$). $x'(t) = -4Ce^{-4t}\cos 2t - \alpha - 2Ce^{-4t}\sin(2t - \alpha)$. From

$x(0) = 5, x'(0) = 4$, we have $5 = C\cos(-\alpha)$, $4 = -4C\cos(-\alpha) - 2C\sin(-\alpha)$. $C = 5/\cos(-\alpha)$ and $-12 = 5\sin(-\alpha)/\cos(-\alpha)$. Thus $\tan(-\alpha) = -12/5$, $\tan(\alpha) = 12/5$. Thus $\alpha = \arctan(12/5)$ and $\cos(\alpha) = 5/13$, $C = 13$. Position function: $x(t) = 13\cos(2t - \arctan(12/5)$.

3.5, 14. Associated homogeneous equation: $y^{(4)} - 2y'' + y = 0$. Characteristic equation: $r^4 - 2r^2 + 1 = 0$ or, equivalently, $(r^2-1)^2 = 0$ or, equivalently, $(r-1)^2(r+1)^2 = 0$. Solutions: $r = \pm 1$ (each with multiplicity 2). General solution: $y_c = c_1e^x + c_2xe^x + c_3e^{-x} + c_4xe^{-x}$.

Since $f(x) = xe^x$, $f'(x) = xe^x + e^x$. A linear combination of $f(x)$ and its derivatives has the form $Ae^x + Bxe^x$. Trial solution: $x^s(Ae^x + Bxe^x)$. Both e^x and xe^x are particular solutions of the associated equation, hence we take $s = 2$. Trial solution: $Ax^2e^x + Bx^3e^x$.

Plug in trial solution: $(Ax^2e^x + Bx^3e^x)^{(4)} - 2(Ax^2e^x + Bx^3e^x)'' + Ax^2e^x + Bx^3e^x = xe^x$.
$A(12e^x + 8xe^x + x^2e^x) + B(24e^x + 36xe^x + 12x^2e^x + x^3e^x) - 2A(2e^x + 4xe^x + x^2e^x) - 2B(6xe^x + 6x^2e^x + x^3e^x)Ax^2e^x + Bx^3e^x = xe^x$.
$(12A + 24B - 4A)e^x + (8A + 36B - 8A - 12B)xe^x + (A + 12B - 2A - 12B + A)x^2e^x + (B - 2B + B)x^3e^x = xe^x$.
Therefore, $8A + 24B = 0$ and $24B = 1$. It follows that $B = 1/24, A = -1/8$, and $y_p = -\frac{1}{8}x^2e^x + \frac{1}{24}x^3e^x$.

3.5, 17. Associated homogeneous equation: $y'' + y = 0$. Characteristic equation: $r^2 + 1 = 0$. Solutions: $r = \pm i$. General solution: $y_c = c_1\cos x + c_2\sin x$.

Since $f(x) = \sin x + x\cos x$, we consider $\sin x$ and $x\cos x$ separately. Derivatives of $\sin x$ are $\pm\cos x$ or $\pm\sin x$, thus the trial solution for $\sin x$ has form $x^s(A\sin x + B\cos x)$. Both $\sin x$ and $\cos x$ are solutions of the associated equations, thus we must take $s = 1$. For $x\cos x$, the linear combinations of all its derivatives will have the form $Cx\sin x + Dx\cos x + E\sin x + F\cos x$. Thus the trial solution here is $x^s(Cx\sin x + Dx\cos x + E\sin x + F\cos x)$. Again, because both $\sin x$ and $\cos x$ are solutions of the associated equations, thus we must take $s = 1$. The sum of both trial solutions gives us the trial solution for $f(x)$: $x(A\sin x + B\cos x) + x(Cx\sin x + Dx\cos x + E\sin x + F\cos x)$. Combining similar terms together (and relabeling undetermined coefficients), we get $ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x$.

Plug in trial solition: $(ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x)'' + ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x = \sin x + x\cos x$.
$((2c - 2b)\sin x + (2a + 2d)\cos x + (-2a - 4d)x\sin x + (-2b + 4c)x\cos x - cx^2\sin x - dx^2\cos x) + ax\sin x + bx\cos x + cx^2\sin x + dx^2\cos x = \sin x + x\cos x$.
$(2c - 2b)\sin x + (2a + 2d)\cos x + (-2a - 4d + a)x\sin x + (-2b + 4c + b)x\cos x = \sin x + x\cos x$.
Therefore, $2c - 2b = 1$, $2a + 2d = 0$, $-a - 4d = 0$, $-b + 4c = 1$. It follows that $a = d = 0$, $b = -1/3$, $c = 1/6$.

3.5, 38. Associated homogeneous equation: $y'' + 2y' + 2y = 0$. Characteristic equation: $r^2 + 2r + 2 = 0$. Solutions: $r = -1 \pm i$. General solution: $y_c = c_1e^{-x}\cos x + c_2e^{-x}\sin x$.

Since $f(x) = \sin 3x$, $f'(x) = 3\cos 3x$. A linear combination of $f(x)$ and its derivatives has the form $A\sin 3x + B\cos 3x$. Trial solution: $x^s(A\sin 3x + B\cos 3x)$. Neither $\sin 3x$ nor $\cos 3x$ are particular solutions of the associated equation, hence we take $s = 0$. Trial solution: $A\sin 3x + B\cos 3x$.

Plug in trial solution: $(A\sin 3x+B\cos 3x)''+2(A\sin 3x+B\cos 3x)'+2(A\sin 3x+B\cos 3x)=\sin 3x$.
$-9A\sin 3x-9B\cos 3x+2(3A\cos 3x-3B\sin 3x)+2(A\sin 3x+B\cos 3x)=\sin 3x$.
$(-9A-6B+2A)\sin 3x+(-9B+6A+2B)\cos 3x=\sin 3x$.
Hence $-7A-6B=1$ and $-7B+6A=0$. Thus $A=-7/85$, $B=-6/85$, and $y_p=-\frac{7}{85}\sin 3x-\frac{6}{85}\cos 3x$.

The general solution is $y(x)=c_1e^{-x}\cos x+c_2e^{-x}\sin x-\frac{7}{85}\sin 3x-\frac{6}{85}\cos 3x$. Thus $y'(x)=-c_1e^{-x}\cos x-c_1e^{-x}\sin x-c_2e^{-x}\sin x+c_2e^{-x}\cos x-\frac{21}{85}\cos 3x+\frac{18}{85}\sin 3x=(c_2-c_1)e^{-x}\cos x+(-c_1-c_2)e^{-x}\sin x-\frac{21}{85}\cos 3x+\frac{18}{85}\sin 3x$.

$2=y(0)=c_1-\frac{6}{85}$ and $0=y'(0)=(c_2-c_1)-\frac{21}{85}$. Then $c_1=\frac{176}{85}$ and $c_2=\frac{197}{85}$. Solution: $y(x)=(176e^{-x}\cos x+197e^{-x}\sin x-7\sin 3x-6\cos 3x)/85$.

4.1, 6. Set $z=x'$, $w=y'$. Then $x''=z'$ and $y''=w$. Answer:

$$\begin{cases} z'-5x+4y=0 \\ w'+4x-5y=0 \\ z=x' \\ w=y' \end{cases}$$

4.1, 17. $y=x'$, thus $y'=x''$. From $y'=6x-y$, $x''=6x-x'$, i.e. $x''+x'-6x=0$. Char eq-n: $r^2+r-6=0$. $r=-3,2$. General solution: $x(t)=c_1e^{-3t}+c_2e^{2t}$, $y(t)=x'(t)=-3c_1e^{-3t}+2c_2e^{2t}$.

$1=x(0)=c_1+c_2$, $2=y(0)=-3c_1+2c_2$. Hence $c_1=0, c_2=1$. Solution: $x(t)=e^{2t}, y(t)=2e^{2t}$.

4.2, 4. $y=3x-x'$, hence the second eq-n becomes $(3x-x')'=5x-3(3x-x')$. Then $x''-4x=0$. Char. eq-n: $r^2-4=0$. $r=\pm 2$, thus $x(t)=c_1e^{2t}+c_2e^{-2t}$ and $y(t)=3x-x'=c_1e^{2t}+5c_2e^{-2t}$.

$1=x(0)=c_1+c_2, -1=y(0)=c_1+5c_2$. Hence $c_1=3/2, c_2=-1/2$. Solution: $x(t)=\frac{3}{2}e^{t}-\frac{1}{2}e^{-t}, y(t)=\frac{3}{2}e^{t}-\frac{5}{2}e^{-t}$.

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5.1, 18. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, P(t) = \begin{bmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{bmatrix}, \mathbf{f}(t) = \mathbf{0}$.

5.1, 26. $W = \begin{vmatrix} 2e^t & -2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & e^{3t} & e^{5t} \end{vmatrix} = 16e^t e^{3t} e^{5t} = 16e^{9t} \neq 0$. General solution: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3}$.

5.2, 6. $A = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 12 = 0$, so $\lambda = 3, 4$. Eigenvector for $\lambda = 3$, $\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$. For $\lambda = 4$, similarly $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. General solution: $c_1 \begin{bmatrix} 5 \\ -6 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$. $x_1 = 5c_1e^{3t} + c_2e^{4t}, x_2 = -6c_1e^{3t} - c_2e^{4t}$. For given initial values, $5c_1 + c_2 = 1, -6c_1 - c_2 = 0$, i.e. $c_1 = -1, c_2 = 6$.

5.2, 9. $A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + 16 = 0$, so $\lambda = \pm 4i$. Eigenvector for $\lambda = 4i$, $\begin{bmatrix} 2-4i & -5 \\ -6 & -2-4i \end{bmatrix}\mathbf{v} = \mathbf{0}$, then $\mathbf{v} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$. Particular complex solution: $\begin{bmatrix} 5 \\ 2-4i \end{bmatrix} e^{4it} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix} (\cos 4t + i\sin 4t) = \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + i\begin{bmatrix} 5\sin 4t \\ -4\cos 4t + 2\sin 4t \end{bmatrix}$. General solution: $c_1 \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin 4t \\ -4\cos 4t + 2\sin 4t \end{bmatrix}$. $x_1 = 5c_1\cos 4t + 5c_2\sin 4t, x_2 = c_1(2\cos 4t + 4\sin 4t) + c_2(-4\cos 4t + 2\sin 4t)$. For given initial values,

5.2, 23. $A = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{vmatrix} = (3 - \lambda)((3-\lambda_(-3-\lambda)+5) = (3-\lambda)(\lambda^2-4) = 0$. $\lambda = -2, 2, 3$. Eigenvectors: for $\lambda = -2$, $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$; for $\lambda = 2$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; for $\lambda = 3$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. General solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$.

5.4, 6. Eigenvalues: $\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = 0$, so $\lambda = 5$ (multiplicity 2). Eigenvector(s) for $\lambda = 5$: $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\mathbf{v} = 0$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Defect of $\lambda = 1$. Generalized eigenvectors: $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so can choose $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\mathbf{v}$ is an eigenvector, gives particular solution $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t}$. $\mathbf{w}$ is not an eigenvector, so use $\begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \mathbf{w} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$ to obtain particular solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} te^{5t}$. Answer $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} te^{5t} \right)$.

5.4, 8. Eigenvalues: $\begin{vmatrix} 25-\lambda & 12 & 0 \\ -18 & -5-\lambda & 0 \\ 6 & 6 & 13-\lambda \end{vmatrix} = (13-\lambda)(\lambda^2 - 20\lambda + 91) = 0$, $\lambda = 7, 13, 13$. Eigenvector for $\lambda = 7$, $\begin{bmatrix} 18 & 12 & 0 \\ -18 & -12 & 0 \\ 6 & 6 & 6 \end{bmatrix} \mathbf{v} = \mathbf{0}$, so $v_1 + v_2 + v_3 = 0$ (3rd equation), $3v_1 + 2v_2 = 0$ (1st or 2nd equations), get $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$ and a particular solution $\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} e^{7t}$. Eigenvector(s) for $\lambda = 13$, $\begin{bmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{bmatrix} \mathbf{v} = \mathbf{0}$, get $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; particular solutions: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{13t}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{13t}$. Answer: $\mathbf{x} = c_1 \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{13t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{13t}$.

5.4, 17. Eigenvalues: $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 18 & 7-\lambda & 4 \\ -27 & -9 & -5-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 2\lambda + 1) = (\lambda - 1)^3$, so $\lambda = 1$ with multiplicity 3. Eigenvector(s): $\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix} \mathbf{v} = \mathbf{0}$, so $9v_1 + 3v_2 + 2v_3 = 0$ (2nd or 3rd eq-n). Eigenvectors: $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix}$ giving particular solutions $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^t$ and $\begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix} e^t$. Defect of $\lambda = 1$. Generalized eigenvectors: $\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix}^2 \mathbf{w} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{w} = \mathbf{0}$. So, every vector is a generalized eigenvector. We need three linearly independent ones, so choose the two original eigenvectors and one vector linearly independent from them, e.g. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Use $\begin{bmatrix} 0 & 0 & 0 \\ 18 & 6 & 4 \\ -27 & -9 & -6 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 0 \\ 18 \\ -27 \end{bmatrix}$ to obtain the particular solution $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 18 \\ -27 \end{bmatrix} te^t$.

Answer: $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 18 \\ -27 \end{bmatrix} te^t \right)$.

5.5, 5. First, find the general solution: $A = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}$. Eigenvalues: $\lambda = \pm 3i$. Eigenvector for $\lambda = 3i$: $\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix} \mathbf{v} = \mathbf{0}$. $\mathbf{v} = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix}$. Particular complex solution: $\begin{bmatrix} -2 \\ 3+3i \end{bmatrix} e^{3it} = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix} (\cos 3t + i \sin 3t) = \begin{bmatrix} -2\cos 3t \\ 3cos3t - 3\sin 3t \end{bmatrix} + i \begin{bmatrix} -2\sin 3t \\ 3\sin 3t + 3\cos 3t \end{bmatrix}$. General solution: $\mathbf{x} = c_1 \begin{bmatrix} -2\cos 3t \\ 3cos3t - 3\sin 3t \end{bmatrix} + c_2 \begin{bmatrix} -2\sin 3t \\ 3\sin 3t + 3\cos 3t \end{bmatrix}$. Fundamental matrix: $\Phi(t) = \begin{bmatrix} -2\cos 3t & -2\sin 3t \\ 3\cos 3t - 3\sin 3t & 3\sin 3t + 3\cos 3t \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} -2 & 0 \\ 3 & 3 \end{bmatrix}$, $\Phi(0)^{-1} = -\frac{1}{6} \begin{bmatrix} 3 & 0 \\ -3 & -2 \end{bmatrix}$. Answer: $\begin{bmatrix} -2\cos 3t & -2\sin 3t \\ 3\cos 3t - 3\sin 3t & 3\sin 3t + 3\cos 3t \end{bmatrix} \cdot -\frac{1}{6} \begin{bmatrix} 3 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2\cos 3t & -2\sin 3t \\ 3\cos 3t - 3\sin 3t & 3\sin 3t + 3\cos 3t \end{bmatrix} \cdot \frac{-1}{6} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3\cos 3t - \sin 3t \\ -3\cos 3t + 6\sin 3t \end{bmatrix}$.

5.5, 25. $\mathbf{X} = \exp\left(\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} t \right) = \exp\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} t \right) \exp\left(\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t \right) = e^{2t} \exp\left(\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t \right) = e^{2t} \left(\mathbf{I} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} t \right)$ (higher powers of $\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$ are zero) $= e^{2t} \begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}$.
$\mathbf{x} = \mathbf{X} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 4e^{2t} + 35te^{2t} \\ 7e^{2t} \end{bmatrix}$.

5.5, 26. $\mathbf{X} = \exp\left(\begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix} t \right) = \exp\left(\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} t \right) \exp\left(\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t \right) = e^{7t} \exp\left(\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t \right) = e^{7t} \left(\mathbf{I} + \begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix} t \right)$ (higher powers of $\begin{bmatrix} 0 & 0 \\ 11 & 0 \end{bmatrix}$ are zero) $= e^{7t} \begin{bmatrix} 1 & 0 \\ 11t & 1 \end{bmatrix} = \begin{bmatrix} e^{7t} & 0 \\ 11te^{7t} & e^{7t} \end{bmatrix}$.
$\mathbf{x} = \mathbf{X} \begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 5e^{7t} \\ e^{7t}(55t - 10) \end{bmatrix}$.

6.1, 5. $1 - y^2 = 0, x + 2y = 0$. Hence, $y = \pm 1$ and, for $y = 1$, $x = -2$; for $y = -1$, $x = 2$. Critical points: $(-2, 1), (2, -1)$.

6.1, 15. $x' = -2x$, thus $x = ae^{-2t}$; $y' = -y$, thus $y = be^{-t}$. As $t \to \infty$, $x \to 0, y \to 0$. Asymptotically stable.

6.1, 17. $x' = y, y' = -x$, thus $x'' = y' = -x$. $x'' + x = 0$. Characteristic equation: $r^2 + 1 = 0$. $x = c_1 \cos t + c_2 \sin t$ and $y = c_1 \sin t - c_2 \cos t$. Stable.

6.2, 5. $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$. $\lambda_1 = \lambda_2 = -1 < 0$. Asymptotically stable.

6.2, 16. Linearization: $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = 0$. $\lambda = 2 \pm i$. Real parts of λ are positive. Unstable.

6.2, 23. Linearization: $\begin{bmatrix} 2 + 3x^2 & -5 \\ 4 & -6 + 4y^3 \end{bmatrix}$. At $(0, 0)$, $\begin{bmatrix} 2 & -5 \\ 4 & -6 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 2-\lambda & -5 \\ 4 & -6-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 8 = 0$. $\lambda = -2 \pm 2i$. Real parts are negative. Asymptotically stable.

6.2, 26. Linearization: $\begin{bmatrix} 3 - 2x & -2 - 2y \\ 2 - 2x & -1 + 4y^3 \end{bmatrix}$. At $(0, 0)$, $\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 3-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0$. $\lambda = 1 > 0$. Unstable.