Math 211 - Introduction to Linear Algebra (Spring 2018)

Instructor: Ben McMillan
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Locataion: MW 2:00pm--3:50pm in Library E4330

Course Information:

The course syllabus is here. Some critical points:

- The midterm will be in class on Monday, March 5.
- My office hours this term are Mondays and Tuesdays 11:30am--12:30pm in the math tower, office 2-116. I will also be available in the MLC (S-235) on Wednesdays 1:00--2:00pm.

Schedule and Homework:

The following is a tentative schedule for the course. As homework is assigned it will be posted here. You are encouraged to work with others, but please make sure to write up solutions in your own words (this will help you on the exams!).

Unless stated otherwise, the homework is from the corresponding section of the book.

All of the problems assigned in week n are due at the beginning of lecture on the Wednesday of week n+1.

Week	Date	Topic(s) Covered	Reading	Homework
1	1/22	Linear systems of equations	Chapter 1.1	1, 7, 8, 11, 17, 21
	1/24	Gauss-Jordan elimination	1.2	1, 2, 5, 18, 22, 30, 31
2	1/29	Counts of solutions and matrix algebra	1.3	1, 2, 4, 9, 10, 14
	1/31	Matrix algebra	1.3	16, 17, 22, 30, 34, 55
3	2/5	Linear transformations	2.1	1, 2, 3, 5, 6, 33, 42
	2/7	Linear geometry	2.2	3, 16, 29, 33, 53
4	2/12	Linear composition is matrix multiplication	2.3	1, 2, 10, 11, 13, 40, 65
	2/14	Linear inverses	2.4	16, 18, 21, 23, 41
5	2/19	Linear inverses continued	2.4	2, 3, 5, 34, 43, 82, 83
	2/21	Image and kernel of linear functions	3.1	1, 4, 14, 17, 19, 25, 49
6	2/26	(Im and Ker are) linear subspaces	3.1 3.2	32, 34, 40, 44 1, 2, 3, 5
	2/28	.	Review	Review
7	3/5	.	Midterm	Review sheet here.
	3/7	Snow Day	.	.
8	3/12	Spring Break!	.	.
	3/14	Spring Break!	.	.
9	3/19	Linear (in)dependence	3.2	8, 15, 16, 21, 22, 34, 36, 37, 39, 50
	3/21	Snow Day	.	.
10	3/26	Basis and dimension	3.2 3.3	27, 29, 43 1, 3, 9, 29, 62, 67
	3/28	The Rank-Nullity Theorem	3.3	22, 27, 31, 33, 34, 37
11	4/2	Linear coordinates	3.4	1, 3, 4, 6, 15, 16, 50
	4/4	Linear maps using coordinates	3.4	25, 27, 30, 33, 37, 43
12	4/9	Abstract Vector Spaces	4.1	1, 2, 7, 9, 16, 20, 25, 47, 48
	4/11	Linear Isomorphisms	4.2	1, 2, 3, 4, 7, 9, 42, 60
13	4/16	Linear map + a basis gives a matrix	4.3	5, 6, 13, 14, 20, 41, 44, 48, 60
	4/18	The determinant	6.1	5, 7, 9, 10, 15, 23, 28, 39, 43
14	4/23	Properties of the determinant	6.2	1, 3, 7, 11, 13, 15, 21, 38, 48, 55
	4/25	Eigenvectors	7.1 7.3	1, 4, 19, 23 1, 4, 16, 34

15	4/30	Geometry, eigenvalues and the determinant	.	None
	5/2	Review	.	Review Sheet
16	5/9	Final Exam	Engineering 143, 8:00am - 10:45am	

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Spring 2018 Stony Brook

Instructor:	Ben McMillan	**Time:**	MW 2:30pm–3:50pm
Email:	bmcmillan@math.stonybrook.edu	**Place:**	Library E4330

Course Page: The primary webpage for this course is

math.stonybrook.edu/~bmcmillan/math211/

where you will find up to date information, homework, and announcements. Please bookmark it and check back regularly.

You will also find announcements and grades on the course blackboard page.

Office Hours: Office hours are an invaluable resource, one that you really should use!

My office hours this term are Mondays and Tuesdays 11:30am–12:30pm in the math tower, office 2-116. I will also be available in the MLC (S-235) on Wednesdays 1:00–2:00pm.

You can find the grader's office hours at math.stonybrook.edu/office-hours

Textbook: The lecture will roughly follow Bretscher's *Linear Algebra with Applications, Fifth Edition*, and I will assign homework problems from the book.

Exams: The midterm will be held *in class* on Monday, March 5. Please let me know if you will need DSS accomodations at least 2 weeks before this.

The final is scheduled for Wednesday, May 9, 8:00am-10:45am. Please ensure NOW that you won't have any scheduling conflicts with the final!

Homework: After each lecture I will post relevant homework questions on the course webpage. At the *beginning* of each Wednesday lecture you will turn in the problems assigned the previous week.

Each homework will be graded out of 30 points, which will be divided into two parts. So, 14 of your points will come from attempting all of the problems. For the remaining 16 points, I will choose 2 problems that the grader will look at more closely and provide more detailed feedback for.

Grading Policy: Homework: 30%, Midterm: 30%, Final: 40%.

Americans with Disabilities Act: If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, Room 128, (631)632–6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. http://studentaffairs.stonybrook.edu/dss/index.html.

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The exam will cover chapters 1 and 2, and section 3.1 as well as linear transformations as we've discussed them in lecture. The following is a list of things I definitely want you to know. It is certainly not exhaustive (I'm not about to retype 100 pages of textbook!), but hopefully gives you some idea. Note that some questions here are to demonstrate an idea, and so are chosen to be simplistic. Others are at what I feel is an appropriate level for the exam. You would do well to review your homeworks, especially the problems that you struggled with.

- Systems of equations and Gauss elimination. You should be able to completely reduce a system of equations. You will also benefit from understanding what you are doing and why in this process.
- You should be comfortable with matrix algebra—adding and rescaling vectors, matrix multiplication, linearity of these operations.
- The definition of a linear transformation is the one thing that I have asked you to memorize in this course. It's that important. That doesn't mean that there will be a question like "state the definition of linearity" on the exam. You obviously need to understand the definition, well enough that you can apply it.

  For example, consider a linear map $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ for which

  $$S[e_1] = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \qquad S[e_2] = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}.$$

  Since S is linear, you can determine it's value for any vector. What is

  $$S\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]?$$

- Linear maps always have a matrix representation: for fixed linear $T\colon \mathbb{R}^n \to \mathbb{R}^m$, there is a corresponding matrix A so that $T[v] = Av$ for all vectors v in $\mathbb{R}^n$. However, linear maps don't always come with the matrix given. A good example from the homework is the linear map $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across a line (2.2.16(b)). That is a perfectly valid and useful description of a linear map, but sometimes it is also useful to know the corresponding matrix. What 2x2 matrix corresponds to R? (See next bullet point for a hint.)
- Homework 3 was a difficult one. I hope that at this point you can look back and find it a little easier, now that we've had some more time to think about linearity. I would reccomend reviewing it specifically, as there was a lot of meat there.

  One particularly useful fact that was helpful for that homework: for fixed linear $T\colon \mathbb{R}^n \to \mathbb{R}^m$, the corresponding matrix A is the matrix whose n columns are given by $T[e_1], \ldots, T[e_n]$. (Recall the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$.) This is one of those facts that sounds complicated, but is not, and knowing it will make your life much easier. To check your understanding, make sure you understand why the linear map S above is represented by the matrix

  $$\begin{pmatrix} 2 & -1 \\ -3 & 2 \\ 6 & -4 \end{pmatrix}.$$

- Another useful theorem: a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is uniquely determined by it's value on the n vectors $e_1, \ldots, e_n$. Why is this true? (Hint: T is a linear map.) Use the theorem and planar geometry to show that the two following linear maps are equal:

1) $R_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule "take v and rotate it by $2\pi/3$ degrees counter-clockwise."

2) $R_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule "take v, reflect across the x-axis, then reflect the result across L", where L is the line through the origin at angle $\pi/3$ from the x-axis

- A linear function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is, like any function, a rule that assigns a vector in $\mathbb{R}^m$ to each vector in $\mathbb{R}^n$. Can you express R_2 from the previous bullet point as the composition of 2 linear maps?
- On the other hand, the composition of linear maps works nicely together with matrix multiplication: If T is represented by A and S represented by B, then TS is represented by AB.
- Invertibility. A linear map is invertible if and only if the representing matrix is invertible. We discussed in lecture how it is easier to show that a linear map *is not* invertible and that a matrix *is* invertible. For example, the linear map $\mathbb{R}^3 \to \mathbb{R}^3$ given by the rule "project v to the xy-plane" is not invertible. Why? On the other hand, the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
is invertible. Why? What is its inverse? (Hint: the inverse is upper triangular.)
- The formula for inverses of 2x2 matrices is useful.
- Recall the method for using Gauss-Jordan elimination to find the inverse of a matrix.
- There will likely be a question involving the image or the kernel of a linear map. I may remind you of the definition, but you will need to understand it well enough to apply it. For example, it would be reasonable for me to ask you to demonstrate that $\ker(T)$ for some given linear map T is a subspace. (You have to check that it is closed under addition and rescaling. We did the proof in class. Don't memorize the proof.)
- Throughout the lecture, there were several theorems, often given with a proof. You should know/understand and be able to apply the theorems. Their purpose is to save you work, because applying a theorem is better than reproving it in a specific instance. You will not be asked to recreate the proofs, but you should know them well enough that you could follow the logic if provided. If I gave you a proof in lecture, it was because there was an idea in it that I think is useful for you to know. Those ideas may well appear on the exam. For example, many of the theorems we've seen shared very similar proofs—essentially, use the two properties of linearity and the decomposition of a vector into the standard basis—and you would do well to understand that general idea. Understanding the proofs is a good way to gain that understanding.

- This review is necessarily not exhaustive. I recommend looking back over the homework, especially the problems that were graded and the ones that you found difficult. I also offered exercises in lecture, which should give you an idea of what I think is interesting to ask, and what I think is reasonable to ask.
- Consider a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ and vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^5$. Suppose that $T[\vec{v}_1], T[\vec{v}_2], T[\vec{v}_3]$ are linearly independent. Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ independent? Hint: use the definition of linear independence.
- Make sure you know and understand the statement of the rank nullity theorem. It's consistently useful, because it often quickly tells you the dimension of either the kernel or the image of a map.
- Knowing the dimension of a space makes finding a basis half as difficult: if $\dim(V) = n$ and vectors $\vec{v}_1, \dots, \vec{v}_n$ either 1) span V or 2) are independent, then both 1 and 2 are true, meaning they form a basis for V.
- Consider the linear map $T\colon P_3 \to P_3$ given by the rule $T[f(t)] := f(t) + tf'(t) + f''(t)$. Use the standard basis of P_3 to express T as a matrix. Draw the coordinate square that gives rise to this matrix.
- Find a basis for the kernel of T from the previous bullet. Find a basis for the image.
- Let P_2 be the vector space of quadratic polynomials. Let $\mathcal{B} = \{1+t, t+t^2, 1+t^2\}$ be a basis for P_2. What are the coordinates of $2+t+2t^2$ in this basis? What are the coordinates of the general quadratic $a_0 + a_1 t + a_2 t^2$?
- Express the matrix

$$A = \begin{pmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{pmatrix}$$

in terms of the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

Draw the coordinate square.
- Write the change of basis matrix from the basis $\mathcal{B}$ in the previous bullet to the basis

$$\mathcal{A} = \left\{ \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \right\}.$$

How about the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$.
- Make sure you understand the criterions for when a linear map T is an isomorphism. There is a handy table on page 183 of the book. Don't memorize it, but do make sure that you understand the pieces.
- Consider the linear map given by reflection across the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. Reason geometrically to find a basis of eigenvectors for this map. (Note that the vector $(1\ 1\ 1)^T$ is perpendicular to this plane.)
- Consider a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ which has a basis consisting of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. What is the matrix of T in this basis? Why?

- What is the determinant of
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix}$$
- Diagonalize $A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$. First you find the eigenvalues, then the eigenvectors, then you use that to form the matrix S whose columns are the eigenvectors. Then $D = S^{-1}AS$ will be diagonal.
- What is A^{10}? Here A is the matrix from the previous problem. Don't multiply it out. Instead, note that D^{10} is easy to express, and check that $A^{10} = (SDS^{-1})^{10} = SD^{10}S^{-1}$.