MAT 200: Logic, Language and Proof

Syllabus

Homework

Due Monday, Sept 12 - Problems 1.1, 1.4, 1.5, 1.7, 1.8

Due Monday, Sept 19 - Problems 1.9, 1.10, 1.11, 1.13, 1.16, 1.21

There is a midterm in class on Monday, Sept 26. Here are some practice problems.

Here are solutions to the practice problems.

Due Monday, Sept 26 - Problems 2.3, 2.4, 2.6, 2.8, 2.9

Due Monday, Oct 10 - Problems 2.11, 2.14, 2.15, 2.16, 2.19

Due Monday, Oct 17 - Problems 3.4, 3.10, 3.11, 3.12, 3.20

Due Monday, Oct 24 - Problems 3.16, 3.17, 3.18

There is a midterm in class on Monday, Oct 24. Here are some practice problems.

Due Monday, Nov 7 - Exercise 14.2, Problems 4.2, 4.6, 4.7, 4.14

Due Monday, Nov 14 - Problems 5.3, 5.4, 5.6, 5.7, 5.15

Due Monday, Nov 21 - Problems 5.16, 5.17, 5.18, 5.19

Due Monday, Dec 5 - Problems 6.2, 6.3, 6.6, 6.12, 6.17

Due Monday, Dec 12 - Extra Credit Homework

MAT 200, Fall 2011
Instructor: Dave Jensen
Office Hours: M 2:30-3:30, 4:20-5:20, W 4:20-5:20
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Class Website: http://www.math.sunysb.edu/∼djensen/mat200.html

COURSE DESCRIPTION

The goal of the course is to introduce the student to logical reasoning and proofs. This course serves as an introduction to rigorous mathematics used in upper-division mathematics courses, but reasoning skills are of course generally useful in life, whatever your pursuits.
We will first discuss logical language and operations, and methods of proof in general. We will then focus on sets and maps between them - the foundational objects of mathematics. Finally we will apply the rigorous language to systematically define and study some notions of number theory and other mathematical topics as time permits.

TEXTBOOK

The official textbook for this class is *An Introduction to Mathematical Reasoning: numbers, sets and functions* by Peter J. Eccles. Reading the assigned chapter of the textbook before lecture will greatly help you!

HOMEWORK

Weekly problem sets will be assigned and collected in class on Mondays. The emphasis of the course is on writing proofs, so please try to write legibly and explain your reasoning clearly and fully. You are encouraged to discuss the homework problems with others, but your write-up must be your own work.

GRADING

There will be two midterm exams and a final exam. In addition to the exams, weekly homework will be assigned and collected. Final course grades will be based on this breakdown:

30 % final exam, 20 % each midterm exam, 30 % homework

The exam dates are as follows:

First Midterm: Monday, September 26, in class
Second Midterm: Monday, October 24, in class
Final Exam: Tuesday, December 13, 5:15 PM - 7:45 PM and Wednesday, December 14, 2:15 PM - 4:45 PM

DISABILITIES

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services at http://studentaffairs.stonybrook.edu/dss/ or (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

http://www.stonybrook.edu/ehs/fire/disabilities.shtml

ACADEMIC INTEGRITY

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

http://www.stonybrook.edu/uaa/academicjudiciary/

CRITICAL INCIDENCE MANAGEMENT

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

1. Prove that, for any pair of integers a and b, $a - b$ is divisible by 3 if and only if $a^3 - b$ is divisible by 3.

2. Prove that, for any pair of real numbers x and y, $\sqrt{x^2 + y^2} \leq |x| + |y|$.

3. Prove that there is no largest odd integer.

4. Prove that there is no real number x such that $sin(x) > \frac{1}{2}sec(x)$.

5. Prove that, for any positive integer N, $4^N + 5$ is divisible by 3.

6. Let F_N be the N^{th} Fibonacci number. Prove that, for all positive integers M and N,

   $$F_{M+N} = F_{M-1}F_N + F_M F_{N+1}.$$

   Use this to show that F_M divides F_{MN}.

7. Prove that, for any sets A, B, and C, $(A \cap C) \backslash B = (A \backslash B) \cap C$.

8. Prove that, for any sets A, B, and C, if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A \backslash C$.

There are many ways to prove each of these. For example, there is no reason to use induction in the first problem, but it is one possible approach. In order to receive credit on the exam, you do not need to prove each statement in the exact same way that I chose, but your argument must be complete, articulate, and logically coherent.

1. Prove that, for any pair of integers a and b, $a - b$ is divisible by 3 if and only if $a^3 - b$ is divisible by 3.

   First, let's show that, $\forall a \in \{0\} \cup \mathbb{N}$, $a^3 - a$ is divisible by 3. We prove this by induction on a. To start, consider the base case $a = 0$. In this case, $a^3 - a = 0 = 3 \cdot 0$. Since 0 is an integer, by definition, $a^3 - a$ is divisible by 3. To prove the inductive step, suppose that for some $k \in \{0\} \cup \mathbb{N}$, $k^3 - k$ is divisible by 3. We must show that $(k+1)^3 - (k+1)$ is divisible by 3 as well. Since $k^3 - k$ is divisible by 3, by definition $\exists q \in \mathbb{Z}$ such that $k^3 - k = 3q$. Hence

   $$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3(k^2 + k)$$
   $$= 3q + 3(k^2 + k) = 3(q + k^2 + k).$$

   Since $q + k^2 + k$ is an integer, by definition $(k+1)^3 - (k+1)$ is divisible by 3. It follows by induction that $a^3 - a$ is divisible by 3 for every $a \in \{0\} \cup \mathbb{N}$.

   This means that, for any $a \in \mathbb{N}$, $\exists q \in \mathbb{Z}$ such that $a^3 - a = 3q$. It follows that $(-a)^3 - (-a) = -(a^3 - a) = 3(-q)$, which is divisible by 3 since $-q$ is an integer. We therefore see that $a^3 - a$ is divisible by 3 $\forall a \in \mathbb{Z}$.

   We now show that, if $a - b$ is divisible by 3, then $a^3 - b$ is divisible by 3. Let a and b be integers such that $a - b$ is divisible by 3. By definition, $\exists q \in \mathbb{Z}$ such that $a - b = 3q$. Moreover, by the above, $\exists p \in \mathbb{Z}$ such that $a^3 - a = 3p$. Hence

   $$a^3 - b = (a^3 - a) - (a - b) = 3p - 3q = 3(p - q).$$

   Since $p - q$ is an integer, it follows by definition that $a^3 - b$ is divisible by 3.

   Finally, we show that if $a^3 - b$ is divisible by 3, then $a - b$ is divisible by 3. Let a and b be integers such that $a^3 - b$ is divisible by 3. By definition, $\exists q \in \mathbb{Z}$ such that $a^3 - b = 3q$. Moreover, by the above, $\exists p \in \mathbb{Z}$ such that $a^3 - a = 3p$. Hence

$$a - b = (a^3 - b) - (a^3 - a) = 3(q - p).$$

Since $q - p$ is an integer, it follows by definition that $a - b$ is divisible by 3.

2. Prove that, for any pair of real numbers x and y, $\sqrt{x^2 + y^2} \leq |x| + |y|$.

   Let x and y be two real numbers. Notice that

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y|.$$

   Since $|x| \geq 0$ and $|y| \geq 0$, we see that $2|x||y| \geq 0$, and hence

$$(|x| + |y|)^2 \geq |x|^2 + |y|^2.$$

   Next, we show that, for any $z \in \mathbb{R}$, $|z|^2 = z^2$. There are two cases to consider. First, if $z \geq 0$, then $|z|^2 = z^2$ by definition. On the other hand, if $z < 0$, then $|z|^2 = (-z)^2 = (-1)^2 z^2 = z^2$. It follows that

$$(|x| + |y|)^2 \geq |x|^2 + |y|^2 = x^2 + y^2.$$

   Finally, recall that we showed in class that, for any positive real numbers a and b, $a \geq b$ if and only if $a^2 \geq b^2$. Setting $a = |x| + |y|$ and $b = \sqrt{x^2 + y^2}$, we see that

$$a^2 = (|x| + |y|)^2 \geq x^2 + y^2 = b^2.$$

   Hence

$$a = |x| + |y| \geq \sqrt{x^2 + y^2} = b.$$

3. Prove that there is no largest odd integer.

   We prove this by contradiction. Suppose that there is a largest odd integer, and call it N. Since $N + 2$ is a larger integer than N, it must be even. Hence by definition $\exists q \in \mathbb{Z}$ such that $N + 2 = 2q$. But then, since $q - 1$ is an integer as well, we see that by definition $N = 2(q - 1)$ is even. This contradicts our assumption that N is odd. There is therefore no largest odd integer.

4. Prove that there is no real number x such that $sin(x) > \frac{1}{2}sec(x)$.

   Oops! This is false. For example, consider the case $x = \frac{3\pi}{4}$. Notice that
$$\frac{\sqrt{2}}{2} = sin(\frac{3\pi}{4}) > \frac{1}{2}sec(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}.$$

5. Prove that, for any positive integer N, $4^N + 5$ is divisible by 3.

   We prove this by induction on N. First, consider the base case $N = 1$. In this case $4^N + 5 = 9 = 3 \cdot 3$ is divisible by 3. To prove the inductive step, assume that for some $k \in \mathbb{N}$, $4^k + 5$ is divisible by 3. We must show that $4^(k+1) + 5$ is divisible by 3. By definition, $\exists q \in \mathbb{Z}$ such that $4^k + 5 = 3q$. Hence
$$4^{k+1} + 5 = 4 \cdot 4^k + 5 = 4 \cdot (4^k + 5) - 15 = 4 \cdot 3q - 15 = 3(4q - 5).$$
   Since $4q - 5$ is an integer, by definition we see that $4^{k+1} + 5$ is divisible by 3. It follows by induction that $4^N + 5$ is divisible by 3 for all positive integers N.

6. Let F_N be the N^{th} Fibonacci number. Prove that, for all positive integers M and N,
$$F_{M+N} = F_{M-1}F_N + F_M F_{N+1}.$$
   Use this to show that F_M divides F_{MN}.

   We prove the first part by strong induction on N. Let $M \in \mathbb{N}$. We consider first the base cases $N = 1$ and $N = 2$. When $N = 1$, note that by definition
$$F_{M+1} = F_{M-1} + F_M = F_{M-1}F_1 + F_M F_2.$$
   Similarly, when $N = 2$, note that by definition
$$F_{M+2} = F_{M+1}+F_M = F_{M-1}+F_M+F_M = F_{M-1}+2F_M = F_2F_{M-1}+F_3F_M.$$
   To prove the inductive step, we assume that for all positive integers $k < N$,
$$F_{M+k} = F_{M-1}F_k + F_M F_{k+1}.$$

We must show that

$$F_{M+N} = F_{M-1}F_N + F_M F_{N+1}.$$

To see this, note that by the inductive hypothesis,

$$F_{M+N} = F_{M+N-1}+F_{M+N-2} = F_{M-1}F_{N-1}+F_M F_N+F_{M-1}F_{N-2}+F_M F_{N-1}.$$

Moreover, by the definition of the Fibonacci numbers, this equals

$$F_{M+N} = F_{M-1}(F_{N-1}+F_{N-2})+F_M(F_N+F_{N-1}) = F_{M-1}F_N+F_M F_{N+1}.$$

Hence, the expression holds for all positive integers M and N.

To see that F_M divides F_{MN}, we again proceed by induction on N. Let $M \in \mathbb{N}$. We consider first the base case $N = 1$. When $N = 1$, we see that clearly F_M divides F_M. To prove the inductive step, we assume that F_M divides F_{Mk} for some positive integer k. We must show that F_M divides $F_{M(k+1)}$. By the above, we know that

$$F_{M(k+1)} = F_{M+kM} = F_{M-1}F_{kM} + F_M F_{kM+1}.$$

By the inductive hypothesis there exists a $q \in \mathbb{Z}$ such that $F_{kM} = qF_M$, hence the expression above equals

$$F_{M(k+1)} == qF_{M-1}F_M + F_M F_{kM+1} = F_M(qF_{M-1} + F_{kM+1}).$$

Since $qF_{M-1}+F_{kM+1}$ is an integer, we see by definition that F_M divides $F_{M(k+1)}$. By induction, we therefore see that F_M divides F_{MN} for every positive integer N.

7. Prove that, for any sets A, B, and C, $(A \cap C)\backslash B = (A\backslash B) \cap C$.

We prove this by mutual containment. Let $x \in (A\cap C)\backslash B$. Then, since $x \in A \cap C$, $x \in A$, and since $x \in (A \cap C)\backslash B$, $x \notin B$. It follows that $x \in A\backslash B$. Similarly, since $x \in A\cap C$, $x \in C$. Therefore, x is an element of both $A\backslash B$ and C. In other words, $x \in (A\backslash B) \cap C$. This proves that $(A \cap C)\backslash B \subseteq (A\backslash B) \cap C$.

To prove the reverse containment, let $x \in (A\backslash B) \cap C$. Then x is an element of both $A\backslash B$ and C. Since $x \in A\backslash B$, $x \in A$ and $x \notin B$. Hence, x is an element of both A and C. In other words, $x \in A \cap C$. Moreover, since $x \notin B$, we see that $x \in (A \cap C)\backslash B$. This shows that $(A\backslash B) \cap C \subseteq (A \cap C)\backslash B$. Since we have shown that each set contains the other, we have $(A \cap C)\backslash B = (A\backslash B) \cap C$.

8. Prove that, for any sets A, B, and C, if $A \cap B \subseteq C$ and $x \in B$, then $x \notin A \backslash C$.

   Let $x \in B$ and suppose that $A \cap B \subseteq C$. If $x \in A$, then $x \in A \cap B$, hence by assumption $x \in C$. It follows that $x \notin A \backslash C$. On the other hand, if $x \notin A$, then by definition $x \notin A \backslash C$. Since this covers all of the possible cases, we see that $\forall x \in B$, $x \notin A \backslash C$.

1. For each of the following functions, determine whether it is bijective, injective but not surjective, surjective but not injective, or none of these.

   (a) $sin : \mathbb{R} \to [-1, 1]$.

   (b) $\alpha : Fun([n], Y) \to Y^n$, $\alpha(f) = (f(1), f(2), \ldots, f(n))$.

   (c) $u : Inj(X, X) \to Inj(X, X)$, $u(f) = f^{-1}$.

   (d) WYD : { people } $\to$ { people }, $WYD(x) = x$'s daddy.

2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that, if $g \circ f$ is injective, then f is injective.

3. Let X and Y be non-empty finite sets. Use the definition of cardinality to prove that $|X \times Y| = |X| \cdot |Y|$.

4. Let X and Y be non-empty finite subsets of a third set Z. Use the definition of cardinality to prove that $|X| = |Y|$ if and only if $|X \backslash Y| = |Y \backslash X|$.

5. Prove by induction that every non-empty subset of $\mathbb{N}$ contains a smallest element.

6. Suppose that $f : [n] \to X$ is a surjection. Prove, by induction on n, that X is a finite set and that $|X| \leq n$.

7. Prove that there are two non-bald people in New York State with exactly the same number of hairs on their head.

8. Let $X = \{n \in \mathbb{N} | n \text{ is a multiple of 5 and } n \leq 200,000\}$. Similarly, let $Y = \{n \in \mathbb{N} | n \text{ is a multiple of 3 and } n \leq 120,000\}$. Prove, without counting the number of elements in X or Y, that $|X| = |Y|$.

9. Each square of a 3×7-checkerboard is colored either purple or yellow. Show that the board must contain a rectangle consisting of at least 4 squares whose 4 corner squares are either all purple or all yellow.

MAT200 Extra Credit Homework

Due Monday, December 12

1. Find *all* integer solutions to the following Diophantine equations:

   (a) $4a^2 + b^2 = c^2$. (Hint: try a change of variables.)

   (b) $2a^2 + b^2 = 4c^2$. (Hint: use modular arithmetic.)

   (c) $2a^2 + b^2 = c^2$.

2. Describe each of the following functions geometrically as a tranformation of the Riemann sphere.

   (a) $f_1(z) = (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)z$.

   (b) $f_2(z) = -2z + 1$.

   (c) $f_3(z) = \frac{z-i}{z+i}$. (Hint: what is the image of the real axis?)

3. For each of the following relations on the set of people, determine whether it is an equivalence relation, a total ordering, or neither. Which of the defining properties of an equivalence relation or a total ordering does it satisfy?

   (a) $A \clubsuit B$ if A is smellier than B.

   (b) $A \diamondsuit B$ if A and B are the same age (to the nearest year).

   (c) $A \heartsuit B$ if A loves B.

   (d) $A \spadesuit B$ if A was born on one of the three days of the week following the day of the week on which B was born.

   (e) $A \star B$ if A is a banker.