MAT 171, Accelerated Single-Variable Calculus

Fall 2006

Instructor: Scott Simon

Mathematics
SUNY Stony Brook

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Office Hours: Monday 10:45-12:45, Wednesday 10:45-11:45.

Time and place: MWF 9:35am-10:30am Old Chem144

This is the accelerated version of MAT 132. It goes faster and is more theoretical (i.e., it includes proofs). There will be two midterms and a final as well as weekly homework assignments and quizzes. It also uses a different text. The main topics we will cover are: transcendental functions, methods of integration and infinite series.

Send me email at: sbsimon@math.sunysb.edu
or to Arthur Popa, the section 1 TA: arthur@math.sunysb.edu
or to Ionel Patu, the section 2. TA: ipatu@grad.physics.sunysb.edu

Syllabus: dvi file, pdf file
Homework assignments: pdf file

First midterm is early to mid-October in class. There are no calculators or notes allowed on the exam.

Here is a formula sheet for the first exam.
Here is a review sheet for the first exam.
Here are the cutoffs for the first exam.

Second midterm is Thursday, November 16 from 7:45pm-9:15pm at PSY A137. There are no calculators or notes allowed on the exam.

Here is a formula sheet for the second exam.
Here is a review sheet for the second exam.
Here are the cutoffs for the second exam.

Here is a formula sheet for the final exam.
Here is a review sheet for the final exam.
Here is an old sample final exam. Some of the questions are about things we haven't covered (such as sinh, etc.), but most of them are relevant.
Here is another old sample final.
Here are answers to one of the sample finals.
There WILL be class on Wednesday the 13th and Friday the 15th of December.

The Final is Wed., Dec 20, 8:00AM-10:30AM in Old Chem 144 (our usual classroom).

MLC hours: M,Tu,Wed 10-9, Th 10-6, F 10- 2, located in the basement of the Math Tower.

MATH 171, ACCELERATED SINGLE VARIABLE CALCULUS
SUNY STONY BROOK
FALL 2006

Lecturer	Scott Simon
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Office	4-121
Office Hours	TBA
Text	*Thomas' Calculus, 11th edition* (silver cover), by M. Weir, J. Hass, and F. Giordano
Web pages	math department: www.math.sunysb.edu course page: www.math.sunysb.edu/~sbsimon/math171/index.html
Syllabus	We will cover parts of chapters 5,6,7,8,9, and 11 with certain omissions and additional material. Homework assignments will be given in class and posted on the web.
Class Days	Classes begin on Wednesday, September 6 and end on Friday, December 15.
Holidays	There are no classes on Yom Kippur (October 2) and Thanksgiving break (November 23-24)
Lecture Times	Monday, Wednesday, and Friday, from 9:35 a.m.- 10:30 a.m.
Lecture Location	Room 144 in the Old Chemistry Building
Recitation Information	Section 1 meets Tu, Th 12:50 p.m.- 1:45 p.m. at Chemistry 123, and is taught by Arthur Popa, Section 2 meets M, W at 3:50 p.m. 4:45 p.m, and is taught by Ionel Patu
Homework	A list of problems from the text or supplementary material will be provided in class and on the web. Material covered in Monday's lecture will be due on the second recitation of the week, and material covered in Wednesday's and Friday's lecture will be due on the first recitation of the following week. The recitation instructor will collect and return homework, as well as give weekly quizzes.

Homework 1: p. 143 # 9, 22; p. 132, # 15; p. 235, # 31, 63; p. 260, # 9; p. 314, # 23; p. 388, # 5, 47; p. 365, #28
For Quiz 1: p. 375 # 19; p. 383 # 3, 45; p. 407 # 35, 43
Homework 2: p. 474 # 34,36,48,49,50; p. 484 # 28,32,42,49,73.
For Quiz 2: p. 493 # 38,54,60,68,71
For # 38 use implicit differentiation.
Homework 3: p. 500, #40,62,92 For # 40: You dont need to use logorithmic diferentiation
p. 530, # 14,74,104,119,127,128,147
Quiz 3: p. 559 # 18,29,38,50,54,66
Homework 4 (first part): p. 568 #1,31,37,44

Due to the short week, there will be no second part of homework 4 (so it will be worth half the points of a normal homework). Also, there will be no quiz, due to the exam. Nevertheless, here are the "quiz practice problems", which are fair game for the exam: p.579, # 12,20,22,45,50,51

Homework 5: p. 585 # 10,16,24,34,45,46; p. 591 # 12,26,41,44
Quiz 4: p. 631 # 36,44,50,58,65,66,67
Homework 6: p. 648 # 10,12,14,19,20,21,22; p. 657 # 6,10,16,26,32,34
Quiz 5: p. 757 # 24,36,46,50,90a
Homework 7: p. 769 # 8,16,26,36,40,66,71,75,77, p. 776 # 2,10,14,22,32,37,38,39,42
Quiz 6: p. 781 # 2,8,14,28,36,38,40
Homework 8: p. 786 # 2,8,14,22,30,46,47, p. 792 # 2,6,14,22,34,54,57,61a
Quiz 7: p. 804 # 8,16,28,39,41,42,46
Homework 9: p. 810 # 2,4,8,21,26,29
Quiz 8: Review homework 9.
Homework 10: p. 819 # 2,5,8,13,25,30a
Since there will be an exam on November 16th, homework 10 will not be collected. All material between exam 1 and homework 10, inclusive, are fair game for the exam.
Due to the short week, Homeworks 11 and 12 will both be collected on the week after break.
Homework 11:
1. cc = cubic centimeters.
Salt water which has a concentration of 0.1 g salt per cc of water is entering a tank at a rate of 10 cc per minute. Initially ($t = 0$) the tank contains a solution of 100 cc of water with 20 grams of salt dissolved in it. Completely mixed solution is exiting the tank at 1 cc per minute. If t represents the number of minutes after the moment when the tank had 100 cc of solution, find a function that describes the amount of salt in the tank as a function of t. Simplify as much as you can, but only as much as you can reasonably easily do. Hint: look at p.655.

2. Write the 2nd order Taylor polynomial approximation with the remainder

in little-o notation:

$$\frac{e^x - 1}{x}$$

3. Use a Taylor series expansion with little-o notation to find the limit:

$$\lim_{x \to 0} \frac{1 - \cos 2x - 2x^2}{x^4}$$

Homework 12: p.831 # 3,14,19,37,39,58,66
Homework 13: p.839 # 3,4,5,16, section 9.3 # 2,3,11,12. Feel free to use a calculator.
Homework 14:
1. Use the formulas $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{w+z} = e^w e^z$ to prove that $\cos(x+y) = \cos x \cos y - \sin x \sin y$, and to prove a similar formula for $\sin(x+y)$.

2. Given the differential equation $ay'' + by' + cy = 0$,
a) show that $y = e^{\lambda x}$ is a solution to the differential equation if $a\lambda^2 + b\lambda + c = 0$.
b)If λ_1 and λ_2 are both roots of the above quadratic equation, then for any complex numbers c_1 and c_2, $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ is also a solution to the differential equation.

3. Let $f(z) = z^2$. Show that f maps a ball of radius 2^{-7} centered at $2e^{\pi i/3}$ into a ball of radius 2^{-4} centered at $4e^{2\pi i/3}$.

Formulas given on the exam

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$2 \sin x \cos x = \sin 2x$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$$

The range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$.
The range of $\cos^{-1}$ is $[0, \pi]$.
The range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$.
The range of $\cot^{-1}$ is $(0, \pi)$.
The range of $\sec^{-1}$ is $[0, \pi/2) \cup (\pi/2, \pi]$.
The range of $\csc^{-1}$ is $[-\pi/2, 0) \cup (0, \pi/2]$.

Review problems: These review problems are for your benefit only. They will not be collected, and they are not comprehensive (i.e. there may be things on the test not covered in the review problems). The exam will cover material through section 8.4

Know the exact definitions of: Derivative, continuity. Know the following definition:

$$\int_a^b f(x)dx = \sum_{\|P\|\to 0} f(c_k)\Delta x_k,$$

and know what each of the symbols (e.g. $c_k, \Delta x_k$) mean.

Know how to use:

Intermediate value theorem
Mean Value theorem
Fundamental theorem of calculus (Parts I and II)
The formula for $(f^{-1})'$

Know how to do the homework/ quiz problems. Here are some extras that we can review in class.

Page	**Problem numbers**
89	47
113	53
122	9
132	55,59
236	64,65
352	1
366	27,67
376	61
389	27
407	37,39
474	13,45
484	43,47
493	25,57
500	57,61
530	57,73,75,93
558	39,43,47,53,57,63
568	5,7,11,25
579	17,21,29

There will be no quiz on the week of Oct. 2, but here are the “quiz questions” to practice for the exam: p.579 # 12,20,22,45,50,51

These cutoffs are advisory only. They can be used to estimate your final grade if you continue to perform at approximately the same level throughout the semester.

Cutoffs:

80-100	A
65-79	B
45-64	C
25-44	D
0-24	F

For scores within 2 points of a cutoff, add a plus or minus to the grade where appropriate.

These formulas will be provided on exam 2

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{R}.$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2k!}, \quad x \in \mathbb{R}.$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad x \in (-1, 1].$$

The remainder term for the nth Taylor polynomial for a function f expanded about a is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Review sheet for exam 2

The exam covers all material covered in class (and homework) from indefinite integrals through Taylor series and error estimates, inclusive.

You can study old homework/ quizzes to prepare for the exam. Here are some other things to look at (but the exam may not be limited to only problems of this type).

Problems:

p. 682 # 17,19,27,29

p. 682 #37,39,43 (slope field part, not the Euler part)

p. 840 #1-68 odd

Memorize: Definition of improper integral, infinite series.

Be able to use sequence and series convergence (or divergence) theorems.

Cutoffs for Exam 2

45-100	A
30-44	B
20-30	C
10-20	D
0-10	F

Scores within 2 points of a boundary: add a + or - to the grade where appropriate. For example, 45-47: A-.

Also, please note that these grades are advisory (to give you an idea of how to interpret your grade).

Formulas For Final Exam

$$\cos^2 x - \sin^2 x = \cos 2x$$
$$e = \lim_{x\to 0}(1+x)^{1/x}$$
$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$$

Euler's method If $y' = f(x,y)$, then

$$y_0 = y(x_0),$$
$$x_{n+1} = x_n + dx,$$
$$y_{n+1} = y_n + f(x_n, y_n)dx.$$

Improved Euler's method

$$y_0 = y(x_0),$$
$$x_{n+1} = x_n + dx,$$
$$z_{n+1} = y_n + f(x_n, y_n)dx,$$
$$y_{n+1} = y_n + \frac{f(x_n, y_n) + f(x_{n+1}, z_{n+1})}{2}dx,$$

Little-o As $x \to a$, we have:

1. $o(g(x)) = \pm o(g(x)) = o(g(x))$.
2. $o(cg(x)) = \pm o(g(x)), \quad c \neq 0$.
3. $f(x) \cdot o(g(x)) = o(f(x)g(x))$.
4. $o(o(g(x))) = o(g(x))$.
5. $(1+g(x))^{-1} = 1 - g(x) + o(g(x))$ if $g(x) \to 0$ as $x \to a$.

Fourier coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx,$$
$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx dx, \quad k \geq 1,$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx dx, \quad k \geq 1.$$

The Fourier series is given by

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

Binomeal coefficients are given by

$$\binom{m}{0} = 1, \quad \binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}, \quad k \geq 1.$$

Review Sheet for Final Exam

These are some review problems which you may study in addition to homework problems and quiz problems. This is not to be considered a comprehensive list of what will be on the exam. Other sources of practice exercises include p. 88,p.547, p.634, p. 682, and p. 840. You will not need to know material which we have not covered, such as hyperbolic sine, etc. Good luck!

page	problem number
406	21,35
493	45,65
500	55,59,69,73
531	91,103,111
558	35,41,45,51,55,61,69,81
568	23,29
579	15,19,27,33,39
585	13,21,31,37
591	27,35
631	33,63
648	17,21,23
657	13,21
664	1 (just the first term)
757	21,83
769	13,21,39,43,49,57
775	29
781	35
786	25,37,43
792	9,43,47
804	31,37
810	7,19,27
819	17,29,37
831	9,13,31,55
838	7,13

1. Let $a = 1 + i, \quad b = 1 - 2i$.
   (a) Find a in polar coordinates.
   (b) Find $a + b$, $a - b$, ab, and a/b in rectangular coordinates.
   (c) Find $\overline{a}$ in polar coordinates.
   (d) Find $\overline{a} + \overline{b}$, $\overline{a - b}$, $\overline{ab}$, and $\overline{a/b}$ in rectangular coordinates.
2. Let $\Re z$ denote the real part of z and $\Im z$ the imaginary part.
   (a) Show that
   $$\Re z = \frac{z + \overline{z}}{2}$$
   and
   $$\Im z = \frac{z - \overline{z}}{2i}.$$

(b) Use part (a) to conclude that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

when z is real.

(c) Use Taylor series to show the same equalities for any complex z.

SAMPLE FINAL MAT 142
FINAL is Friday, December 16, 2005,
11:30 to 1:00 in Physics P-112

1. Place the letter corresponding to the correct answer in the box next to each question.

(i) $\square$ The sequence $\{a_n\} = \{1 + (-1)^n \frac{1}{n}$ converges to
(a) 0 **(b)** -1 **(c)** 1 **(d)** $\frac{1}{2}$ **(e)** $-\frac{1}{2}$ **(f)** it diverges

(ii) $\square$ The sequence $\{a_n\} = \{(-1)^n(1 - \frac{1}{n})\}$ has least upper bound equal to
(a) -1 **(b)** 0 **(c)** 1 **(d)** 2 **(e)** $\frac{1}{2}$ **(f)** it has no upper bound

(iii) $\square$ Define a sequence by $a_0 = 1$, $a_n = \frac{3}{2}a_{n-1}$. Then the sequence converges to
(a) 0 **(b)** 1 **(c)** 2 **(d)** 4 **(e)** $\frac{3}{2}$ **(f)** the sequence diverges

(iv) $\square$ The infinite series $\sum_{n=0}^{\infty} 3^{-n}$ converges to
(a) 0 **(b)** $\frac{1}{3}$ **(c)** $\frac{2}{3}$ **(d)** $\frac{3}{4}$ **(e)** 1 **(f)** none of these

(v) $\square$ The infinite series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges to
(a) 0 **(b)** 1 **(c)** e **(d)** e^2 **(e)** 2 **(f)** none of these

(vi) $\square$ What is the inverse function of $y = \sqrt{1 - x^2}$ on $(0, 1)$?
(a) $x^2 - 1$ **(b)** $\sqrt{1 - x^2}$ **(c)** $x^2 + 1$ **(d)** $\sqrt{1 + x^2}$ **(e)** $\sqrt{1 - x}$ **(f)** none of these

(vii) $\square$ $\int_0^2 \frac{2x}{x^2 - 5} dx =$
(a) $\ln 2$ **(b)** $\ln 5$ **(c)** $-\ln 5$ **(d)** $-\ln 2$ **(e)** 0 **(f)** none of these

(viii) $\square$ $\frac{d}{dx} 2^{x^2} =$
(a) 2^{x^2} **(b)** $2^{x^2}2x$ **(c)** $2^{x^2} \ln 2$ **(d)** $2^{x^2}2x \ln 2$ **(e)** $2^{x^2}x^2 \ln 2$ **(f)** none of these

(ix) $\square$ Find the limit $\lim_{x \to 0^+} (1 + x)^{1/x}$.
(a) 0 **(b)** $1/e$ **(c)** 1 **(d)** e **(e)** ∞ **(f)** none of these

(x) $\square$ What is $\frac{d}{dx} \sin^{-1}(x)$,$|x| < 1$?
(a) $x/\sqrt{1 + x^2}$ **(b)** $1/\sqrt{1 + x^2}$ **(c)** $1/\sqrt{1 - x^2}$ **(d)** $-1/\sqrt{1 - x^2}$ **(e)** $1/(|x|\sqrt{x^2 - 1})$ **(f)** none of these

2. Evaluate each of the following integrals. You may use the table of integrals at the end of the book.

    (i) $\int \sin^3(x)dx$

    (ii) $\int \frac{dx}{1+\sin 3x}$

    (iii) $\int \sqrt{x^2-1}dx$

    (iv) $\int \frac{dx}{\sqrt{4+x^2}}$

    (v) $\int \frac{\sqrt{x+2}}{x}dx$

3. State whether each series diverges or converges. Explain your answer.

    (i) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

    (ii) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

    (iii) $\sum_{n=1}^{\infty}(-1)^n n^{1/2}$

    (iv) $\sum_{n=1}^{\infty} \frac{n^4}{2^n}$

    (v) $\sum_{n=1}(-1)^n \frac{n^2}{1+n^3}$

4. Solve each of the following differential equations.

    (i) $y' = e^{x-y}$

    (ii) $y' = 3x^2e^{-y}$

    (iii) $xy' + 3y = \frac{\sin x}{x^2}, x > 0$

    (iv) $xy' + 2y = 1 - \frac{1}{x}, x > 0$

    (v) $2y'e^{x/2} + y$

5. Write out the Taylor series at $x = 0$ up to order x^4 for each of the following functions.

    (i) $\sin(x^2)$

    (ii) $e^x \cos(x)$

    (iii) $\frac{1+x^2}{1-x}$

    (iv) $\sqrt{1+x}$

    (v) $\sin^2(x)e^{x^2+1}(1-\cos(x))$

6. Write out the Maclaurin series for $\sin x$ and $\tan x$ up to the third power. Use these series to evaluate

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}.$$

7. Prove that

$$1 + \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 3 + \ldots + \frac{n}{2^{n-1}} + \ldots = 4.$$

8. Give the definition of the hyperbolic functions $\sinh x$ and $\cosh x$. Using these definitions, show $\sinh 2x = 2\sinh x \cosh x$.

9. Quote Taylor's theorem and use it to show the Taylor series for $\sin(x)$ converges to $\sin(x)$ for all real numbers.

MAT 142 Spring 2003
FINAL

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

THERE ARE 11 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE.
SHOW YOUR WORK!!!

1	40	
2	30	
3	30	
4	20	
5	30	
6	30	
7	40	
8	dropped	
9	30	
10	20	
11	30	
Total	300	

1. Solve the following differential equation with initial value:

$$\frac{dy}{dx} = x + y, \qquad\qquad y(0) = 0.$$

2. A semicircular plate $2\,ft$ in diameter is submerged vertically into a fluid with the diameter along the surface. The weight-density of the fluid is $60\,lb/ft^3$.
Find the force exerted by the fluid on one side of the plate.

3. Use the definition of the natural logarithm

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0,$$

to prove that, for $a > 0$:

$$\ln\, ax = \ln a + \ln x.$$

4. Integrate:

a)

$$\int \frac{1}{\sqrt{4x + x^2}}\, dx.$$

b)

$$\int \cos{(\ln t)}\, dt.$$

5.

a) Find the Taylor series generated by $f(x) = x^5$ at $x = 1$.

b) Use a), or any other method, to write the following expression using partial fractions:

$$y = \frac{x^5}{(x-1)^6}.$$

6. Find a value c that makes the function $f(x)$ continuous at $x = 0$, where

$$f(x) = \frac{\sin(-7x)}{\tan(11x)}, \quad x \neq 0, \qquad f(0) = c.$$

7. Study the convergence of the following series.

a)

$$\sum_{n=10}^{\infty} \frac{(\ln n)^2}{n^4}$$

b)

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

c)

$$\sum_{n=1}^{\infty} (-1)^{n+1}(1+\frac{1}{n})$$

8. Find a sequence a_n such that all three conditions below are satisfied:

1) $\lim_{n\to\infty} a_n = 0$,

2) $a_n \geq 0$ for all n and

3) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ diverges.

9. Study the following power series (radius of convergence, interval of convergence, convergence at the endpoints).

a)

$$\sum_{n=0}^{\infty} \frac{2^n\, x^n}{n!}.$$

b)

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\sqrt{n}\,\ln n}.$$

10. Find the Maclaurin series of the following function

$$f(x) = x \ln (1 + x^2).$$

11. Prove, using (ϵ, N), that

$$\lim_{n\to\infty} (-1 - \frac{1}{n^{\frac{2}{3}}}) = -1.$$

MAT 142 Spring 2003
FINAL

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

THERE ARE 11 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE.
SHOW YOUR WORK!!!

1	40	
2	30	
3	30	
4	20	
5	30	
6	30	
7	40	
8	dropped	
9	30	
10	20	
11	30	
Total	300	

1. Solve the following differential equation with initial value:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 0.$$

$$y' - y = x$$

$$y' + Py = Q.$$

$$v = e^{\int P} = e^{-x}$$

$$y'e^{-x} - ye^{-x} = xe^{-x}$$

$$(ye^{-x})' = xe^{-x}$$

$$ye^{-x} = \int xe^{-x}dx$$

$$= -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + C$$

$$y = -x - 1 + Ce^{x}$$

$$0 = -1 + C \Rightarrow C = 1$$

$$y = e^{x} - x - 1$$

Check: $y' = e^{x} - 1 = (e^{x} - x - 1) + x$ ✓

$y(0) = 1 - 0 - 1 = 0$ ✓

2. A semicircular plate $2\,ft$ in diameter is submerged vertically into a fluid with the diameter along the surface. The weight-density of the fluid is $60\,lb/ft^3$.
Find the force exerted by the fluid on one side of the plate.

Not Covered in Class

3. Use the definition of the natural logarithm

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0,$$

to prove that, for $a > 0$:

$$\ln ax = \ln a + \ln x.$$

$$\frac{d}{dx} \ln ax = \frac{d}{dx}\int_1^{ax} \frac{dt}{t}, \qquad u = ax \quad \frac{du}{dx} = a$$

$$y = \ln ax = \int_1^{ax} \frac{dt}{t}.$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \quad \frac{dy}{du} = \frac{d}{du}\int_1^u \frac{dt}{t} = \frac{1}{u} = \frac{1}{ax}.$$

$$\frac{dy}{dx} = \frac{1}{ax}\cdot a = \frac{1}{x}.$$

Also! $\frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x}$. Therefore,

$\ln ax - \ln x = C$. Plugging in $x=1 \Rightarrow$

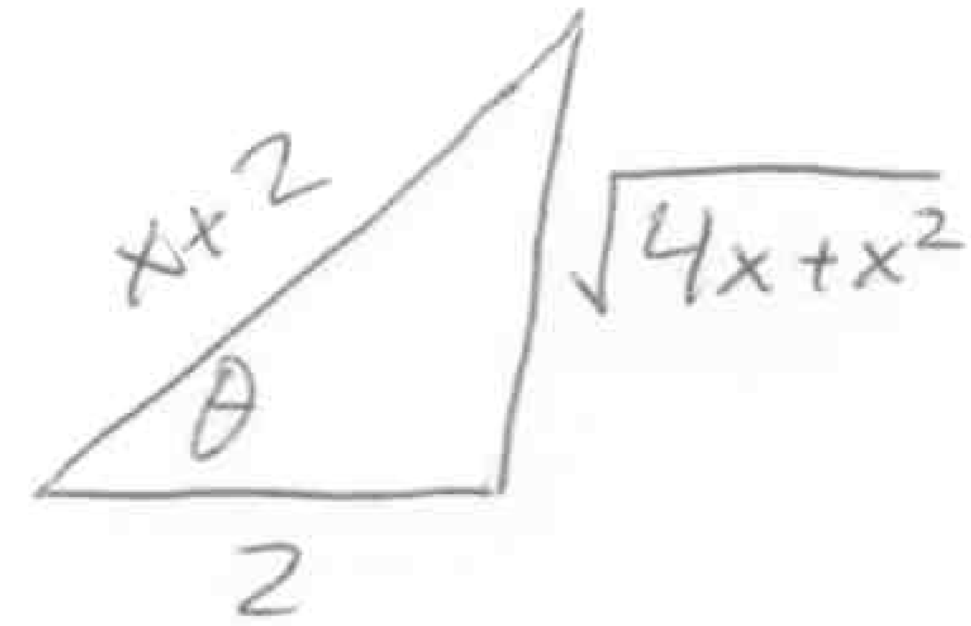
$\ln a - \underbrace{\cancel{\ln 1}}_{0} = C \Rightarrow C = \ln a.$

$\therefore \ln ax = \ln a + \ln x.$

4. Integrate:

a)

$$\int \frac{1}{\sqrt{4x+x^2}}\,dx.$$



$$x^2+4x = (x+2)^2-4.$$

$$2\sec\theta = x+2$$
$$2\sec\theta\tan\theta\,d\theta = dx$$
$$2\tan\theta = \sqrt{4x+x^2}$$

$$\int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$
$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$
$$= \ln|\sec\theta+\tan\theta| + C$$
$$= \ln\left|\frac{x+2}{2}+\frac{\sqrt{4x+x^2}}{2}\right| + C$$

b)

$$\int \cos(\ln t)\,dt.$$

$$u=\ln t,\quad du=\frac{dt}{t} \Leftrightarrow t\,du=dt \Leftrightarrow e^u du = dt$$
$$e^u = t$$

$$I = \int e^u \cos u\,du = e^u \sin u - \int e^u \sin u\,du$$
$$= e^u\sin u + e^u\cos u - \int e^u \cos u\,du$$
$$= e^u\sin u + e^u\cos u - I \Rightarrow$$
$$2I = e^u\sin u + e^u\cos u + C \Rightarrow$$
$$I = \frac{e^u}{2}(\sin u+\cos u) + C'.$$

5.

a) Find the Taylor series generated by $f(x) = x^5$ at $x = 1$.

$f'(x) = 5x^4$, $f''(x) = 5\cdot 4x^3$, $f'''(x) = 5\cdot 4\cdot 3x^2$, $f^{(4)}(x) = 5\cdot 4\cdot 3\cdot 2x$

$f^{(5)}(x) = 5!$, $f^{(n)}(x) = 0$, $n \geq 6$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3 + 5(x-1)^4 + (x-1)^5$$

(or use $((x-1)+1)^5 = \sum_{k=0}^{5} \binom{5}{k}(x-1)^k$).

b) Use a), or any other method, to write the following expression using partial fractions:

$$y = \frac{x^5}{(x-1)^6}.$$

$$y = \frac{1}{(x-1)^6} + \frac{5}{(x-1)^5} + \frac{10}{(x-1)^4} + \frac{10}{(x-1)^3} + \frac{5}{(x-1)^2} + \frac{1}{x-1}$$

6. Find a value c that makes the function $f(x)$ continuous at $x = 0$, where

$$f(x) = \frac{\sin(-7x)}{\tan(11x)}, \quad x \neq 0, \qquad f(0) = c.$$

$$\lim_{x \to 0} \frac{\sin(-7x)}{\tan(11x)} = \lim_{x \to 0} \frac{-7\cos(-7x)}{11\sec^2(11x)} = -\frac{7}{11}.$$

$$c = -\frac{7}{11}$$

7. Study the convergence of the following series.

a)

$$\sum_{n=10}^{\infty} \frac{(\ln n)^2}{n^4}$$

$\ln n < n \quad \forall n \in \mathbb{N}.$

$\therefore \quad \frac{(\ln n)^2}{n^4} < \frac{n^2}{n^4} = \frac{1}{n^2}. \quad \sum \frac{1}{n^2}$ conv $\Rightarrow \sum_{n=10}^{\infty} \frac{(\ln n)^2}{n^4}$

converges by the comparison test.

b)

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

Ratio test!

$$\frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)}.$$

as $n \to \infty$,

the ratio goes to 0.

$\therefore$ Converges.

c)

$$\sum_{n=1}^{\infty} (-1)^{n+1} (1 + \frac{1}{n})$$

terms do not go to 0.

diverges.

8. Find a sequence a_n such that all three conditions below are satisfied:

1) $\lim_{n\to\infty} a_n = 0$,

2) $a_n \geq 0$ for all n and

3) $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ diverges.

$$a_n = \begin{cases} 0, & n \text{ odd} \\ \frac{n}{2}, & n \text{ even}. \end{cases}$$

$$\Rightarrow S_k = \sum_{n=1}^{k} (-1)^{n+1} a_n =$$

$$0 - 1 + 0 - \frac{1}{2} + 0 - \frac{1}{3} + 0 - \frac{1}{4} + \cdots\cdots$$

$$= -\sum_{n=1}^{k} \frac{1}{n} \quad \text{diverges.}$$

9. Study the following power series (radius of convergence, interval of convergence, convergence at the endpoints).

a)

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

ratio test: $\dfrac{2^{n+1}|x|^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n |x|^n} = \dfrac{2|x|}{n+1} \to 0 \ \forall x \in \mathbb{R}$.

$\therefore$ Interval of convergence is $(-\infty, \infty)$,

radius of conv. $= \infty$.

(or: observe $= e^{2x}$)

b)

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\sqrt{n}\ln n}.$$

ratio: $\dfrac{|x-1|^{n+1}}{\sqrt{n+1}\,\ln(n+1)} \cdot \dfrac{\sqrt{n}\,\ln n}{|x-1|^n} = \dfrac{\sqrt{n}}{\sqrt{n+1}}\,\dfrac{\ln n}{\ln(n+1)}\,|x-1|$

$\dfrac{\sqrt{n}}{\sqrt{n+1}} = \dfrac{1}{\sqrt{1+\frac{1}{n}}} \to 1$. $\lim\limits_{y\to\infty} \dfrac{\ln y}{\ln(y+1)} = \lim\limits_{y\to\infty} \dfrac{\frac{1}{y}}{\frac{1}{y+1}}$

$= \lim\limits_{y\to\infty} \dfrac{y+1}{y} = 1$. $\therefore$ ratio $\to |x-1|$.

radius of conv. $= 1$, $x = 0$: $\sum \dfrac{(-1)^n}{\sqrt{n}\ln n}$ conv.

by alt. ser. test $x = 2$: $\sum \dfrac{1}{\sqrt{n}\ln n}$?

$\dfrac{1}{\sqrt{n}\ln n} \geq \dfrac{1}{n\ln n}$. Claim: $\sum \frac{1}{n\ln n}$ diverges.

pf: $\displaystyle\int_2^b \frac{dx}{x\ln x}$ $\quad u = \ln x$, $du = \dfrac{dx}{x}$.

$=$

$\displaystyle\int_{\ln 2}^{\ln b} u^{-1}\,du = \ln u \Big|_{\ln 2}^{\ln b} = \ln(\ln b) - \ln(\ln 2)$.

Taking the limit as $b \to \infty$, the $\int$ diverges, so the sum diverges, and the other sum diverges by the comp. test.

$\therefore$ Interval of conv $= [0, 2)$.

10. Find the Maclaurin series of the following function

$$f(x) = x \ln(1 + x^2).$$

$$\ln(1+x) = \int_1^x \frac{dt}{1+t} = \int_1^x \sum_{n=0}^{\infty} (-1)^n t^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

$$x \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{n}$$

11. Prove, using (ϵ, N), that

$$\lim_{n\to\infty} \left(-1 - \frac{1}{n^{\frac{2}{3}}}\right) = -1.$$

Not required (not done in class)