MAT 160: Mathematical Puzzles and Games

MAT 160, Spring 2011
Instructor: Dave Jensen
Office Hours: MF 10-11:30
Office: Math Tower 4-120
Class Website: http://www.math.sunysb.edu/∼djensen/mat160

COURSE DESCRIPTION

This course aims to help students sharpen their problem solving skills and their ability to formulate and express mathematical ideas, through challenging (and hopefully entertaining) puzzles and problems. This is a discovery-style learning course, which means that the majority of time in "lecture" will be spent on **student** presentations.

TEXTBOOK

There is no required textbook for this course.

HOMEWORK

Homework is an essential part of this class. A problem set will be assigned each week. You will be expected to think about all of them and come up with ideas on how to tackle them. You are strongly encouraged to collaborate on these problems, but you must write down your solutions yourself. You should also be prepared to **present** your solutions during class. The purpose of these presentations is to help you develop your ability to communicate mathematical ideas. You do not have to present your solutions all by yourself, nor do your solutions need to be complete and correct in order for you to present, but you must participate in presentations in order to earn a passing grade.

GRADING

This class is a 1-credit course with pass/fail grading. Your grade will be determined by class attendance, participation in presentations, and the effort you demonstrate on the homework. A student with more than three unexcused absences will not pass the class. There will be no exams.

DISABILITIES

If you have a physical, psychological, medical, or learning disability that may affect your course work, please contact Disability Support Services at http://studentaffairs.stonybrook.edu/dss/ or (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website:

http://www.stonybrook.edu/ehs/fire/disabilities.shtml

ACADEMIC INTEGRITY

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at

http://www.stonybrook.edu/uaa/academicjudiciary/

CRITICAL INCIDENCE MANAGEMENT

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn.

MAT118 Homework 1

Due Monday, February 7

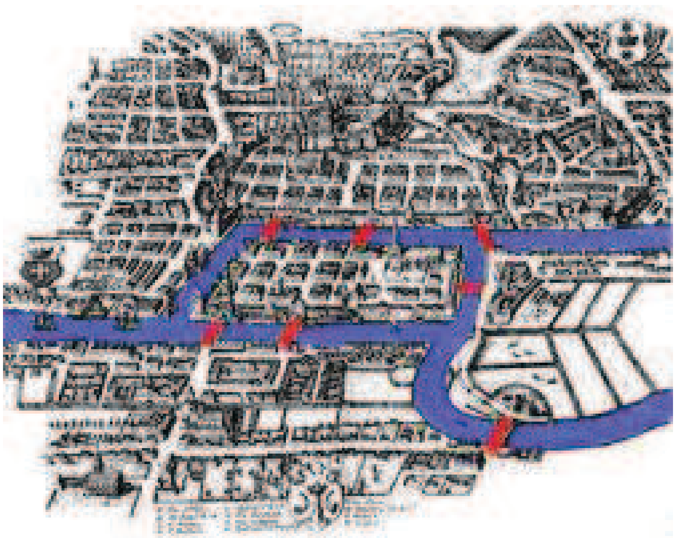
1. Ten people stand back-to-front in a line each wearing either a white or red hat. Starting at the back of the line with the person who can see nine hats, each person will make a guess as to the color of his own hat. People who guess correctly will be freed. Those that guess incorrectly will be immediately executed. The people will hear the guesses made behind them, but are otherwise unable to communicate. Knowing that they are about to play this game, what strategy could the people agree upon to ensure the survival of a maximal number of people? How many people can be guaranteed to survive?

2. A house has one entrance and many rooms. Every room has 1, 2 or 4 doors, and these doors lead directly to other rooms or to the outside. The rooms with 1 door are precisely the bathrooms. Prove that this house must have an odd number of bathrooms.

3. On the table in front of you is a collection of coins, some of which are heads up and some of which are tails up. Your goal is to separate the coins into two collections, turning over whichever coins you wish, so that both collections contain the same number of heads-up coins. Unfortunately, you are blindfolded. After a few moments, a person walks by and tells you the number of coins that are lying heads up. How can you use this information to accomplish your goal?

4. A regular 8×8 checkerboard is missing two of its squares – the lower left square and the upper right square. That leaves 62 squares on the board. You are given 31 dominoes, each of which is a 2×1 rectangle that can cover exactly two squares on the checkerboard. Can you place the dominoes on the checkerboard so that all 62 squares are covered?

5. The city of Konigsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. A map of the city is pictured below. Is there a way to walk through the city crossing each bridge once and only once? The islands cannot be reached by any route other than the bridges.



MAT160 Homework 2

Due Monday, February 14

1. Show that in any finite gathering of people, there are at least two people who know the same number of people at the gathering (assume that "knowing" is a mutual relationship).

2. A group of people are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table and this platform can rotate. Each person ordered a different entree and it turns out that no one has the correct dish in front of him or her. Show that it is possible to rotate the platform so that at least two people will have the correct entree.

3. There are currently 24,594 students enrolled at SUNY Stony Brook. Show that there is a collection of 68 or more students who all have the same birthday.

4. A **lattice point** in the plane is a point (x, y) such that both x and y are integers. Given 5 lattice points in the plane prove that you can find a pair amongst them such that the line segment joining them has a third lattice point on its interior (not necessarily from the same set).

5. (a) Consider any seven points in a 3×4 rectangle. Prove that some pair of points must be separated by a distance of less than or equal to $\sqrt{5}$.

   (b) Now consider any **six** points in a 3×4 rectangle. Prove that some pair of points must be separated by a distance of less than or equal to $\sqrt{5}$.

MAT160 Homework 3

Due Monday, February 21

1. Eight people sit around a lunch table. As it happens, each person's age is the average of the two persons' ages on his/her right and left. Show that all their ages are equal.

2. Zack and his ten buddies sit around a dinner table at John Harvard's. Each person's age differs from the two persons' ages on his/her right and left by at most one year. Zack is 26 years old. Can he order a pitcher of beer to share?

3. The integers $1, 2, \ldots, n^2$ are placed (without duplication) in any order onto an $n \times n$ chessboard, with one integer per square. Show that there exist two (horizontally, vertically, or diagonally) adjacent squares whose values differ by at least $n + 1$.

4. Show that there do not exist 2 positive integers a and b such that $a^2 = 2b^2$.

5. An odd number of people are standing in a field in such a way that their mutual distances are all different. Everybody shoots the person closest to him. Show that:

   (a) At least one person survives.

   (b) Nobody is hit by more than five bullets.

   (c) The paths of the bullets do not cross.

MAT160 Homework 4

Due Monday, February 28

1. At a party with N people, each person shook every other person's hand exactly once. How many total handshakes were there?

2. Show that the sum of the internal angles of a convex polygon with n sides is always $(n-2)\pi$ radians.

3. Let S_n be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive integers. So, for example, if $n = 2$, then $\{1\}, \{2\}$ and the empty set $\{\}$ are acceptable but $\{1, 2\}$ is not, so $S_2 = 3$. Determine S_n.

4. A rectangular piece of paper is divided into various different regions by N randomly drawn straight lines. Show that, using only two colors, the paper can be colored in such a way that each individual region is entirely one color, and any two adjacent regions are different colors. (Note: here, two regions are "adjacent" if they meet along a line segment. If the intersection of the two regions is just a point, they are not adjacent.)

5. $N + 1$ numbers are chosen randomly from amongst the $2N$ natural numbers $1, 2, 3, \ldots, 2N$. Show that, among the chosen numbers, there is a pair such that one of them is divisible by the other.

MAT160 Homework 5

Due Monday, March 7

1. Show that the sum of the digits of any multiple of 9 is a multiple of 9. (This implies, for example, that 13401 is a multiple of 9 because $1 + 3 + 4 + 0 + 1 = 9$.)

2. Show that the alternating sum of the digits of any multiple of 11 is a multiple of 11. (This implies, for example, that 1232 is a multiple of 11 because $1 - 2 + 3 - 2 = 0$, which is a multiple of 11.)

3. Consider the equations below:

$$11 = 2 + 3^2 \qquad 1111 = 22 + 33^2 \qquad 111111 = 222 + 333^2$$

   Make a conjecture about the pattern and prove that it holds.

4. A ten-digit number uses each of the digits 1,2,3,4,5,6,7,8,9,0 exactly once. The number is divisible by 10. If you remove the last digit from the right end of the number, the remaining number is divisible by 9. If you remove the last digit from the right end of *this* number, the remaining number is divisible by 8, and so on, so that each of the numbers in turn is divisible by 7,6,5,4,3,2,1. What is the number?

5. Consider the following sequence of numbers:

$$101, 10101, 1010101, 101010101, \ldots$$

   How many of the numbers in this sequence are prime? (Hint: try multiplying the numbers by 99. What do you notice?)

MAT160 Homework 6

Due Monday, March 14

1. Show that, when written in base 8, the sum of the digits of any multiple of 7 is a multiple of 7. (This implies, for example, that $2051 = 4003_8$ is a multiple of 7 because $4 + 0 + 0 + 3 = 7$.)

2. The *hexidecimal*, or base 16, system uses the 16 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, and f. Show that a number, when written in hexidecimal, is a multiple of 4 if and only if its last digit is 0,4,8, or c.

3. If you were to place a mirror at the top of this page, the digits 0,1, and 8 would appear the same in the mirror as they do on the page. All the other digits – 2,3,4,5,6,7, and 9 – would look different. Any number that uses only these digits, such as 18 or 801, for example, would also appear the same in the mirror as they do on the page. Let's call such numbers *mirror numbers.*

   (a) What is the last mirror number before 800?

   (b) What is the 100th mirror number?

4. Consider the sequence of numbers whose only digits are 0,1, and 3:

$$1, 3, 10, 11, 13, 30, 31, 33, 100, \ldots$$

   Show that the 6560th term in this sequence is divisible by 3.

5. Consider the following sequence of numbers *written in base k*:

$$101_k, 10101_k, 1010101_k, 101010101_k, \ldots$$

   (a) Show that none of the numbers in this sequence, except possibly the first one, are prime.

   (b) Conclude that the polynomial

$$k^8 + k^6 + k^4 + k^2 + 1$$

   can be factored.

MAT160 Homework 7

Due Monday, March 21

1. If today is Monday, what day of the week will it be 311 days from now? What does this have to do with modular arithmetic?

2. Show that the sum of the digits of a number (mod 9) is equal to the number itself (mod 9).

3. What is the last digit of 3^{1000}?

4. Show that the number $x^2 + 1$ is not divisible by 7 for any integer x.

5. Show that, for any integer x, the number $x^5 - x$ is always divisible by five.

MAT160 Homework 8

Due Monday, March 28

1. Show that, in New York City, there are at least two non-bald people with the same number of hairs on their head.
2. 25 students are sitting at their desks, which are arranged in a 5×5 grid. Their teacher asks them to all stand up and move to an adjacent desk. Each student may move to the desk immediately to her right, immediately to her left, immediately in front of her, or immediately behind her. Show that it is impossible for the students to do what the teacher asked.
3. Show that, in any group of six people, there are either three people who are all friends with each other, or there are three people, none of whom are friends with either of the other two.
4. Let's call a finite set of integers *self-excluding* if the number of elements in the set is not an element of the set. So, for example, $\{1, 3\}$ is a self-excluding set, since it has 2 elements and 2 is not an element of the set, while $\{1, 2, 3\}$ is not a self-excluding set, since it has 3 elements and 3 is an element of the set. How many self-excluding subsets of $\{1, 2, 3, \ldots, N\}$ are there?
5. Ten people stand back-to-front in a line. This time, each one is wearing either a hat that is one of **three** different colors – red, white or blue. Starting at the back of the line with the person who can see nine hats, each person will make a guess as to the color of his own hat. People who guess correctly will be freed. Those that guess incorrectly will be immediately executed. The people will hear the guesses made behind them, but are otherwise unable to communicate. Knowing that they are about to play this game, what strategy could the people agree upon to ensure the survival of a maximal number of people? (Hint: can you rephrase your original solution to this problem using modular arithmetic?)

MAT160 Homework 9
Due Monday, April 4

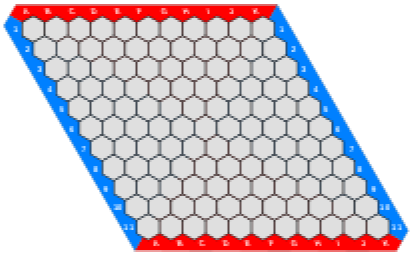
1. In the game of Go, two players alternate turns. On a player's turn, she may either place a stone on the board, or she may pass. At then end of the game, the score is tallied and one of the two players wins. Show that the first player has a winning strategy. (Hint: for this problem, it is not important how the score is calculated.)

2. In the game of Divisor Chomp, two players alternate turns. On a player's turn, she must name a divisor of 900. She cannot name a number that has already been mentioned, or any multiple of a number that has already been mentioned. The player who names the number 1 loses. Show that the first player has a winning strategy.

3. This less interesting version of the game Nim is played with two piles of matchsticks. One of the piles contains 20 matchsticks, and the other contains 21. The two players alternate turns. On a player's turn, she may remove as many matchsticks as she likes, as long as they are all from the same pile. The player who takes the last matchstick wins. Show that the first player has a winning strategy. What is this winning strategy?

4. The game of Hex is played on a hexagonal grid. (See the attached illustration.) Players take turns placing a stone of their color on a single cell within the overall playing board. The goal is to form a connected path of your stones linking the opposing sides of the board marked by your colors, before your opponent connects his or her sides in a similar fashion. The first player to complete his or her connection wins the game. The four corner hexagons each belong to both adjacent sides. Show that one of the two players must win the game (there are no draws).

5. Show that the first player in Hex has a winning strategy.



MAT160 Homework 10

Due Monday, April 11

1. (a) Suppose that, in an effort to make soccer more exciting, FIFA decides that goals are now worth 5 points each. Show that this new game is equivalent to soccer.

   (b) Suppose that, in an effort to make baseball more exciting, MLB decides to implement the following scoring system: each run is worth 1 point, and, at the end of each inning, a team is awarded a number of bonus points equal to the number of runs they scored in that inning. Show that this new game is equivalent to baseball.

2. Where does the game of Divisor Chomp get its name? Consider the game of Divisor Chomp from the previous homework, except that this time the starting number is $4000 = 2^5 \cdot 5^3$. Show that this game is equivalent to the game of Chomp we played in class. Conclude that the first player has a winning strategy.

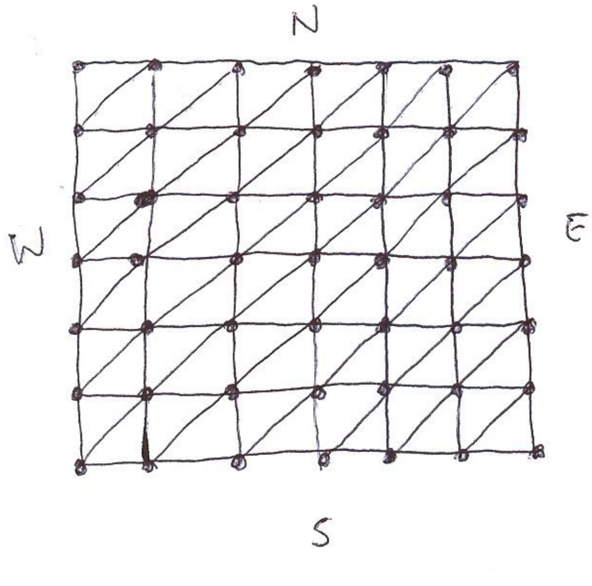
3. Consider the following game: two players alternate naming numbers. On a player's turn, she may either divide the last number named by a power of 2 or by a power of 5. The first player to say the number 1 wins the game. Suppose that the first player begins the game by naming the number $10^{20} = 2^{20} \cdot 5^{20}$. Show that this game is equivalent to the game of Nim from the previous homework. Describe a winning strategy for this game.

4. The game of Set is played with a deck of 81 cards. Each card contains one, two or three shapes. On a given card, these shapes are all either ovals, diamonds, or squiggles. Also, on a given card, these shapes are all either red, green, or purple. Finally, on a given card, all of the shapes are either transparent, opaque, or partially shaded. A **set** consists of three cards such that, for each of these four properties (number, shape, color, and shading), either all of the cards are the same or they are all different. The game is played by dealing twelve cards face-up on the table. The first player to identify a set wins the round.

   The Finite Geometry Game is also played with a deck of 81 cards. Each card contains four numbers – one in each corner. Each of these numbers is either a 1, 2 or 3. A **set** consists of three cards such that the sum of the numbers in the upper-left corners of the cards is a multiple of 3, as is the sum of the numbers in the upper-right corners, the lower-left corners, and the lower-right corners. The game is played by dealing twelve cards face-up on the table. The first player to identify a set wins the round.

Show that Set and the Finite Geometry Game are equivalent.

5. The game board below represents the downtown street plan of a city. Two players represent groups of gangsters. Player 1 controls the areas to the north and to the south of the city. Player 2 controls the areas to the east an west. The nodes in the street plan represent street intersections. The players take turns in labeling nodes that have not already been labeled. A player who manages to label both ends of a street controls the street. Player 1 wins if he links the north and south with a route that he controls. Player 2 wins if she links the east and west. Show that this game is equivalent to the game of Hex. Conclude that the first player has a winning strategy.



MAT160 Homework 10

Due Monday, April 25

1. (a) Recall the game of 24 from class. Suppose that you are now allowed to take 1,2,3, or 4 stones. Which player is in a better position at the start of the game? Describe a winning strategy for this player.

   (b) Suppose that you are now allowed to take 1,2, or 7 stones. Which player is in a better position at the start of the game? Describe a winning strategy for this player.

2. The following game is played on an empty chessboard with one king. The king is placed on a square at random, and then two players take turns moving the king (in the standard way that a chess king moves). They are not allowed to move the king to a square that it has already visited. The first player who is unable to move loses. Which player is in a better position at the start of the game? Find a winning strategy for this player.

3. The game of Fibonacci Nim is also played with a pile of 24 stones. The first player may take as many stones as she likes, provided that she does not take the whole pile. Thereafter, each player may take any number of stones, up to twice the number of stones removed by her opponent on the previous move. Which player is in a better position at the start of the game? Describe a winning strategy for this player. (Hint: the game was not named accidentally.)

4. The game of Nim is played with 3 heaps of matchsticks, one with 3 matchsticks, one with 4 matchsticks, and one with 5 matchsticks. On a player's turn, she may remove as many matchsticks as she likes, as long as they are all from the same pile. The player who takes the last matchstick wins. In this exercise, we devise a winning strategy for the first player.

   Write the number of matchsticks in each pile in binary. So the starting position is:

$$\begin{array}{c} 011 \\ 100 \\ 101 \end{array}$$

We will call a position **balanced** if there is an even number of 1's in each column, and **unbalanced** otherwise.

(a) Show that it is impossible to move from a balanced position to a balanced position.

(b) Show that it is always possible to move from an unbalanced position to a balanced position.

(c) Find a winning strategy for the first player.

5. The SOS game is played on a row of N squares, initially empty. On a player's turn, she may place either an S or an O (her choice) in one of the empty squares. The first player who succeeds in completing SOS in consecutive squares wins the game.

   (a) Suppose that $N = 4$ and the first player places an S in the first square. Show that the second player has a winning strategy.

   (b) Suppose that $N = 7$. Show that the first player has a winning strategy.

MAT160 Homework 12

Due Monday, May 2

1. The payoff matrix for the game of Chicken is depicted below. Find all of the Nash equilibria in this game. Explain in your own words why each is a Nash equilibrium.

	Swerve	Don't Swerve
Swerve	2,2	1,3
Don't Swerve	3,1	4,4

2. **Stag Hunt** – Two employees at a small company are upset with the new policy regarding vacation time. Each is considering sending a memo to their boss outlining their complaints. Each employee knows that, if the boss receives two memos, he will have to take the complaint seriously – this is the best possible outcome. If the boss receives only one memo, however, he may simply fire the person who complained. This is the worst outcome for the person who complained, but the second-best for the person who kept quiet; at least he knows that the boss is aware of the problem. If both employees keep quiet, of course, then nothing will happen.

   (a) Draw the payoff matrix for this game.

   (b) What are the Nash equilibria in this game? Why?

3. **Gift of the Magi** – A couple wants to go see a movie together. The man wants to see a bloody horror flick, while the woman wants to go to the feel-good romantic comedy of the season. The two movies are playing at separate theaters, and the man has forgotten his phone, so they are unable to communicate. Their only hope, therefore, is to show up at one of the theaters and hope that the other person chooses the same movie. It is more important to this couple to spend the evening together than to see their preferred movie, but, as long as they're together, each person would rather go to the movie of their choice than the other movie.

   (a) Draw the payoff matrix for this game.

   (b) What are the Nash equilibria in this game? Why?

4. In a two-player combinatorial game, if a player with a winning strategy uses that winning strategy, and the other player uses whatever strategy she wishes, they are in Nash equilibrium. Explain why.

5. **Prisoner's Dilemma** – Two criminals are caught by the police and placed in separate interrogation rooms. The district attorney has enough evidence to convict them for a minor crime, but in order to convict them of something more significant, he needs a witness. Each of the two prisoners has the option of either keeping quiet or of ratting the other person out. If they both keep quiet, they will both serve 3 years in prison for the minor crime. If one stays quiet while the other talks, the DA will reward the witness by letting him go free, while the other will spend 10 years in prison. If they both talk, they will both serve 7 years.

   (a) Draw the payoff matrix for this game.

   (b) What are the Nash equilibria in this game? Why?

MAT160 Homework 13

Due Monday, May 9

The following finite automata were all designed to play the iterated Prisoner's Dilemma. If you don't remember how this game is played, you can find it on last week's homework.

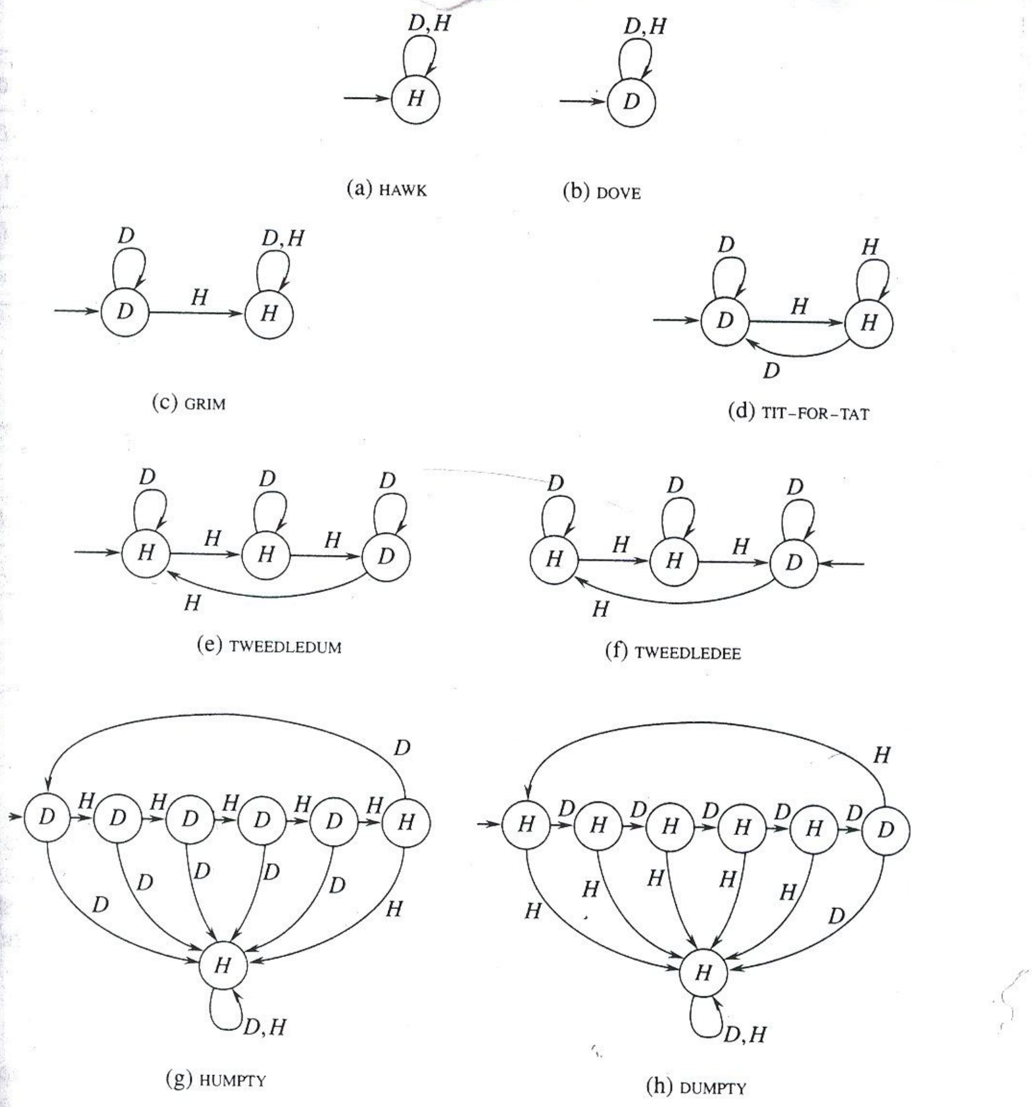


Figure 8.5 Some finite automata.

1. (a) Why is the third automaton called GRIM?
   (b) Suppose that you know that your opponent is GRIM. What would be your best strategy?
   (c) Why is the fourth automaton called TIT-FOR-TAT?
   (d) Suppose that you know that your opponent is TIT-FOR-TAT. What would be your best strategy?
2. Suppose that you know that your opponent is DUMPTY. What would be your best strategy?
3. An automaton is called "nice" if it is never the first to play the strategy *hawk* and "nasty" if it is never the first to play the strategy *dove*. (Note: it is possible for an automaton to be neither nasty nor nice.)
   (a) Which of the automata above are nice?
   (b) Which of the automata above are nasty?
   (c) Describe an automaton with more than one state that is nasty.
4. Suppose that TIT-FOR-TAT plays against TWEEDLEDUM for 8 iterations. How many years would each of the two players spend in prison?
5. Suppose that TWEEDLEDEE plays against TWEEDLEDUM for 8 iterations. How many years would each of the two players spend in prison?