MAT 142 - Analysis II

Welcome to Mat 142! The aim of the course is to further develop the rigorous theory of single variable calculus after the Analysis I course.

Click on the top for more information:

The Info section contains times and locations of the lectures and recitations, information about the textbook, etc.

You will find information about office hours and ways to contact your instructors in the Instructors section.

The week-by-week progress of the lectures and the weekly homework assignments are posted in the Schedule & Homework section.

Information about the exams is contained in the Exams section.

Info

Times and places:

Lectures	MW 5.30-6.50pm	Physics P112	Lorenzo Foscolo
Recitations	M 4-4.53pm	Physics P128	Jordan Rainone

Important dates are on the university Spring 2017 academic calendar.

Textbook:

Notes for each lecture will be made available on the Schedule & Homework page.

The basic textbook for the course is:

[A] *Calculus. Vol. 1: One-variable calculus with an introduction to linear algebra*, by T. Apostol,

This has already been used in Analysis I and will serve as a reference for the basic definitions and results. However, most of the topics we will discuss during the course are not included in this book. Specific references will be given during the course.

Besides Apostol's book, two classic books on one-variable calculus that is worth consulting from time to time are:

[R] *Principles of Mathematical Analysis* by W. Rudin, Mc Graw-Hill

[S] *Calculus* by M. Spivak, Publish or Perish

Note: in the homework and notes for the lectures I will use [A], [R], [S] to refer to these books.

Prerequisites:

C or higher in MAT 141 or permission of the Advanced Track Committee.

Main topics covered:

The main topics we will cover in the course are: applications of integrals to geometry (length of parametrised plane curves, the Isoperimetric Problem); convergence, approximation and compactness results for sequences of functions; existence and uniqueness of solutions to first-order differential equations; Fourier Series and applications to mathematical physics.

Lectures and office hours:

You are expected to attend lectures and recitations every week. Lectures give some basic understanding of the topics covered in the course. Recitations build your problem-solving skills. They are very important because one learns mathematics only by doing it. The time and location of the lectures and recitations are given above.

The lecturer and the recitation instructors hold office hours every week. The times and locations are on the Instructors page, as well as contact details of all the instructors. You are encouraged to see your lecturer or recitation instructor to discuss homework and other questions.

Homework:

Homework is assigned weekly. It is due at the recitation meeting the following week and must be handed in to the recitation instructor. No late homework will be accepted. Every week 10/15 problems will be assigned and 4 of these will be graded.

Grading policy:

There will be two midterm exams worth 20% of the final grade each, a final exam (40%) and weekly homework (20%). Check the Exams page for the dates of the exams and make sure to be available at those times.

If you need math help:

We are happy to help! Come to our office hours with questions on homework and lectures. Additional help is also available at the Math Learning Center.

DSS advisory:

If you have a physical, psychiatric, medical, or learning disability that could adversely affect your ability to carry out assigned course work, we urge you to contact the Disabled Student Services office (DSS), Educational Communications Center (ECC) Building, room 128, (631) 632-6748. DSS will review your situation and determine, with you, what accommodations are necessary and appropriate. All information and documentation regarding disabilities will be treated as strictly confidential.
Students for whom special evacuation procedures might be necessary in the event of an emergency are encouraged to discuss their needs with both the instructor and with DSS. Important information regarding these issues can also be found at the following web site: http://ws.cc.stonybrook.edu/ehs/fire/disabilities.shtml

Academic Integrity:

Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology and Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at: http://www.stonybrook.edu/uaa/academicjudiciary

Instructors

Lorenzo Foscolo

Room 2-121, Math Tower
E-mail: lorenzo.foscolo@stonybrook.edu

Office hours:

M 4-5pm in Math Tower 2-121
Tue 1.30-2.30pm in the MLC
Tue 2.30-3.30pm in Math Tower 2-121

Jordan Rainone

Room S-240A, Math Tower
E-mail: jordan.rainone@stonybrook.edu

Office hours:

M 1.30-3.30pm in MLC
F 12-1pm in MLC

Schedule & Homework

Week 1, Jan 23-29

Problem sheet 1.

(This homework will not be graded. Solutions will be discussed during the lecture of Wednesday Jan 25.)

Week 2, Jan 30 - Feb 5

Review: definition of integrals §§ 1.9, 1.12, 1.16, 1.17 in [A]

the Fundamental Theorem of Calculus §§ 5.1-5.3 in [A]

complex numbers §§ 9.1-9.7 in [A]

Notes: Notes1

Reading: vectors, dot product, norm: §§ 12.1-12.9 in [A]

polar coordinates: §§ 2.9-2.10 in [A]

parametrized curves: Appendix to Chapter 12 in [S]

Homework: HW1 (due on Feb 6 at the recitation meeting)

Week 3, Feb 6 - Feb 12

Reading: uniform continuity and integrability: §§ 3.17-3.18 in [A]

parametrized curves and their length: pp. 135-137 in [R]

curvature: §§ 2.1-2.3 in [P]

For a reference to the topics on curves (length, curvature and the Isoperimetric Inequality) we are studying you can have a look at sections 1.1, 1.2, 1.3, 2.1, 2.2, 3.1 and 3.2 of

[P] *Elementary Differential Geometry*, by A. Pressley, Springer Undergraduate Mathematics Series.

Notes: Notes1 and Notes2

Homework: HW2 (due on Feb 13 at the recitation meeting)

Week 4, Feb 13 - Feb 19

Reading: the Isoperimetric Inequality: §§ 3.1-3.2 in [P]

sequences: §§ 10.2-10.3 in [A] and Chapter 3 in [R]

Notes: Notes2 and Notes3

Homework: HW3 (due on Feb 20 at the recitation meeting)

Week 5, Feb 20 - 26

Review and First Midterm Exam

Week 6, Feb 27 - Mar 5

Reading: Newton's Method: exercises 16-17-18 on p. 81 in [R]

Uniform convergence: §§11.1-11.4 in [A] and pp. 143-154 in [R]

Notes: Notes3 and Notes4

Homework: HW4 (due on Mar 6 at the recitation meeting)

Week 7, Mar 6-12

Reading: Uniform convergence: Chapter 7, pp. 143-160 in [R]

Notes: Notes4

Homework: HW5 (due on Mar 20 at the recitation meeting)

Week 8, Mar 20-26

Reading: Weierstrass Approximation Theorem: Chapter 7, pp. 159-160 in [R]

Taylor polynomials: Chapter 7, §§ 7.1-7.8 in [A]

Series of functions: Chapter 11, §§ 11.6-11.16 in [A]

Notes: Notes4

Homework: HW6 (due on Mar 27 at the recitation meeting)

Week 9, Mar 27 - Apr 2

Reading: Power series, Taylor series: Chapter 11, §§ 11.6-11.13 in [A]

Homework: HW7 (due on Apr 3 at the recitation meeting)

Week 10, Apr 3-9

Second Midterm Exam

Reading: First-order differential equations. Notes5 provides a guide to readings and exercises and contains detailed references to sections in Chapter 8 of [A].

Homework: problems 1, 2.(a), 2.(b).iv, 2.(c) in Notes5

(due on Apr 9 at the recitation meeting)

Week 11, Apr 10-16

Reading: First-order differential equations. Notes5 provides a guide to readings and exercises and contains detailed references to sections in Chapter 8 of [A].

Homework: problems 10-11 in §8.5 in [A], 2 and 10 in §8.24 in [A], 4.(d) and 5 in Notes5

(due on Apr 17 at the recitation meeting)

Week 12, Apr 17-23

Reading: First and second-order differential equations. Notes5 provides a guide to readings and exercises and contains detailed references to sections in Chapter 8 of [A].

Homework: problems 6.(e), 7.(a) and (c) for exercises 3 and 5 in §8.22 of [A], 7.(b) for exercise 6 in §8.26 of [A], 8 in Notes5

(due on Apr 24 at the recitation meeting)

Week 13, Apr 24-30

Reading: Second-order differential equations. Notes5 provides a guide to readings and exercises and contains detailed references to sections in Chapter 8 of [A].

Homework: exercises 15, 17, 19, 20 in §8.14 of [A]

exercises 6, 7, 12, 22 in §8.17 of [A]

problems 14.(b) and 14.(c) in Notes5

(due on May 1 at the recitation meeting)

Week 14, May 1-7

Reading: §14.20 in [A].

Review.

Exams

Midterm I: Wednesday Feb 22, 5.30-6.50pm, Physics P112

The Review Sheet 1 contains pointers to all the topics we have covered so far and that you should expect to find on the exam. The exam will contain 3 problems, two on curves and one on sequences.

Midterm II: Monday April 3, 5.30-6.50pm, Physics P112

The exam will cover:
1) Newton’s Method and existence of fixed points
2) Uniform convergence of sequences of functions (uniform convergence and continuity/integration/differentiation, Dini’s Theorem)
3) Arzelà-Ascoli Theorem
4) Weierstrass Approximation Theorem
5) Taylor polynomials and integral formula for the remainder
6) Uniform convergence of series of functions
7) Power series and radius of convergence
8) Taylor series

Final exam: Thursday May 11, 8.30-11pm, Physics P112

The exam will cover everything we have seen during the semester, with an emphasis on differential equations. You can expect
- a bunch of questions of the form “solve this differential equation/initial value problem”
- a more “theoretical” question about differential equations (such as problems 8, 9 and 10 in Notes 5 about existence and uniqueness of solutions)
- a couple of questions about uniform convergence and/or Taylor series
- a couple of questions about curves (length, curvature) and polar coordinates

In order to prepare for the exam, review past homework assignments, online notes, your personal notes, textbooks and do plenty of exercises (including reproving some of the results we studied).

I will hold office hours as follows:

Monday May 8, 4-5pm, Math Tower 2-121

Tuesday May 9, 11am-1pm, Math Tower 2-121
1.30-2.30pm in MLC

If you need help outside of these times, write me an email and we will arrange a time to meet.

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I. THE PLANE

§1. Vectors

Definition

The vector space of n-tuples $\mathbb{R}^n$ is the space of all n-tuples $\underline{v} = (v_1, \dots, v_n)$ of real numbers with the following operations of addition and scalar multiplication:

$$\forall \quad \underline{v} = (v_1, \dots, v_n) \,,\; \underline{w} = (w_1, \dots, w_n) \in \mathbb{R}^n \text{ and } k \in \mathbb{R}$$

$$\underline{v} + \underline{w} = (v_1 + w_1, \dots, v_n + w_n) \in \mathbb{R}^n$$

$$k\underline{v} = (kv_1, \dots, kv_n) \in \mathbb{R}^n$$

Theorem (Properties of vector addition and scalar multiplication)

$\forall \; \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$ we have:

(i) $\underline{u} + \underline{w} = \underline{w} + \underline{u}$

(ii) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$

(iii) The vector $\underline{0} = (0, \dots, 0)$ is an "additive identity": $\underline{0} + \underline{v} = \underline{v} + \underline{0} = \underline{v}$

(iv) $a(b\underline{u}) = (ab)\underline{u}$

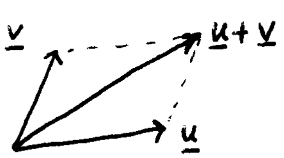
(v) $a(\underline{u} + \underline{w}) = a\underline{u} + a\underline{w}$

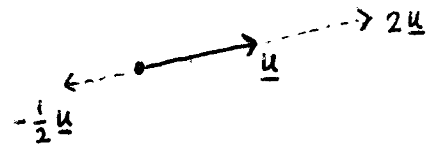
(vi) $(a+b)\underline{u} = a\underline{u} + b\underline{u}$

(vii) $0\underline{u} = \underline{0}$ and $1\underline{u} = \underline{u}$

(viii) $-\underline{u} = -1 \cdot \underline{u}$ is the additive inverse of $\underline{u}$: $(\underline{u} + \underline{v}) - \underline{u} = \underline{v}$

Remark (Geometric interpretation of vector addition and scalar multiplication)





§1.1 The dot product

Definition

The dot product of two vectors $\underline{u} = (u_1, \dots, u_n)$ and $\underline{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$ is

$$\underline{u} \cdot \underline{v} = u_1 v_1 + \dots + u_n v_n$$

Theorem (Properties of the dot product)

$\forall$ $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ we have:

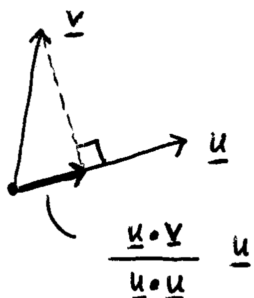
(i) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ (commutative law)

(ii) $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$ (distributive law)

(iii) $k(\underline{u} \cdot \underline{v}) = (k\underline{u}) \cdot \underline{v} = \underline{u} \cdot (k\underline{v})$ (homogeneity)

(iv) $\underline{u} \cdot \underline{u} \geq 0$ and $\underline{u} \cdot \underline{u} = 0$ iff $\underline{u} = \underline{0}$ (positivity)

Remark (Geometric interpretation of the dot product)



Theorem (Cauchy-Schwarz Inequality)

$\forall$ $\underline{u}, \underline{v} \in \mathbb{R}^n$ we have $(\underline{u} \cdot \underline{v})^2 \leq (\underline{u} \cdot \underline{u})(\underline{v} \cdot \underline{v})$

Moreover equality holds iff $\exists$ $k \in \mathbb{R}$ s.t. $\underline{u} = k\underline{v}$ or $\underline{v} = k\underline{u}$.

proof

Wlog we can assume that $\underline{u} \neq \underline{0} \neq \underline{v}$ (otherwise the result is trivial)

~~Consider the case~~ Using property (iii) we can ~~factor~~ further assume $\underline{u} \cdot \underline{u} = 1$.

Indeed, we can replace $\underline{u}$ with $\dfrac{\underline{u}}{\sqrt{\underline{u} \cdot \underline{u}}}$.

Consider the vector $\underline{w} = \underline{v} - (\underline{u} \cdot \underline{v})\underline{u}$. By property (iv) we have

$\underline{w} \cdot \underline{w} \geq 0$ with equality iff $\underline{w} = \underline{0}$, that is $\underline{v} = k\underline{u}$ with $k = \underline{u} \cdot \underline{v}$.

Now, $\underline{w} \cdot \underline{w} = \underline{v} \cdot \underline{w} - 2(\underline{u} \cdot \underline{v})^2 + (\underline{u} \cdot \underline{v})^2$ since $\underline{u} \cdot \underline{u} = 1$.

Here we used properties (i), (ii) and (iii).

Rearranging, $\underline{w}\cdot\underline{w} \geq 0 \iff (\underline{u}\cdot\underline{v})^2 \leq \underline{v}\cdot\underline{v} = \underbrace{(\underline{u}\cdot\underline{u})}_{=1}(\underline{v}\cdot\underline{v})$ ∎

<u>Definition</u>

The norm $\|\underline{u}\|$ of a vector $\underline{u} \in \mathbb{R}^n$ is $\|\underline{u}\| = \sqrt{\underline{u}\cdot\underline{u}}$

<u>Theorem</u> (Properties of the norm)

$\forall\, \underline{u}, \underline{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ we have:

(i) $\|\underline{u}\| \geq 0$ and $\|\underline{u}\| = 0$ iff $\underline{u} = \underline{0}$ (positivity)

(ii) $\|k\underline{u}\| = |k|\,\|\underline{u}\|$ (homogeneity)

(iii) $2\|\underline{u}\|^2 + 2\|\underline{v}\|^2 = \|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2$ (parallelogram law)

(iv) $4\,\underline{u}\cdot\underline{v} = \|\underline{u}+\underline{v}\|^2 - \|\underline{u}-\underline{v}\|^2$ (polarization identity)

(v) $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ (triangle inequality)

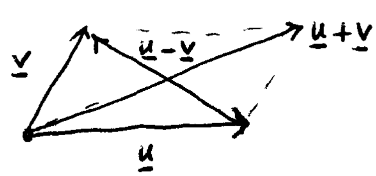
<u>proof</u> of (v)

$$\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\,\underline{u}\cdot\underline{v}$$

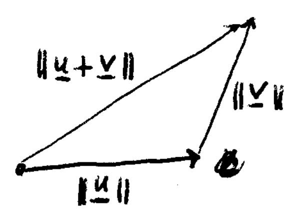
$$\left(\|\underline{u}\| + \|\underline{v}\|\right)^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\|\underline{u}\|\,\|\underline{v}\|$$

Cauchy-Schwarz Inequality: $\underline{u}\cdot\underline{v} \leq |\underline{u}\cdot\underline{v}| \leq \|\underline{u}\|\,\|\underline{v}\|$ ∎

<u>Remark</u> (geometric interpretation)



parallelogram law



triangle inequality

<u>Exercise</u> When does equality holds in the triangle inequality?

<u>Definition</u>

The distance between two points $\underline{u}, \underline{v}$ in $\mathbb{R}^n$ is

$$d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$$

~~Theorem~~

Theorem (Properties of the distance)

$\forall\ \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$ we have:

(i) $d(\underline{u}, \underline{v}) = d(\underline{v}, \underline{u})$

(ii) $d(\underline{u}, \underline{v}) = 0$ iff $\underline{u} = \underline{v}$

(iii) $d(\underline{u}, \underline{w}) \leq d(\underline{u}, \underline{v}) + d(\underline{v}, \underline{w})$

§2. Complex numbers

Definition

The set of complex numbers $\mathbb{C}$ is $\mathbb{R}^2$ endowed with vector sum $(a, b) + (c, d) = (a+c, b+d)$ and the product

$$(a, b)(c, d) = (ac - bd,\ ad + bc).$$

(*)
To connect this to the standard complex notation $z = a + bi$ we need three observations:

(i) every vector (a,b) in $\mathbb{R}^2$ can be written as

$$(a,b) = a(1,0) + b(0,1)$$

(ii) $(1,0)$ is the multiplicative identity in $\mathbb{C}$: $(1,0)(a,b) = (a,b)$ $\forall\ (a,b) \in \mathbb{C}$. By abuse of notation we then write $1 = (1,0)$

(iii) $(0,1)(0,1) = -1$ and we set $i = (0,1)$

(*) **Theorem**

$\mathbb{C}$ is an algebra over $\mathbb{R}$, that is scalar multiplication by a real number and of vector addition, the product satisfy

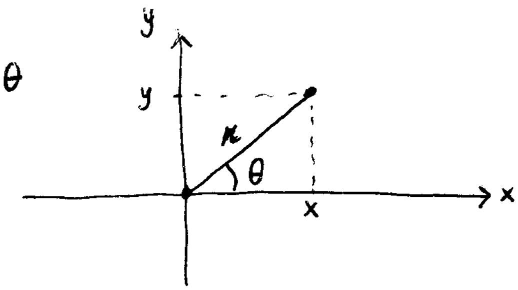
(i) $(\underline{u} + \underline{v})\underline{w} = \underline{u}\,\underline{w} + \underline{v}\,\underline{w}$

(ii) $\underline{u}(\underline{v} + \underline{w}) = \underline{u}\,\underline{v} + \underline{v}\,\underline{w}$

(iii) $k(\underline{u}\,\underline{v}) = (k\underline{u})\underline{v} = \underline{u}(k\underline{v})$

$\forall\ \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^2$ and $k \in \mathbb{R}$

§3. Polar coordinates



Recall: polar form of a complex number $z = re^{i\theta}$

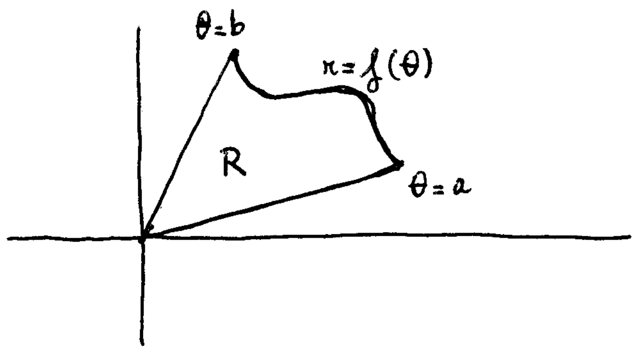
Polar coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

§3.1 Area in polar coordinates

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function such that

- $f(x) \geq 0 \quad \forall\, x \in [a,b]$
- $0 \leq b - a \leq 2\pi$

The polar equation $\quad r = f(\theta)\,, \quad \theta \in [a,b] \quad$ describes a curve in the plane in polar coordinates.



Examples

(i) $r = 2\sin\theta\,, \ \theta \in [0,\pi] \qquad \left(x^2 + (y-1)^2 = 1 \right)$

(ii) $r = \dfrac{1}{\cos\theta} \qquad (x = 1)$

(iii) (spiral of Archimedes) $\quad r = \theta$

Given a polar equation $r = f(\theta)\,, \ \theta \in [a,b]$ consider the radial set R of f over $[a,b]$:

$$R = \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \;\middle|\; 0 \leq r \leq f(\theta), \ \theta \in [a,b] \right\}$$

Theorem

If f^2 is integrable on $[a,b]$ then R is measurable and

$$a(R) = \frac{1}{2}\int_a^b f^2(\theta)\,d\theta$$

proof.

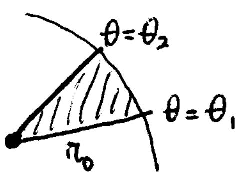
Let s, t be two step functions such that

$$0 \le s(\theta) \le f(\theta) \le t(\theta) \qquad \forall\ \theta \in [a,b]$$

Let S and T be the radial sets of s & t, respectively, over $[a,b]$.

Since $s \le f \le t$ we have $S \subseteq R \subseteq T$ and therefore by the monotone property of area $a(S) \le a(R) \le a(T)$.

Now observe that the area of a circular sector $\left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} 0 \le r \le r_0, \\ \theta \in [\theta_1, \theta_2] \end{array} \right\}$



is $\frac{1}{2} r_0^2 (\theta_2 - \theta_1)$

We conclude that $a(S) = \frac{1}{2}\int_a^b s^2(\theta)\,d\theta$ and $a(T) = \frac{1}{2}\int_a^b t^2(\theta)\,d\theta$ (why?)

Hence:

$$\int_a^b s^2(\theta)\,d\theta \le 2a(R) \le \int_a^b t^2(\theta)\,d\theta.$$

Since f^2 is integrable and s^2, t^2 are arbitrary step functions with $s^2 \le f^2 \le t^2$ on $[a,b]$, we have $2a(R) = \int_a^b f^2(\theta)\,d\theta$. ∎

II. PLANE CURVES

§1. Parametrized curves

Definition

A parametrized (plane) curve is a vector-valued function

$$\underline{c} = (u,v) : [a,b] \longrightarrow \mathbb{R}^2$$

Example

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function s.t. $f(t) \geq 0 \ \forall\, t \in [a,b]$ and $0 \leq b-a \leq 2\pi$.

Then $\underline{c}(t) = \big(f(t)\cos t,\ f(t)\sin t\big)$ is a parametrization of the curve described by the polar equation $r = f(\theta)$, $\theta \in [a,b]$.

Remark: The distinction between a parametrized curve and its "trace" should be clear: $\underline{c}(t) = \big(\cos(2t), \sin(2t)\big)$, $t \in [0,\pi]$ has the same trace as

$$\underline{d}(t) = \big(\cos(t), \sin(t)\big),\ t \in [0, 2\pi] \quad \text{and}$$

$$\underline{e}(t) = \big(\cos(t), \sin(t)\big),\ t \in [0, 4\pi]$$

Given two parametrized curves $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$, $\underline{d} : [a,b] \longrightarrow \mathbb{R}^2$ and a function $\alpha : [0,t] \longrightarrow \mathbb{R}$ we define

$$\underline{c}+\underline{d} : [a,b] \longrightarrow \mathbb{R}^2 \quad \text{by} \quad (\underline{c}+\underline{d})(t) = \underline{c}(t) + \underline{d}(t) \quad \text{(using vector addition)}$$

$$\alpha\underline{c} : [a,b] \longrightarrow \mathbb{R}^2 \quad \text{by} \quad (\alpha\underline{c})(t) = \alpha(t)\,\underline{c}(t) \quad \text{(using scalar multiplication)}$$

Definition

Let $\underline{c} = (u,v)$ be a parametrised curve. Then the symbols

$\lim_{t \to t_0} \underline{c}(t)$ and $\underline{c}'(t)$ mean

$$\lim_{t\to t_0} \underline{c}(t) = \Big(\lim_{t\to t_0} u(t),\ \lim_{t\to t_0} v(t)\Big)$$

$$\underline{c}'(t) = \big(u'(t),\ v'(t)\big)$$

Remark There is a different equivalent definition of limit, see HW1

One could also define: $\underline{c}'(t) = \lim_{h\to 0} \dfrac{\underline{c}(t+h) - \underline{c}(t)}{h}$ (Exercise: Show that this is an equivalent definition)

Remarks: $\underline{c}, \underline{d} : [a,b] \longrightarrow \mathbb{R}^2 \qquad \alpha : [a,b] \longrightarrow \mathbb{R}$

(i) $(\underline{c} + \underline{d})' = \underline{c}' + \underline{d}'$

(ii) $(\alpha \underline{c})' = \alpha' \underline{c} + \alpha \underline{c}'$

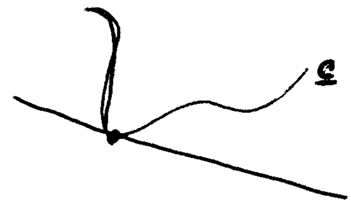
(iii) Define a function $\underline{c} \bullet \underline{d} : [a,b] \longrightarrow \mathbb{R}$ by $(\underline{c} \bullet \underline{d})(t) = \underline{c}(t) \bullet \underline{d}(t)$

Then $(\underline{c} \bullet \underline{d})'(t) = \underline{c}'(t) \bullet \underline{d}(t) + \underline{c}(t) \bullet \underline{d}'(t)$. [Exercise: prove this formula]

Definition

A parametrized curve $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ is regular at $t_0 \in [a,b]$ if $\underline{c}'(t_0)$ exists and $\underline{c}'(t_0) \neq \underline{0}$.

If $\underline{c}$ is regular at t_0 we define the tangent line of $\underline{c}$ at $\underline{c}(t_0)$ as the set of points of the form $\underline{c}(t_0) + s\,\underline{c}'(t_0)$



Remark ~~the tangent line could not be defined~~

The graph of $f(x) = |x|$ can be parametrised by a parametrized curve $\underline{c} : \mathbb{R} \longrightarrow \mathbb{R}^2$ s.t. $\underline{c}'(t)$ exists for all $t \in \mathbb{R}$, but $\underline{c}'(0) = \underline{0}$, see HW1

Note how one cannot define the tangent line to the origin $\underline{0}$.

Definition

Let $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ be a parametrized curve, and $p : [\alpha, \beta] \longrightarrow [a,b]$ a function. The parametrized curve $\underline{c} \circ p : [\alpha, \beta] \longrightarrow \mathbb{R}^2$ defined by $(\underline{c} \circ p)(t) = \underline{c}(p(t))$ is called a reparametrization of $\underline{c}$.

Lemma (Chain Rule)

Assume that $\underline{c}'(t)$ and $p'(t)$ exist. Then $(\underline{c} \circ p)'(t)$ also exists and $(\underline{c} \circ p)'(t) = p'(t)\,\underline{c}'(p(t))$

proof.

$\underline{c}(t) = (u(t), v(t)) \qquad (\underline{c} \circ p)(t) = \big(u(p(t)), v(p(t))\big)$

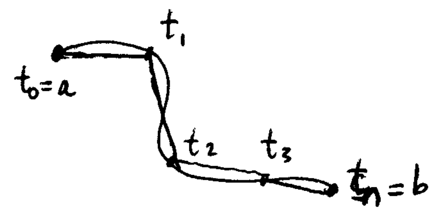
$(\underline{c} \circ p)'(t) = \big((u \circ p)'(t), (v \circ p)'(t)\big) = \big(u'(p(t))\,p'(t), v'(p(t))\,p'(t)\big)$

§2. Length

Let $\underline{c}: [a,b] \longrightarrow \mathbb{R}^2$ be a regular parametrized curve.

Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a,b]$.

~~Define~~ $\cancel{\ell(c,P) = \sum_{i=1}^{n} \sqrt{[x(t_i) - x(t_{i-1})]^2}}$



Define:

$$\ell(\underline{c}, P) = \sum_{i=1}^{n} \| \underline{c}(t_i) - \underline{c}(t_{i-1}) \|$$

Lemma

(i) If $\underline{c}$ parametrizes a straight segment then $\ell(\underline{c}, P) = \| \underline{c}(b) - \underline{c}(a) \|$ for every partition P.

(ii) If $\underline{c}$ does not parametrize a straight segment then there exists a partition $P = \{a, t_1, b\}$ s.t. $\ell(\underline{c}, P) > \| \underline{c}(b) - \underline{c}(a) \|$

proof.

(i) $\underline{c}(t) = \underline{c}(a) + \dfrac{t-a}{b-a} \left(\underline{c}(b) - \underline{c}(a) \right)$

$$\Rightarrow \quad \| \underline{c}(t_i) - \underline{c}(t_{i-1}) \| = |t_i - t_{i-1}| \, \frac{\| \underline{c}(b) - \underline{c}(a) \|}{b-a}$$

$$\Rightarrow \quad \sum_{i=1}^{n} \| \underline{c}(t_i) - \underline{c}(t_{i-1}) \| = \frac{\| \underline{c}(b) - \underline{c}(a) \|}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \| \underline{c}(b) - \underline{c}(a) \|$$

(ii) Since the trace of $\underline{c}$ is not contained in a line, $\exists$ $t_1 \in (a,b)$ s.t. $\underline{c}(t_1) - \underline{c}(a)$ is not parallel to $\underline{c}(b) - \underline{c}(a)$. Then the Triangle Inequality is strict: $\| \underline{c}(b) - \underline{c}(a) \| < \| \underline{c}(t_1) - \underline{c}(a) \| + \| \underline{c}(b) - \underline{c}(t_1) \|$ ∎

Remark Part (ii) says that a straight line is the shortest curve between two points.

It also says that $\ell(\underline{c}, P)$ is increasing if we refine the partition.

Definition

The length of the curve $\underline{c}$ is $\ell(\underline{c}) = \sup_P \ell(\underline{c}, P)$, provided this number exists. In this case we say that $\underline{c}$ is rectifiable.

Goal: We want to find ~~[scribble]~~ a formula for $\ell(\underline{c})$, at least when $\underline{c}'$ is continuous.

Recall:

~~Theorem~~ Theorem (Theorem 3.14 in [A])

Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then f is integrable on $[a,b]$.

proof.

Given a partition $P=\{t_0=a, t_1,\dots,t_n=b\}$ of $[a,b]$ consider the step functions s and S defined as follows:

$\forall\, i=1,\dots,n$ set $\quad m_i=\min\limits_{[t_{i-1},t_i]} f(t) \qquad M_i=\max\limits_{[t_{i-1},t_i]} f(t)$

$s(t)=m_i$ on $[t_{i-1},t_i] \qquad S(t)=M_i$ on $[t_{i-1},t_i]$

Define $\underline{I}(f,P)=\int_a^b s(t)\,dt=\sum_{i=1}^n m_i(t_i-t_{i-1})$

$$\overline{I}(f,P)=\int_a^b S(t)\,dt=\sum_{i=1}^n M_i(t_i-t_{i-1})$$

Recall that every continuous function on $[a,b]$ is uniformly continuous (Theorem 3.13 in [A]): $\forall\, \varepsilon>0\ \exists\, \delta>0$ s.t.

$$|x-y|<\delta \implies |f(x)-f(y)|$$

Fix $\varepsilon>0$ and choose the partition P so that $t_i-t_{i-1}<\delta\ \ \forall\, i=1,\dots,n$

Then $M_i-m_i<\varepsilon$ and therefore

$$\overline{I}(f,P)-\underline{I}(f,P)<\varepsilon\sum_{i=1}^n(t_i-t_{i-1})=(b-a)\,\varepsilon.$$

Now, if $\underline{I}(f)$ and $\overline{I}(f)$ are the lower and upper integrals of f we have

$$\underline{I}(f,P)\le\underline{I}(f)\le\overline{I}(f)\le\overline{I}(f,P)$$

Thus: $\overline{I}(f)-\underline{I}(f)\le\overline{I}(f,P)-\underline{I}(f,P)<(b-a)\,\varepsilon.$

Since ε is arbitrary we conclude that $\overline{I}(f)=\underline{I}(f)$. ∎

Theorem

Let $\underline{c}: [a,b] \longrightarrow \mathbb{R}^2$ be a parametrized curve with $\underline{c}': [a,b] \longrightarrow \mathbb{R}^2$ continuous. Then $\underline{c}$ is rectifiable and

$$\ell(\underline{c}) = \int_a^b \|c'(t)\|\, dt$$

proof.

Write $\underline{c}(t) = \big(u(t), v(t)\big)$ for functions $u, v: [a,b] \longrightarrow \mathbb{R}$ with continuous derivatives.

Note that $\|c'(t)\| = \sqrt{[u'(t)]^2 + [v'(t)]^2}$ is a continuous function on $[a,b]$.

Therefore $\int_a^b \|c'(t)\, dt\|$ exists. Moreover, from the proof of the previous theorem

$\forall\, \varepsilon > 0,\ \exists\, \delta_1 > 0$ s.t. if P is a partition of $[a,b]$ with $|t_i - t_{i-1}| < \delta_1 \ \forall i = 1, .., n$

then

$$\overline{I}\big(\|c'\|, P\big) - \underline{I}\big(\|c'\|, P\big) < \varepsilon. \qquad (*)$$

Now consider $\|\underline{c}(t_i) - \underline{c}(t_{i-1})\|$. By the Mean Value Theorem applied to u and v we can find $\xi_i, \zeta_i \in [t_{i-1}, t_i]$ s.t.

$$\|\underline{c}(t_i) - \underline{c}(t_{i-1})\| = \sqrt{[u(t_i) - u(t_{i-1})]^2 + [v(t_i) - v(t_{i-1})]^2}$$

$$= \sqrt{[u'(\xi_i)]^2 + [v'(\zeta_i)]^2}\ (t_i - t_{i-1}) \qquad (**)$$

<u>Claim</u>: $\forall\, \varepsilon > 0,\ \exists\, \delta_2 > 0$ s.t. if $|t_i - t_{i-1}| < \delta_2$ then

$$\Big|\, \|\underline{c}(t_i) - \underline{c}(t_{i-1})\| - \|c'(t)\|\,(t_i - t_{i-1}) \,\Big| < \varepsilon \quad \forall\, t \in [t_{i-1}, t_i] \qquad (***)$$

proof. of Claim:

First set $M = \sup_{[a,b]} \|c'(t)\|$ and note that $|u'(t)| \le M$ and $|v'(t)| \le M$ for all $t \in [a,b]$. [Exercise: Why is M finite?]

Since the function $x \longmapsto \sqrt{x}$ is uniformly continuous on $[0, 2M^2]$ [why?]

$\forall\, \varepsilon > 0,\ \exists\, \delta_{\sqrt{\ }} > 0$ s.t.

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{b-a} \quad \forall\, x, y \in [0, 2M^2] \text{ with } |x - y| < \delta_{\sqrt{\ }}$$

On the other hand, u' and v' are also uniformly continuous on $[a,b]$. Thus $\exists\ \delta_2 > 0$ s.t. if $|t_i - t_{i-1}| < \delta$ then

$$\left| [u'(x)]^2 - [u'(y)]^2 \right| \leq |u'(x) + u'(y)| \; |u'(x) - u'(y)|$$

$$\leq 2M \; |u'(x) - u'(y)|$$

$$\leq 2M \cdot \frac{\delta_{\sqrt{}}}{4M} = \frac{1}{2}\delta_{\sqrt{}} \qquad \forall\ x,y \in [t_{i-1}, t_i]$$

and similarly for v'.

Hence: $\left| \left([u'(\xi_i)]^2 + [v'(\xi_i)]^2\right) - \left([u'(t)]^2 + [v'(t)]^2\right) \right| < \delta_{\sqrt{}} \qquad \forall\ t \in [t_{i-1}, t_i]$

and we conclude that

$$\left| \sqrt{[u'(\xi_i)]^2 + [v'(\zeta_i)]^2} - \sqrt{[u'(t)]^2 + [v'(t)]^2} \right| < \frac{\varepsilon}{b-a}$$

Using (**) and the obvious estimate $t_i - t_{i-1} \leq b-a$, the Claim is proved ∎

~~Using (***) we~~ Now fix $\varepsilon > 0$ and a partition P with $|t_i - t_{i-1}| < \min\{\delta_1, \delta_2\}$ for all $i = 1, \dots, n$.

Then (***) implies that

$$\underline{I}(\|c'\|, P) - \varepsilon \leq \ell(c, P) \leq \overline{I}(\|c'\|, P) + \varepsilon$$

Hence

$$-2\varepsilon < \underline{I}(\|c'\|, P) - \overline{I}(\|c'\|, P) + \varepsilon \leq \int_a^b \|c'(t)\|\,dt - \ell(c, P) \leq \overline{I}(\|c'\|, P) - \underline{I}(\|c'\|, P) + \varepsilon < 2\varepsilon$$

and therefore $\ell(c)$ exists and satisfies

$$\int_a^b \|c'(t)\|\,dt - 2\varepsilon < \ell(c) < \int_a^b \|c'(t)\|\,dt + 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary the Theorem is proved. ∎

Remark [R], Theorem 6.27, gives a slightly different proof.

Example

$\underline{c}(t) = \big(f(t)\cos t,\ f(t)\sin t \big)$, $t \in [a,b]$ with f' continuous.

$$\ell(\underline{c}) = \int_a^b \sqrt{f^2(t) + [f'(t)]^2}\, dt$$

Definition

Let $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ be a regular parametrized curve with continuous $\underline{c}'$. Fix $t_0 \in [a,b]$. The arc length of $\underline{c}$ from t_0 is

$$s(t) = \int_{t_0}^{t} \| \underline{c}'(\tau) \| \, d\tau$$

Remark Let $t_1 < t_2$ be points in $[a,b]$. Denote by $\underline{c}|_{[t_1,t_2]}$ the curve $\underline{c} : [t_1,t_2] \longrightarrow \mathbb{R}^2$. Note that

$$\ell\left(\underline{c}|_{[t_1,t_2]}\right) = \int_{t_1}^{t_2} \|c'(t)\|\, dt = \int_{t_0}^{t_2} \|c'(t)\|\, dt - \int_{t_0}^{t_1} \|c'(t)\|\, dt = s(t_2) - s(t_1).$$

Definition

We say that $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ is parametrised by arc length if

$s(t) = t - t_0$ for some $t_0 \in [a,b]$.

Lemma

$\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ is parametrised by arc length iff $\| \underline{c}'(t) \| \equiv 1$.

Proof

By the Fundamental Theorem of Calculus,

$$s(t) = t - t_0 \text{ for some } t_0 \in [a,b] \iff \underset{\forall\, t \in [a,b]}{s'(t) = 1} \iff \|c'(t)\| \equiv 1 \quad \forall\, t \in [a,b]$$

Def. of arc length

∎

Theorem

Every regular parametrised curve with continuous derivative can be parametrised by arc length.

Problem sheet 1 MAT 142, Spring 2017

1. Prove the parallelogram law and the polarization identity: for every pair of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ we have

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}^2, \qquad \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 4\,\mathbf{u}\cdot\mathbf{v},$$

2. For a vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ define

$$\|\mathbf{u}\|_1 = |u_1| + |u_2|, \qquad \|\mathbf{u}\|_\infty = \max_{i=1,2} |u_i|.$$

Determine whether $\|\cdot\|_1$ and $\|\cdot\|_\infty$ satisfy each of the following properties:

(a). Positivity: $\|\mathbf{u}\| \geq 0$ for all $\mathbf{u} \in \mathbb{R}^2$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$;

(b). Homogeneity: $\|c\,\mathbf{u}\| = |c|\,\|\mathbf{u}\|$.

(c). Triangle Inequality: $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.

Draw the sets of points in the plane with $\|\mathbf{u}\|_1 \leq 1$ and $\|\mathbf{u}\|_\infty \leq 1$ respectively.

3. (Exercises 1 and 3 in Chapter 4, Appendix 1 of [S])

Given a point $\mathbf{v}$ in $\mathbb{R}^2$ let $R_\theta(\mathbf{v})$ be the point obtained by rotating $\mathbf{v}$ through an angle θ in anticlockwise direction around the origin.

(a). Show that

$$R_\theta(1,0) = (\cos\theta, \sin\theta), \qquad R_\theta(0,1) = (-\sin\theta, \cos\theta).$$

(b). It should be clear that

$$R_\theta(\mathbf{u}+\mathbf{v}) = R_\theta(\mathbf{u}) + R_\theta(\mathbf{v}), \qquad R_\theta(c\,\mathbf{v}) = c\,R_\theta(\mathbf{v})$$

for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Deduce the formula

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

(c). Show that

$$R_\theta(\mathbf{u}) \cdot R_\theta(\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$$

for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.

(d). Let $\mathbf{e}$ be the vector $\mathbf{e} = (1,0)$ and $\mathbf{w} = R_\theta(\mathbf{e}) = (\cos\theta, \sin\theta)$. Observe that $\|\mathbf{e}\| = 1$, $\|\mathbf{w}\| = 1$ and $\mathbf{e}\cdot\mathbf{w} = \cos\theta$. Deduce that for every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ we have

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

4. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be two vectors and define

$$\mathbf{u} \times \mathbf{v} = u_1 v_2 - u_2 v_1.$$

(a). How does $\times$ behaves with respect to the operations of addition and scalar multiplication? What happens if one interchanges the order of $\mathbf{u}$ and $\mathbf{v}$?

(b). Show that $R_\theta(\mathbf{u}) \times R_\theta(\mathbf{v}) = \mathbf{u} \times \mathbf{v}$.

(c). Argue in a similar way as in Problem 2 to show that

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(d). Deduce that $|\mathbf{u} \times \mathbf{v}|$ is the area of the parallelogram with vertices $\mathbf{0}, \mathbf{u}, \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$.

5. Let (r_1, θ_1) and (r_2, θ_2) be the polar coordinates of two points in the plane. Show that the distance d between the two points is given by

$$d^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_1 - \theta_2).$$

6. The *cardiod* is the curve with polar equation $r = 1 - \sin\theta$, $\theta \in [0, 2\pi)$.

(a). Sketch the graph of the cardiod.

(b). Show that it can be described by the equation

$$(x^2 + y^2 + y)^2 = x^2 + y^2.$$

(Take some care in motivating your choice of sign when taking square roots!)

(c). Calculate the area of the region enclosed by the cardiod.

7. Find a parametrized curve that runs clockwise twice around the unit circle centered at the origin.

8. (Parametrization of an interval; exercise 2 in Chapter 4 of [S])

There is a very useful way of describing the points of the closed interval $[a, b]$ (where we assume, as usual, that $a < b$).

(a). First consider the interval $[0, b]$, for $b > 0$. Prove that if $x \in [0, b]$, then $x = tb$ for some t with $0 \leq t \leq 1$. What is the significance of the number t? What is the mid-point of the interval $[0, b]$?

(b). Now prove that if $x \in [a, b]$, then $x = (1 - t)a + tb$ for some t with $0 \leq t \leq 1$. (Hint: This expression can also be written as $a + t(b - a)$.) What is the midpoint of the interval $[a, b]$? What is the point $1/3$ of the way from a to b?

(c). Prove, conversely, that if $0 \le t \le 1$, then $(1-t)a + tb$ is in $[a, b]$.

(d). Prove that the points of the open interval (a, b) are those of the form $(1-t)a + tb$ for $0 < t < 1$.

9. Let $f(t)$ be the function

$$f(t) = \begin{cases} t^2 & \text{if } t \ge 0 \\ -t^2 & \text{if } t \le 0. \end{cases}$$

Let $\mathbf{c}(t)$ be the parametrised curve $\mathbf{c}(t) = (f(t), t^2)$.

(a). Show that f is differentiable.

(b). Calculate $\mathbf{c}'(t)$.

(c). Show that the trace of $\mathbf{c}$ is the same of the trace of the parametrized curve $s \mapsto (s, |s|)$.

(This problem shows why it makes sense to insist that $\mathbf{c}'(t) \neq \mathbf{0}$ in the definition of a regular parametrized curve.)

10. We say that a parametrized curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ has a *weak tangent* at t if the unit vector $\frac{\mathbf{c}(t+h)-\mathbf{c}(t)}{\|\mathbf{c}(t+h)-\mathbf{c}(t)\|}$ has a limit when $h \to 0$. Prove that the *cuspidal cubic* $\mathbf{c}(t) = (t^3, t^2)$, $t \in \mathbb{R}$, has a weak tangent at the origin but it is not regular there. Make a sketch.

11. Consider a curve given by the polar equation $r = f(\theta)$, $\theta \in [a, b]$. We can parametrize the curve by

$$\mathbf{c}(t) = (f(t)\cos t, f(t)\sin t)\,, t \in [a, b].$$

(a). Find a formula for the slope of the tangent line of the curve at the point with polar coordinates $(f(t), t)$.

(b). Calculate the slope of the tangent lines to the *spiral of Archimedes* $r = \theta$, $\theta \ge 0$, at the point with $\theta = \frac{\pi}{4}$. Make a sketch of the spiral and the tangent line.

12. There are two natural ways of defining limits of vector-valued functions. Let $\mathbf{c}_0 = (x_0, y_0)$ be a point in $\mathbb{R}^2$ and $\mathbf{c}(t) = (x(t), y(t))$ be a vector-valued function $\mathbf{c} : [a, b] \to \mathbb{R}^2$.

- **Definition 1.**

  We say that $\lim_{t\to t_0} \mathbf{c}(t) = \mathbf{c}_0$ if $\lim_{t\to t_0} x(t) = x_0$ and $\lim_{t\to t_0} y(t) = y_0$.

- **Definition 2.**

  We say that $\lim_{t\to t_0} \mathbf{c}(t) = \mathbf{c}_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

  $$\|\mathbf{c}(t) - \mathbf{c}_0\| < \varepsilon$$

  whenever $|t - t_0| < \delta$.

Prove that the two definitions are equivalent.

MAT 142 - ANALYSIS II : NOTES 2 (WEEK 3 - WEEK 4)

II. PLANE CURVES

§3. Curvature

Definition

Let $\underline{c}: [a,b] \to \mathbb{R}^2$ be a regular parametrized curve. The unit tangent vector to $\underline{c}$ is the unit vector $\underline{\tau}(t) = \dfrac{\underline{c}'(t)}{\|\underline{c}'(t)\|}$, $\forall\, t \in [a,b]$.

Remark: Suppose that $\underline{c}: [a,b] \to \mathbb{R}^2$ is parametrized by arc-length $s \in [a,b]$. Then $\|\underline{c}'(s)\| = 1$ and therefore $\underline{\tau}(s) = \underline{c}'(s)$.

Lemma 1

Assume that $\underline{\tau}'(t)$ exists. Then $\underline{\tau}'(t) \bullet \underline{\tau}(t) = 0$.

proof.

$$\|\underline{\tau}\|^2 = 1 \Rightarrow \frac{d}{dt}\left(\underline{\tau} \bullet \underline{\tau}\right) = 2\, \underline{\tau}' \bullet \underline{\tau} = 0$$

■

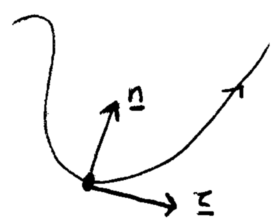
Definition

The unit normal vector of $\underline{c}$ is the unique vector $\underline{n}(t)$, $t \in [a,b]$ such that:

(i) $\|\underline{n}(t)\| = 1$

(ii) $\underline{n}(t) \bullet \underline{\tau}(t) = 0$

(iii) $\underline{\tau}(t) \times \underline{n}(t) > 0$



Remark: If $\underline{c}: [a,b] \to \mathbb{R}^2$ has unit normal $\underline{n}$ and $p: [a,b] \to [a,b]$ is defined by $p(t) = a + b - t$, the the unit normal of $\underline{c} \circ p$ is $-\underline{n}$.

~~By Lemma 1, we can write $\underline{\tau}'(t) = \kappa(t)\underline{n}(t)$ for $\kappa: [a,b] \to \mathbb{R}$~~

~~Definition~~

~~The function $\kappa: [a,b] \to \mathbb{R}$ is called the curvature of $\underline{c}$.~~

Definition Let $\underline{c}: [a,b] \to \mathbb{R}^2$ be a regular curve parametrized by arc-length and assume that $\underline{\tau}'$ exists. By Lemma 1 $\underline{\tau}'(s) = \kappa(s)\, \underline{n}(s)$ for some $\kappa: [a,b] \to \mathbb{R}$. κ is called the curvature of $\underline{c}$.

Proposition

Let $\underline{c}:[a,b]\to\mathbb{R}^2$ be a regular parametrized curve, not necessarily parametrized by arc-length. Then, when it exists, the curvature κ of $\underline{c}$ satisfies:

$$|\kappa| = \frac{|\underline{c}' \times \underline{c}''|}{\|\underline{c}'\|^3}$$

proof.

Arc-length of $\underline{c}$: $\quad s(t) = \int_{t_0}^{t} \|c'(u)\|\, du \;\Rightarrow\; \frac{ds}{dt} = \|c'\|$

$$\Rightarrow \frac{d}{ds} = \frac{1}{\|c'\|}\frac{d}{dt}$$

$$\underline{\tau} = \frac{\underline{c}'}{\|\underline{c}'\|} \;\Rightarrow\; \frac{d}{ds}\underline{\tau} = \frac{1}{\|\underline{c}'\|}\frac{d}{dt}\left(\frac{\underline{c}'}{\|\underline{c}'\|}\right) = \frac{\underline{c}''}{\|\underline{c}'\|^2} - \frac{\underline{c}'}{\|\underline{c}'\|^3}\frac{d}{dt}\|\underline{c}'\|$$

Now, $\quad \frac{d}{dt}\left(\|\underline{c}'\|\right) = \frac{d}{dt}\left(\sqrt{\underline{c}'\cdot\underline{c}'}\right) = \frac{\underline{c}'\cdot\underline{c}''}{\|\underline{c}'\|}$

Therefore:

$$\frac{d\underline{\tau}}{ds} = \frac{1}{\|\underline{c}'\|^2}\left(\underline{c}'' - \frac{(\underline{c}'\cdot\underline{c}'')}{\underline{c}'\cdot\underline{c}'}\,\underline{c}'\right) \Rightarrow$$

$$\kappa^2 = \frac{1}{\|\underline{c}'\|^4}\left\| \underline{c}'' - \frac{(\underline{c}'\cdot\underline{c}'')}{(\underline{c}'\cdot\underline{c}')}\,\underline{c}'\right\|^2 = \frac{1}{\|\underline{c}'\|^4}\left[\|\underline{c}''\|^2 - 2\frac{(\underline{c}'\cdot\underline{c}'')^2}{(\underline{c}'\cdot\underline{c}')} + \frac{(\underline{c}'\cdot\underline{c}'')^2}{(\underline{c}'\cdot\underline{c}')^2}\|\underline{c}'\|^2\right]$$

$$\frac{\underline{c}'\cdot\underline{c}''}{\|\underline{c}'\|\,\|\underline{c}''\|} = \cos\theta \quad\longrightarrow\quad = \frac{\|\underline{c}'\|^2\|\underline{c}''\|^2(1-\cos^2\theta)}{\|\underline{c}'\|^6}$$

$$= \frac{\|\underline{c}'\|^2\|\underline{c}''\|^2\sin^2\theta}{\|\underline{c}'\|^6}$$

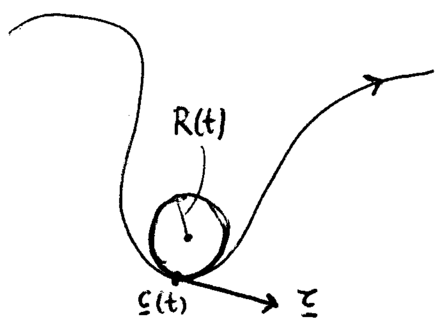
$$\frac{\underline{c}'\times\underline{c}''}{\|\underline{c}'\|\,\|\underline{c}''\|} = \sin\theta \quad\longrightarrow\quad = \frac{(\underline{c}'\times\underline{c}'')^2}{\|\underline{c}'\|^6}$$

$\blacksquare$

Remark: In fact $\kappa = \dfrac{\underline{c}' \times \underline{c}''}{\|\underline{c}'\|^3}$ thanks to our choice of direction for $\underline{n}$.

Definition

Let $\underline{c} : [a,b] \to \mathbb{R}^2$ be a regular parametrized curve and fix $t \in [a,b]$. For $h \in \mathbb{R}$ sufficiently small let $R(t,h)$ be the radius of the circle passing through the points $\underline{c}(t)$, $\underline{c}(t+h)$, $\underline{c}(t-h)$. Then the osculating radius of $\underline{c}$ at $t \in [a,b]$, if it exists, is $R(t) = \lim\limits_{h \to 0} R(t,h)$.

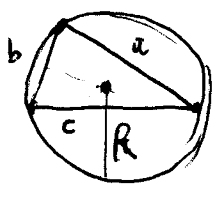


Theorem (Geometric Interpretation of curvature)

Suppose that $R(t)$ and $\kappa(t)$ both exist. Then $R(t) = \dfrac{1}{|\kappa(t)|}$.

proof.

We are going to use the formula for the radius of the circle that circumscribes a triangle:

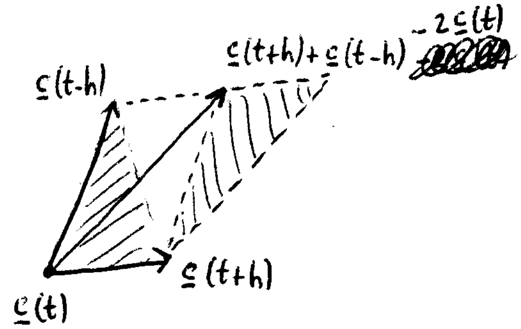


$$R = \frac{abc}{4 \text{ Area}}$$

where a, b, c are the sides of the triangle and Area is its area.

We use this formula to calculate $R(t,h)$: the triangle we are interested in has vertices $\underline{c}(t)$, $\underline{c}(t+h)$, $\underline{c}(t-h)$. Hence

$$R(t,h) = \frac{\|\underline{c}(t+h) - \underline{c}(t)\| \cdot \|\underline{c}(t) - \underline{c}(t-h)\| \cdot \|\underline{c}(t+h) - \underline{c}(t-h)\|}{2\left|\left[\underline{c}(t+h) - \underline{c}(t)\right] \times \left[\underline{c}(t+h) + \underline{c}(t-h) - 2\underline{c}(t)\right]\right|}$$



Here we used the fact that the area of the triangle with vertices $\underline{c}(t), \underline{c}(t+h), \underline{c}(t-h)$ is $\frac{1}{2} \times$ area of parallelogram w/ sides $\underline{c}(t+h) - \underline{c}(t)$ & $\underline{c}(t-h) - \underline{c}(t)$

$= \frac{1}{2} \times$ area of parallelogram w/ sides $\underline{c}(t+h) - \underline{c}(t)$ & $\underline{c}(t+h) - \underline{c}(t) + \underline{c}(t-h) - \underline{c}(t)$

Now,

$$\lim_{h \to 0^+} \frac{\underline{c}(t+h) - \underline{c}(t)}{h} = \underline{c}'(t) \qquad \lim_{h \to 0^+} \frac{\underline{c}(t) - \underline{c}(t-h)}{h} = \underline{c}'(t)$$

$$\lim_{h \to 0^+} \frac{\underline{c}(t+h) - \underline{c}(t-h)}{h} = 2\underline{c}'(t) \qquad \lim_{h \to 0^+} \frac{\underline{c}(t+h) + \underline{c}(t-h) - 2\underline{c}(t)}{h^2} = \underline{c}''(t).$$

(for example, one can use l'Hospital Rule on each coordinate function; our assumptions on $\underline{c}$ guarantee the existence of $\underline{c}'$ & $\underline{c}''$)

Hence: $$\lim_{h \to 0^+} R(t,h) = \frac{\|\underline{c}'\|^3}{|\underline{c}' \times \underline{c}''|}$$ ∎

__Definition__

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a transformation of the plane. We say that T is

(i) the translation by $\vec{v} = (a,b) \in \mathbb{R}^2$ if
$$T(x,y) = (x+a, y+b)$$

(ii) the rotation of angle $\theta \in \mathbb{R}$ if
$$T(x,y) = (\cos\theta\, x - \sin\theta\, y,\ \sin\theta\, x + \cos\theta\, y)$$

(iii) a rigid motion if
$$T(x,y) = (\cos\theta\, x - \sin\theta\, y + a,\ \sin\theta\, x + \cos\theta\, y + b)$$
for some $\theta \in \mathbb{R}$, $(a,b) \in \mathbb{R}^2$.

Lemma

Arc-length and curvature of a parametrised curve are invariant under rigid motions.

proof.

$\underline{c} : [a,b] \longrightarrow \mathbb{R}^2 \qquad \tilde{\underline{c}}(t) = R_\theta \, \underline{c}(t) + \underline{v}$

$$\tilde{\underline{c}}'(t) = R_\theta \, \underline{c}'(t) \qquad \tilde{\underline{c}}''(t) = R_\theta \, \underline{c}''(t)$$

$$\Rightarrow \quad \tilde{s}(t) = \int_{t_0}^{t} \| \tilde{\underline{c}}'(u) \| \, du = \int_{t_0}^{t} \| R_\theta \, \underline{c}'(u) \| \, du = \int_{t_0}^{t} \| \underline{c}'(u) \| \, du = s(t)$$

$$\tilde{\kappa}(t) = \frac{\tilde{\underline{c}}'(t) \times \tilde{\underline{c}}''(t)}{\| \tilde{\underline{c}}'(t) \|^3} = \frac{R_\theta \underline{c}'(t) \times R_\theta \underline{c}''(t)}{\| R_\theta \underline{c}'(t) \|^3} = \frac{\underline{c}'(t) \times \underline{c}''(t)}{\| \underline{c}'(t) \|^3} = \kappa(t)$$

since $\quad R_\theta \underline{u} \cdot R_\theta \underline{v} = \underline{u} \cdot \underline{v} \quad$ and $\quad R_\theta \underline{u} \times R_\theta \underline{v} = \underline{u} \times \underline{v}$.

■

Theorem (Fundamental Theorem of the Local Theory of Plane Curves)

Given a differentiable function $\kappa : [a,b] \to \mathbb{R}$, there exists a regular parametrized curve $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ parametrized by arc-length such that $\kappa(t)$ is ~~the~~ its curvature. Moreover, if $\tilde{\underline{c}}$ is another such curve then $\tilde{\underline{c}}(s) = T(\underline{c}(s))$, $\forall s \in [a,b]$ for some rigid motion T.

proof.

~~First note that if $\underline{c} : [a,b] \to \mathbb{R}^2$~~

First note that if $\underline{c} : [a,b] \longrightarrow \mathbb{R}^2$ is parametrized by arc-length, then $\underline{c}'(s) = \big(\cos\theta(s), \sin\theta(s) \big)$ for some function $\theta : [a,b] \longrightarrow \mathbb{R}$

Then $\quad \underline{c}''(s) = \theta'(s) \big(-\sin\theta(s), \cos\theta(s) \big) = \theta'(s) \, \underline{n}(s) \quad$ and therefore

$\theta'(s) = \kappa(s)$.

Thus define $\qquad \theta(s) = \int_a^s \kappa(u) \, du + \theta_0 \qquad$ for some $\theta_0 \in \mathbb{R}$ and

$$\underline{c}(s) = \left(\int_a^s \cos\theta(t) \, dt + v_1 , \int_a^s \sin\theta(t) \, dt + v_2 \right) \quad \text{for some } (v_1, v_2) \in \mathbb{R}^2.$$

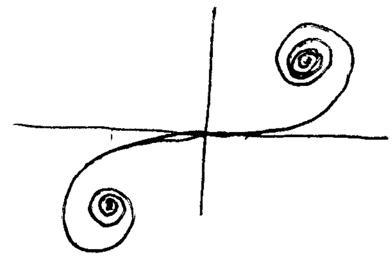
The possible different choices of θ_0 and (v_1, v_2) correspond to moving $\underline{c}$ by a rigid motion.

■

Example (Cornu's Spiral)

$\kappa(s) = s$

$$\underline{c}(s) = \left(\int_0^s \cos\left(\frac{t^2}{2}\right) dt \,,\ \int_0^s \sin\left(\frac{t^2}{2}\right) dt \right)$$



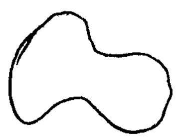
§4. Length & Area: the Isoperimetric Inequality

Definition

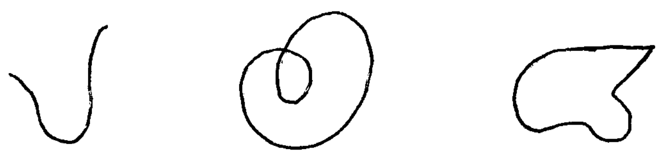
Let $T \in \mathbb{R}$ be a positive constant. A simple closed curve in $\mathbb{R}^2$ with period T is a regular parametrized curve $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ such that

$$\underline{c}(t) = \underline{c}(t') \iff t' = t + kT \text{ for some } k \in \mathbb{Z}.$$

Examples



simple closed curve



non-simple closed curves

Theorem (Jordan Curve Theorem)

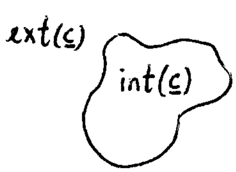
Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve. Then $\mathbb{R}^2 \setminus \{ \underline{c}(t) \mid t \in \mathbb{R} \}$ is the disjoint union of two subsets $\mathrm{int}(\underline{c})$ and $\mathrm{ext}(\underline{c})$ such that:

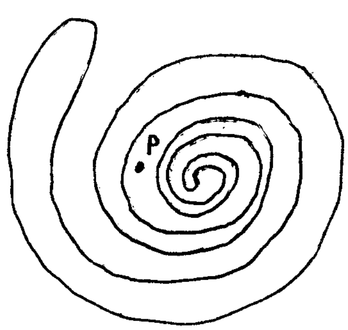
(i) $\mathrm{int}(\underline{c})$ is bounded: $\mathrm{int}(\underline{c})$ is contained inside a circle of sufficiently large radius;

(ii) $\mathrm{ext}(\underline{c})$ is unbounded;

(iii) $\mathrm{int}(\underline{c})$ and $\mathrm{ext}(\underline{c})$ are connected: any two points in $\mathrm{int}(\underline{c})$ ($\mathrm{ext}(\underline{c})$) can be joined by a curve contained entirely in $\mathrm{int}(\underline{c})$ ($\mathrm{ext}(\underline{c})$, respectively)

Examples





Is p in $\mathrm{int}(\underline{c})$ or $\mathrm{ext}(\underline{c})$?

Definition

Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve with period T and continuous $\underline{c}'$. The length $\ell(\underline{c})$ of $\underline{c}$ is

$$\ell(\underline{c}) = \int_0^T \|\underline{c}'(t)\| \, dt$$

Lemma

Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve with period T and continuous $\underline{c}'$. If $\underline{c}$ is parametrized by arc length then $T = \ell(\underline{c})$.

proof.

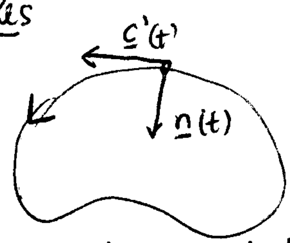
$$\ell(\underline{c}) = \int_0^T \|\underline{c}'(s)\| \, ds = \int_0^T ds = T$$

$\blacksquare$

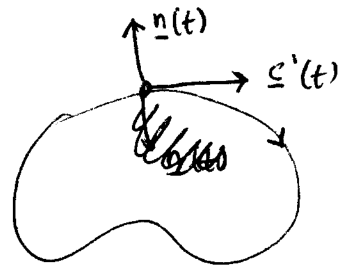
Definition

Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve. We say that $\underline{c}$ is positively oriented if the unit normal $\underline{n}(t)$ points into $\mathrm{int}(\underline{c})$ $\forall\, t \in \mathbb{R}$.

Examples



positively oriented



negatively oriented

Goal: Compare $\ell(\underline{c})$ with the area of $\mathrm{int}(\underline{c})$.

Isoperimetric Problem: which simple closed curve with length ℓ bounds the most area?

Lemma Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve. Assume that $\ell(\underline{c})$ and $a(\mathrm{int}(\underline{c}))$ exist. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a rigid motion and consider the curve $\tilde{\underline{c}}(t) = T(\underline{c}(t))$. Then $\tilde{\underline{c}}$ is a simple closed curve and $\ell(\tilde{\underline{c}}) = \ell(\underline{c})$, $a(\mathrm{int}(\tilde{\underline{c}})) = a(\mathrm{int}(\underline{c}))$.

Remark This says that we can move curves around by rigid motions in the most convenient position. For example we can always assume that:

(i) $\underline{0} \in \mathrm{int}(\underline{c})$: if $\underline{v} = (a,b) \in \mathrm{int}(\underline{c})$ consider the curve $\underline{c} - \underline{v}$

or

(ii) $\underline{0} = \underline{c}(0)$: consider the curve $\underline{c} - \underline{c}(0)$

Theorem (Isoperimetric Inequality)

Let $\underline{c}$ be a simple closed curve with continuous $\underline{c}'$! Then

$$a\left(\operatorname{int}(\underline{c})\right) \leq \frac{1}{4\pi}\, \ell(\underline{c})^2 .$$

Moreover, equality holds iff $\underline{c}$ is a circle.

Definition

A simple closed curve $\underline{c}$ is convex if the straight line segment joining any two points in $\operatorname{int}(\underline{c})$ is entirely contained in $\operatorname{int}(\underline{c})$.

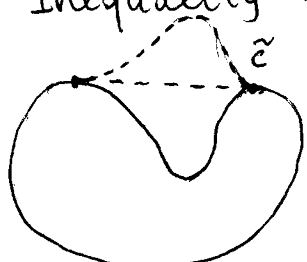
Examples:

~~Definition~~

~~Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a convex simple closed curve~~

Remark ~~Thinking about convex curves is not~~

Informally, knowing the Isoperimetric Inequality for convex curves yields the Isoperimetric Inequality for any curve:

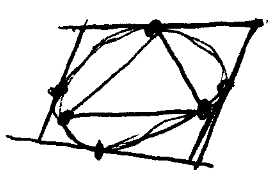


Definition

Let $\underline{c}: \mathbb{R} \to \mathbb{R}^2$ be a convex simple closed curve with period T. For every partition $P = \{t_0 = 0, t_1, \ldots, t_n = T\}$, let $a(\operatorname{int}\underline{c}, P)$ be the area bounded by the polygonal curve with vertices $\underline{c}(t_0), \underline{c}(t_1), \ldots, \underline{c}(t_n) = \underline{c}(t_0)$.

We then set $a(\operatorname{int}\underline{c}) = \sup_P a(\operatorname{int}\underline{c}, P)$ and say that $\operatorname{int}\underline{c}$ is measurable if $a(\operatorname{int}\underline{c})$ exists.

Remark: This is slightly cheating, we should define area using inscribed & circumscribed polygons:



Theorem (Green's Formula for Area)

Let $\underline{c}(t) = (x(t), y(t))$ be a positively-oriented simple closed curve with period T and continuous $\underline{c}'$. Then $\mathrm{int}(\underline{c})$ is measurable and

$$a\left(\mathrm{int}(\underline{c})\right) = \frac{1}{2}\int_0^T xy' - yx' \, dt$$

proof.

Assume without loss of generality that $\underline{0} \in \mathrm{int}(\underline{c})$.

First note that $x, x', y, y' : [0,T] \to \mathbb{R}$ are continuous functions and therefore:

(i) $\exists\, M > 0$ s.t. $|x(t)|, |y(t)|, |x'(t)|, |y'(T)| \leq M \quad \forall\, t \in [0,T]$

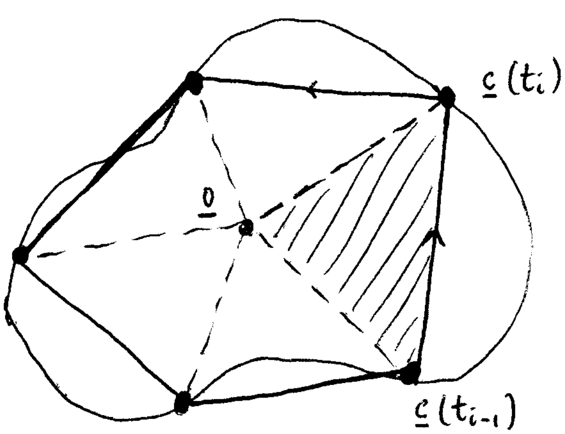
(ii) $\forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t. $|t - s| < \delta \Longrightarrow \begin{cases} |x(t) - x(s)| < \varepsilon \\ |y(t) - y(s)| < \varepsilon \\ |x'(t) - x'(s)| < \varepsilon \\ |y'(t) - y'(s)| < \varepsilon \end{cases}$

(uniform continuity)

In particular, it follows that $xy' - yx'$ is continuous on $[0,T]$ and therefore $\frac{1}{2}\int_0^T xy' - x'y \, dt$ exists. Moreover, $\forall\, \varepsilon > 0, \exists\, \delta_1 > 0$ s.t. if $P = \{t_0 = 0, t_1, \ldots, t_n = T\}$ is a partition of $[0,T]$ with $|t_i - t_{i-1}| < \delta_1$ then

$$\overline{I}\left(\frac{xy' - x'y}{2}, P\right) - \underline{I}\left(\frac{xy' - x'y}{2}, P\right) < \varepsilon \qquad (*)$$

Now, given a partition $P = \{t_0 = 0, t_1, \ldots, t_n = T\}$ of $[0,T]$ consider the polygonal curve with vertices $\underline{c}(t_0), \underline{c}(t_1), \ldots, \underline{c}(t_n) = \underline{c}(t_0)$



Let $a(\mathrm{int}\,\underline{c}, P)$ denote the area of the region bounded by this polygonal curve.

Then $a(\mathrm{int}\,\underline{c}) = \sup_P a(\mathrm{int}\,\underline{c}, P)$ if this sup exists.

Observe that $a(\mathrm{int}\,\underline{c}, P) = \sum_{i=1}^{n} \frac{1}{2}\, \underline{c}(t_i) \times \left[\underline{c}(t_i) - \underline{c}(t_{i-1})\right]$

i.e. $a(\operatorname{int} c, P) = \frac{1}{2}\sum_{i=1}^{n}\left[x(t_i)\left(y(t_i) - y(t_{i-1})\right) - y(t_i)\left(x(t_i) - x(t_{i-1})\right)\right]$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[x(t_i)\, y'(\zeta_i) - y(t_i)\, x'(\xi_i)\right](t_i - t_{i-1})$$

for some $\zeta_i, \xi_i \in [t_{i-1}, t_i]$ by the Mean Value Theorem.

<u>Claim</u>: $\forall\, \varepsilon > 0, \exists\, \delta_2 > 0$ s.t. if $|t_i - t_{i-1}| < \delta_2$ then

$$\left| \left[x(t_i)y'(\zeta_i) - y(t_i)\,x'(\xi_i)\right] - \left[x(t)y'(t) - y(t)x'(t)\right]\right| < \varepsilon \qquad \forall\, t \in [t_{i-1}, t_{i+1}]$$

<u>proof of Claim</u>:

We use boundedness and uniform continuity of $x, y, x', y' : [0,T] \to \mathbb{R}$:

~~if~~ $\exists\, \delta_2 > 0$ s.t.

$$\left| \left[x(t_i)\, y'(\zeta_i) - y(t_i)\, x'(\xi_i)\right] - \left[x(t)\, y'(t) - y(t)\, x'(t)\right]\right| =$$

$$= \left| \left[x(t_i) - x(t)\right] y'(\zeta_i) + x(t)\left[y'(\zeta_i) - y'(t)\right] - \left[y(t_i) - y(t)\right] x'(\xi_i) - y(t)\left[x'(\xi_i) - x'(t)\right]\right| \leq$$

$$\leq |x(t_i) - x(t)|\,|y'(\zeta_i)| + |x(t)|\,|y'(\zeta_i) - y'(t)| + |y(t_i) - y(t)|\,|x'(\xi_i)| + |y(t)|\,|x'(\xi_i) - x'(t)|$$

$$\leq 4M\varepsilon' < \varepsilon \quad \text{if } \varepsilon' < \frac{\varepsilon}{4M}.$$ ∎

Now choose a partition P with $|t_i - t_{i-1}| < \min\{\delta_1, \delta_2\}$. Then by the Claim

$$\underline{I}\left(\frac{xy' - x'y}{2}, P\right) - \varepsilon \leq a(\operatorname{int} c, P) \leq \overline{I}\left(\frac{xy' - x'y}{2}, P\right) + \varepsilon$$

and therefore by (*)

$$\left| a(\operatorname{int} c, P) - \frac{1}{2}\int_0^T xy' - x'y \, dt\right| < \varepsilon$$

Since ε is arbitrary, the Theorem is proved.

∎

Theorem (Wirtinger's Inequality)

Let $F : [0,T] \longrightarrow \mathbb{R}$ be a function with continuous derivative such that $F(0) = 0 = F(T)$. Then

$$\int_0^T F(t)^2\,dt \leq \frac{T^2}{\pi^2}\int_0^\pi (F'(t))^2\,dt.$$

Moreover, equality holds iff and only if $F(t) = A\sin\left(\frac{\pi t}{T}\right)$ fore some $A \in \mathbb{R}$.

proof.

Write $F(t) = G(t)\sin\left(\frac{\pi}{T}t\right)$.

Then $F' = G'\sin\left(\frac{\pi}{T}t\right) + \frac{\pi}{T}G(t)\cos\left(\frac{\pi}{T}t\right)$.

Hence $\int_0^T (F')^2\,dt = \int_0^T (G')^2\sin^2\left(\frac{\pi}{T}t\right) + 2\frac{\pi}{T}GG'\sin\left(\frac{\pi}{T}t\right)\cos\left(\frac{\pi}{T}t\right) + \frac{\pi^2}{T^2}G^2\cos^2\left(\frac{\pi}{T}t\right)dt$

Now,

$$2\frac{\pi}{T}\int_0^T GG'\sin\left(\frac{\pi}{T}t\right)\cos\left(\frac{\pi}{T}t\right)dt = \frac{\pi}{T}\; G^2\sin\left(\frac{\pi}{T}t\right)\cos\left(\frac{\pi}{T}t\right)\Big|_0^T$$

$$+\frac{\pi^2}{T^2}\int_0^T G^2\left[\sin^2\left(\frac{\pi}{T}t\right) - \cos^2\left(\frac{\pi}{T}t\right)\right]dt$$

$$= \frac{\pi^2}{T^2}\int_0^T G^2\left[\sin^2\left(\frac{\pi}{T}t\right) - \cos^2\left(\frac{\pi}{T}t\right)\right]dt$$

Therefore:

$$\int_0^T (F')^2\,dt = \int_0^T (G')^2\sin^2\left(\frac{\pi}{T}t\right) + \frac{\pi^2}{T^2}\int_0^T G^2\sin^2\left(\frac{\pi}{T}t\right)dt$$

$$\geq \frac{\pi^2}{T^2}\int_0^T G^2\sin^2\left(\frac{\pi}{T}t\right)dt = \frac{\pi^2}{T^2}\int_0^T F^2\,dt \qquad \&\text{ "=" iff } G' = 0$$

$\blacksquare$

Theorem (Isoperimetric Inequality for Convex Curves)

Let $\underline{c} : \mathbb{R} \longrightarrow \mathbb{R}^2$ be a simple closed convex curve with period T and continuous $\underline{c}'$. Then

$$a(\operatorname{int}\underline{c}) \leq \frac{1}{4\pi}\,\ell(\underline{c})^2$$

and equality holds iff $\underline{c}$ is a circle.

proof.

Wlog we can assume that $\underline{c}$ is parametrised by arc length, positively oriented and satisfies $\underline{c}(0) = \underline{0} = \underline{c}(T)$. Recall that $T = \ell(\underline{c})$ since $\underline{c}$ is parametrized by arc length.

We use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

<u>Notation</u>: $\dot{f} = \frac{d}{dt} f$

We calculate:

$$1 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \qquad x\dot{y} - \dot{x}y = r^2\dot{\theta}$$

We need to use a trick:

$$\frac{T^2}{4\pi} = \frac{T}{4\pi}\int_0^T \dot{r}^2 + r^2\dot{\theta}^2\, dt.$$

Consider:

$$I := \frac{4T}{\pi}\left(\frac{\ell^2(\underline{c})}{4\pi} - a(\mathrm{int}(\underline{c}))\right) = \int_0^T \frac{T^2}{\pi^2}\dot{r}^2 + \frac{T^2}{\pi^2}r^2\dot{\theta}^2 - 2\frac{T}{\pi}r^2\dot{\theta}\quad dt$$

$$= \int_0^T \left(\frac{T^2}{\pi^2}\dot{r}^2 - r^2\right) + r^2\left(1 - \frac{T}{\pi}\dot{\theta}\right)^2\, dt$$

$$\geqslant 0 \qquad \text{by Wirtinger's Inequality.}$$

Moreover $I = 0$ iff

$$\begin{cases} r(t) = A\sin\left(\frac{\pi}{T}t\right) & \text{for some } A \in \mathbb{R} \\ \dot{\theta} = \frac{\pi}{T} \end{cases}$$

$\Longleftrightarrow \quad r = A\sin(\theta - c)$ which is the polar equation of a circle. ∎

Problem sheet 2 MAT 142, Spring 2017

1. Let $\mathbf{c}(t) = (e^{-t}\cos t, e^{-t}\sin t)$, $t \in [0, \infty)$.

(a). Prove that $\lim_{t\to\infty} \mathbf{c}(t) = \mathbf{0}$. Draw a sketch of the trace of $\mathbf{c}$.

(b). Prove that $\lim_{t\to\infty} \int_0^t \|\mathbf{c}'(\tau)\|\,d\tau$ exists and justify the claim that $\mathbf{c}$ has finite length over $[0,\infty)$.

In the following problems 2–5, $\mathbf{c} = (u, v) : [a, b] \to \mathbb{R}^2$ is a regular parametrized curve with continuous derivative $\mathbf{c}'$.

2. For every $[t_1, t_2] \subset [a, b]$ define

$$\int_{t_1}^{t_2} \mathbf{c}(t)\,dt = \left(\int_{t_1}^{t_2} u(t)\,dt, \int_{t_1}^{t_2} v(t)\,dt\right) \in \mathbb{R}^2.$$

Prove the Fundamental Theorem of Calculus for this notion of integral, i.e. prove that

$$\mathbf{c}(t_2) - \mathbf{c}(t_1) = \int_{t_1}^{t_2} \mathbf{c}'(t)\,dt.$$

3. You are going to prove that the straight line between $\mathbf{c}(a)$ and $\mathbf{c}(b)$ is shorter than $\mathbf{c}$.

(a). Prove that for every $\mathbf{x} \in \mathbb{R}^2$ we have

$$\big(\mathbf{c}(b) - \mathbf{c}(a)\big) \cdot \mathbf{x} \le \|\mathbf{x}\| \int_a^b \|\mathbf{c}'(t)\|\,dt.$$

(b). Take $\mathbf{x} = \mathbf{c}(b) - \mathbf{c}(a)$ and deduce that $\|\mathbf{c}(b) - \mathbf{c}(a)\| \le \ell(\mathbf{c})$. When does equality hold?

4. Define a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ by

$$x_1 = \int_{t_1}^{t_2} u(t)\,dt, \qquad x_2 = \int_{t_1}^{t_2} v(t)\,dt.$$

Note that we can write

$$\|\mathbf{x}\|^2 = x_1 \int_{t_1}^{t_2} u(t)\,dt + x_2 \int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} \big(x_1 u(t) + x_2 v(t)\big)\,dt.$$

Prove that $\|\mathbf{x}\|^2 \le \|\mathbf{x}\| \int_{t_1}^{t_2} \|\mathbf{c}(t)\|\,dt$. Deduce that

$$\left\|\int_{t_1}^{t_2} \mathbf{c}(t)\,dt\right\| \le \int_{t_1}^{t_2} \|\mathbf{c}(t)\|\,dt.$$

5. Prove the Mean Value Inequality: there exists $t \in [t_1, t_2]$ such that

$$\|\mathbf{c}(t_2) - \mathbf{c}(t_1)\| \leq \|\mathbf{c}'(t)\|(t_2 - t_1).$$

6. This is an example of a non-rectifiable curve. Define $\mathbf{c} : [0,1] \to \mathbb{R}^2$ by

$$\mathbf{c}(t) = \left(1, t \sin\left(\frac{\pi}{t}\right)\right)$$

if $t > 0$ and $\mathbf{c}(0) = \mathbf{0}$.

(a). Show that $\mathbf{c}$ is continuous.

(b). Consider the arc $\mathbf{c}_n$ of $\mathbf{c}$ over the interval $\frac{1}{n+1} \leq t \leq \frac{1}{n}$. Since $\mathbf{c}$ is regular with continuous derivative away from $t = 0$, $\mathbf{c}_n$ is rectifiable for every $n \geq 1$. Use Problem 5 to show that

$$\ell(\mathbf{c}_n) \geq \frac{4}{2n+1}.$$

(c). Consider the length of $\mathbf{c}$ over the interval $\frac{1}{N+1} \leq t \leq 1$ and deduce that $\mathbf{c}$ is not rectifiable.

7. The hyperbolic cosine and sine are the functions $\mathbb{R} \to \mathbb{R}$ defined by

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2}.$$

(a). Show that $\cosh^2 t - \sinh^2 t = 1$. Observe that $\cosh t > 0$ for all $t \in \mathbb{R}$.

(b). Show that the derivative of $\cosh t$ is $\sinh t$ and the derivative of $\sinh t$ is $\cosh t$.

(c). The *catenary* is the curve $\mathbf{c} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\mathbf{c}(t) = (t, \cosh t).$$

Show that the curvature of the catenary is

$$\kappa(t) = \frac{1}{\cosh^2 t}.$$

MAT 142 - ANALYSIS II : NOTES 3 (WEEK 4-6)

III. SEQUENCES

§1. Convergent sequences

Definition: A sequence (of real numbers) is a function $f: \mathbb{N} \longrightarrow \mathbb{R}$.

Notation: $\{f(n)\}$ or $\{a_n\}$

Definition A sequence $\{a_n\}$ is said to converge to a if $\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ s.t.

$$|a_n - a| < \varepsilon \quad \forall \ n \geq N.$$

Theorem

Let $\{a_n\}$ be a sequence.

(a) $\{a_n\}$ converges to a if $\forall \ \varepsilon>0$ the interval $(a-\varepsilon, a+\varepsilon)$ contains all but finitely many values a_n.

(b) The limit of a sequence is unique: if $\lim_{n\to\infty} a_n = a$ & $\lim_{n\to\infty} a_n = a'$ then $a=a'$.

(c) If $\{a_n\}$ converges then $\{a_n\}$ is bounded: $\exists M>0$ s.t. $|a_n| \leq M \ \forall n \in \mathbb{N}$.

Definition

A sequence $\{a_n\}$ is

(i) increasing if $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$

(ii) decreasing if $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$

and monotonic if it is either increasing or decreasing.

Theorem

Every bounded monotonic sequence converges.

proof.

Assume $\{a_n\}$ is increasing and let a be $\sup_n a_n$. Note that a exists since $\{a_n\}$ is bounded.

By definition of sup, $\forall \ \varepsilon>0 \quad \exists N \in \mathbb{N}$ s.t. $a_N > a - \varepsilon$.

Then $\forall \ n \geq N \qquad |a - a_n| = a - a_n \leq a - a_N < \varepsilon$ ∎

§. 2 Subsequences

Definition Given a sequence $\{a_n\}_n$ and a sequence $\{n_k\}_k$ of positive integers s.t. $n_1 < n_2 < n_2 < \dots$, the sequence $\{a_{n_k}\}_k$ is called a subsequence of $\{a_n\}$.

Example $a_n = (-1)^n$ $\quad n_k = 2k \quad \Rightarrow \quad a_{n_k} = 1 \quad \forall\ k \in \mathbb{N}$

Theorem (Bolzano-Weierstrass Theorem)

Every bounded sequence contains a ~~bounded and~~ convergent subsequence.

proof.

Let $\{x_n\}_n$ be a bounded sequence. Then there exist $a_1 < b_1$ s.t. $x_n \in [a_1, b_1] \quad \forall\ n \in \mathbb{N}$. Set $n_1 = 1$.

Divide the interval $I_1 = [a_1, b_1]$ into two halves $I_1 = I_1' \cup I_1''$

If I_1' contains infinitely many x_n's set $I_2 = I_1'$ otherwise I_1'' contains infinitely many x_n's and we set $I_2 = I_1''$.

Write $I_2 = [a_2, b_2]$ with $a_1 \leq a_2 < b_2 \leq b_1$ and $b_2 - a_2 = \frac{b_1 - a_1}{2}$.

~~If we proceed in this way we obtain two sequences~~

Let n_2 the smallest natural number ≥ 2 s.t. $x_{n_2} \in I_2$. Note that $a_2 \leq x_{n_2} \leq b_2$.

If we proceed in this way we find sequences $\{a_k\}, \{b_k\}, \{x_{n_k}\}$ s.t.

$$a_1 \leq a_2 \leq \dots \leq a_k \leq x_{n_k} \leq b_k \leq \dots \leq b_2 \leq b_1 \quad \& \quad 0 \leq b_k - a_k = \frac{b_1 - a_1}{2^k}$$

By the convergence of monotonic sequences we have limits

$$\lim_{k \to \infty} a_k = a \quad \& \quad \lim_{k \to \infty} b_k = b.$$

Since $|b_k - a_k| \leq \dfrac{b_1 - a_1}{2^k}$ we have in fact $x := a = b$.

Then also $\lim_{k \to \infty} x_{n_k} = x$. ∎

§3. Cauchy sequences

Def A sequence $\{a_n\}$ is a Cauchy sequence if $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t.

$$|a_n - a_m| < \varepsilon \quad \forall\ n, m \geq N.$$

~~Theorem (Cauchy)~~

Theorem (Cauchy Criterion)

A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.

proof.

If $\lim_{n\to\infty} a_n = a$ then $\forall\, \varepsilon$ $\exists\, N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon/2$ $\forall\, n \geq N$.

Then if $n,m \geq N$ we have $|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a - a_m| < \varepsilon$.

Suppose now that $\{a_n\}$ is a Cauchy sequence.

First of all we show that $\{a_n\}$ is bounded. Indeed, $\exists\, N$ s.t.

$$|a_n - a_m| < 1 \qquad \forall\, n,m \geq N.$$

Then

$$|a_k| \leq \max\left\{ |a_1|, \dots, |a_{N-1}|,\ |a_N + 1|,\ |a_N - 1| \right\}$$

If $\{a_n\}$ is bounded, by the Bolzano-Weierstrass Theorem there exists a convergent subsequence $\{a_{n_k}\}_k$ w/ $\lim_{k\to\infty} a_{n_k} = a$.

Now, $\forall\, \varepsilon > 0$ $\exists\, N', K$ s.t.

$$\begin{matrix} n,m \geq N' \\ k \geq K \end{matrix} \implies \begin{matrix} |a_n - a_m| < \varepsilon/2 \\ |a - a_{n_k}| < \varepsilon/2 . \end{matrix}$$

Set $N = \max\{N', N_K\}$. Then $\forall\, n \geq N$ ~~& $k \geq K$ we have~~

we have $|a - a_n| = |a - a_{n_k} + a_{n_k} - a_n| \leq |a - a_{n_k}| + |a_{n_k} - a_n| < \varepsilon$

provided k is sufficiently large so that $n_k \geq N'$. ∎

§4. Newton's Method

~~Fundamental Theorem of Algebra~~

Example (Newton, 1671)

We want to find a **root** of the polynomial $P(x) = x^3 - 2x - 5$

Note that $x_0 = 2$ satisfies $P(x_0) = -1$

Consider a new polynomial P_1 defined by

$$P_1(x) = P(2+x) = x^3 + 6x^2 + 10x - 1$$

If x is small, $x^3 + 6x^2$ is much smaller than $10x - 1$.

Set $x_1 = 2 + \frac{1}{10}$

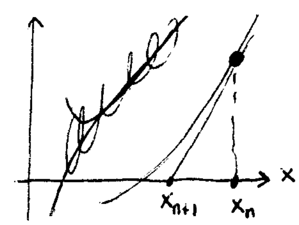
Define a new polynomial P_2 by

$$P_2(x) = P(x_1 + x) = x^3 + 6.3x^2 + 11.23x + 0.061$$

If x is small, $x^3 + 6.3x^2$ is much smaller than $11.23x + 0.061$

We then set $x_2 = 2 + \frac{1}{10} - 0.054 = 2.046$

We calculate $P(x_2) \simeq -0.527206664$



Newton's Method

Let $g : [a,b] \longrightarrow \mathbb{R}$ be a function w/ $g'(x) \neq 0 \ \forall x \in [a,b]$

Fix $x_0 \in [a,b]$ and consider the sequence $\{x_n\}$ defined by

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}, \quad x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}, \ \ldots, \ x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

Q1: When does $\{x_n\}$ converge?

Q2: Assume that $\lim_{n\to\infty} x_n = x$. Does x satisfy $g(x) = 0$?

~~Theorem~~ ~~Let~~ **Remark**: Given g as above define ~~the~~ a function $f : [a,b] \longrightarrow \mathbb{R}$ by $f(x) = x - \frac{g(x)}{g'(x)}$. Then a zero of g (a point x s.t. $g(x) = 0$) correspond to a fixed point of f (a point x s.t. $f(x) = x$).

Note that $x_1 = f(x_0)$, $x_2 = f(x_2)$, ..., $x_{n+1} = f(x_n)$ and that in terms of f we have:

Q1: When does $\{x_n\}$ converge?

Q2: Assume $\lim_{n\to\infty} x_n = x$. Does x satisfy $f(x) = x$.

<u>Theorem</u>

~~Let f be a~~ Let $f : [a,b] \to [a,b]$ be a continuous function. Fix $x_0 \in [a,b]$ and define a sequence $\{x_n\}$ by

$$x_1 = f(x_0), \quad x_2 = f(x_1), \ldots, \quad x_{n+1} = f(x_n).$$

Assume that $\lim_{n\to\infty} x_n = x$. Then $f(x) = x$.

<u>proof.</u>

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x) \quad \text{since } f \text{ is continuous.} \qquad \blacksquare$$

<u>Example</u>:

$$f(x) = [x] + 1 - \frac{1}{2}\left([x]+1-x\right)^2 \quad \text{where } [x] \text{ is the largest integer} \leq x$$

$x_0 = 1$

$x_1 = f(x_0) = 2 - \frac{1}{2} = 1.5$

$x_2 = f(x_1) = 2 - \frac{1}{2} \times \left(\frac{1}{2}\right)^2 = 1.875$

$x_3 = f(x_2) = 1.9921875$

$x_4 = f(x_3) = 1.9999695$

$\vdots$

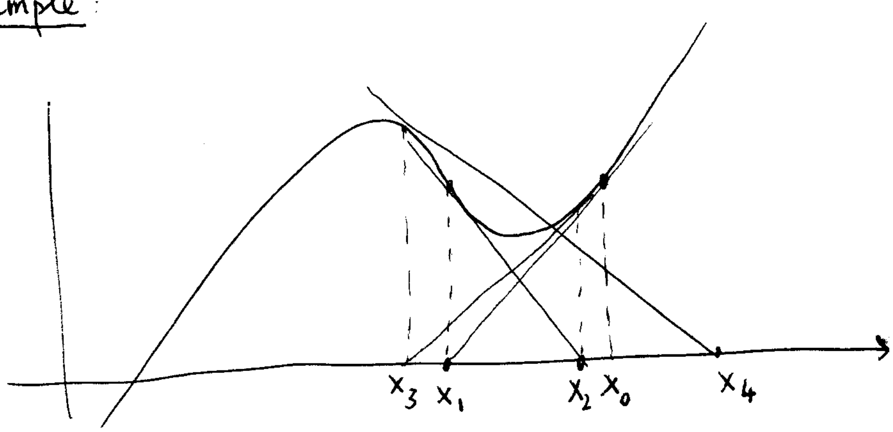
One can check that $\lim_{n\to\infty} x_n = 2$. However,

$$f(x_n) = 2 - \frac{1}{2}(2 - x_n)^2 \xrightarrow[n\to\infty]{} 2 \neq 2.5 = f(2)$$

This shows that the assumption f continuous is necessary.

We now return to Q1: When does $\{x_n\}$ converge?

<u>Example</u>:



~~Example~~ Exercise (Exercise 16, Chap. 3 in [R])

Fix $\alpha > 0$. ~~Choose~~ $g(x) = x^2 - \alpha \rightsquigarrow f(x) = x - \frac{g(x)}{g'(x)} = \frac{1}{2}\left(x + \frac{\alpha}{x}\right)$

Fix $x_0 \geq \sqrt{\alpha}$ and consider the sequence $\{x_n\}$ defined by

$$x_1 = f(x_0), \quad x_2 = f(x_1), \quad \dots, \quad x_{n+1} = f(x_n)$$

(i) Prove that $\{x_n\}$ is decreasing and ~~and~~ conclude that

$$\lim_{n\to\infty} x_n = \sqrt{\alpha}$$

(ii) Set $\varepsilon_n = x_n - \sqrt{\alpha}$ and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

Hence setting $\beta = 2\sqrt{\alpha}$ we have

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_0}{\beta}\right)^{2^{n+1}} \quad \forall \; n \geq 0$$

(iii) If $\alpha = 3$ & $x_0 = 2$ show that $\frac{\varepsilon_0}{\beta} < \frac{1}{10}$ and therefore

$$\varepsilon_4 < 4 \cdot 10^{-16} \qquad \varepsilon_5 < 4 \cdot 10^{-32}$$

<u>Definition</u> A function $f : [a,b] \longrightarrow [a,b]$ is a contraction if $\exists\; c \in (0,1)$ s.t.

$$|f(x) - f(y)| \leq c\,|x - y| \qquad \forall\; x, y \in [a,b].$$

<u>Theorem</u>

Let $f : [a,b] \longrightarrow [a,b]$ be a contraction. ~~Then there~~

Then f has a unique fixed point x_*. Moreover, for every $x_0 \in [a,b]$ consider the sequence $\{x_n\}$ defined by $x_1 = f(x_0)$, $x_{n+1} = f(x_n)$. Then $\lim_{n\to\infty} x_n = x_*$.

proof.

Fix $x_0 \in [a,b]$ and consider the sequence $\{x_n\}$ defined by $x_1 = f(x_0)$, $x_{n+1} = f(x_n)$. We first show that $\{x_n\}$ is a Cauchy sequence. Indeed,

~~$|x_{n+1} - x_n|$~~ note that

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| < |x_{n-1} - x_{n-2}| < \dots < c^n\,|x_1 - x_0| = c^n\,|f(x_0) - x_0|$$

Hence, $\forall\ n,m \geq 1$

$$|x_{n+m} - x_n| = \left| \sum_{k=0}^{m-1} (x_{n+k+1} - x_{n+k}) \right| \leq |f(x_0) - x_0| \sum_{k=0}^{m-1} c^{n+k} = c^n |f(x_0) - x_0| \frac{1-c^m}{1-c}$$

$$\leq c^n |f(x_0) - x_0| \frac{1}{1-c}$$

Since $c<1$, $\lim_{n\to\infty} c^n = 0$ and therefore $\forall\ \varepsilon > 0$

$\exists\ N$ s.t. $c^n |f(x_0) - x_0| \frac{1}{1-c} < \varepsilon \quad \forall\ n \geq N$.

By the Cauchy Criterion, $\{x_n\}$ is convergent. Set $x_* = \lim_{n\to\infty} x_n$.

Now observe that f is continuous:

$\forall\ \varepsilon>0$ if $|x-y| < \frac{\varepsilon}{c}$ we have $|f(x) - f(y)| < c|x-y| < \frac{\varepsilon}{c} \cdot c = \varepsilon$

Hence

$$x_* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x_*)$$

i.e. x_* is a fixed point of f.

Finally, suppose that x_*' is another fixed point. Then

$$|x_* - x_*'| = |f(x_*) - f(x_*')| \leq c\,|x_* - x_*'| \Rightarrow (1-c)|x_* - x_*'| \leq 0$$

$$\Rightarrow \quad x_* = x_*' \quad \text{since } 0<c<1$$

■

<u>Theorem</u>

Let $f: [a,b] \longrightarrow [a,b]$ be a differentiable function. Then f is a contraction iff $\exists\ c \in (0,1)$ s.t. $|f'(x)| \leq c \quad \forall\ x \in (a,b)$.

proof.

If f is a contraction then $\exists\ c \in (0,1)$ s.t.

$$|f(x+h) - f(x)| \leq c|h| \quad \text{and therefore} \quad \left| \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \right| \leq c$$

Conversely, if $|f'(\xi)| \leq c \quad \forall\ \xi \in (a,b)$ we have $\forall\ x,y \in [a,b]$

$$f(x) - f(y) = f'(\xi)(x-y) \quad \text{for some } \xi \in (a,b). \text{ by the Mean Value Theorem}$$

Hence:

$$|f(x) - f(y)| \leq |f'(\xi)|\,|x-y| \leq c\,|x-y|.$$

■

Problem sheet 3 MAT 142, Spring 2017

1. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its parametrization

$$\mathbf{c}(t) = (a\cos t, b\sin t), \quad t \in [0, 2\pi].$$

(a). Calculate the curvature of the ellipse.

(b). Calculate the area of the region enclosed by the ellipse using Green's Formula.

(c). Show that

$$\int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt \geq 2\pi\sqrt{ab}$$

and that equality holds if and only if $a = b$.

2. Let $\mathbf{c}$ be the parametrized curve

$$\mathbf{c}(t) = \left((1 + 2\cos t)\cos t, (1 + \cos t)\sin t\right).$$

Show that $\mathbf{c}(t + 2\pi) = \mathbf{c}(t)$ but $\mathbf{c}$ is not a simple closed curve. Draw a sketch.

3. Does there exist a simple closed curve 4 ft long and bounding an area of 2 ft^2?

4. Consider the sequence $\{a_n\}$ defined by

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

Find all numbers $a \in \mathbb{R}$ such that there exists a subsequence of $\{a_n\}$ converging to a.

5. The Euler number γ is defined as

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right).$$

Show that γ is well-defined. (Hint: show that the sequence $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$ is decreasing.)

6. Let $\{a_n\}$ a bounded sequence. Define two new sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}, \qquad y_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

(a). Prove that $\{x_n\}$ is decreasing and $\{y_n\}$ is increasing and deduce that both sequences have a limit. Set

$$\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} x_n, \qquad \overline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} y_n.$$

(b). Calculate $\overline{\lim}_{n\to\infty} a_n$ when $a_n = \frac{1}{n}$.

(c). Prove that

$$\underline{\lim}_{n\to\infty} a_n \leq \overline{\lim}_{n\to\infty} a_n.$$

(d). Prove that $\{a_n\}$ is convergent if and only if $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$ and that in this case $\lim_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$.

7. Prove that every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous using the Bolzano–Weierstrass Theorem.

MAT 142 — NOTES 4

III. FUNCTIONS

§1. Continuous functions on a closed bounded interval

Definition Let $[a,b]$ a closed bounded interval in $\mathbb{R}$. Set

$$C([a,b]) = \{ f : [a,b] \to \mathbb{R} \text{ s.t. } f \text{ is continuous} \}$$

On $C([a,b])$ we have the following operations:

- sum: $f, g \in C([a,b]) \Rightarrow f+g \in C([a,b])$ where $(f+g)(x) = f(x)\, g(x) \quad \forall x \in [a,b]$
- scalar multiplication: $k \in \mathbb{R}$, $f \in C([a,b]) \Rightarrow kf \in C([a,b])$, where $(kf)(x) = k\, f(x)$ $\forall x \in [a,b]$
- product: $f, g \in C([a,b]) \Rightarrow fg \in C([a,b])$, where $(fg)(x) = f(x)\, g(x) \quad \forall x \in [a,b]$.

These operations satisfy the obvious compatibility, distributivity, commutativity & associativity properties, e.g. $(fg)h = f(gh) \quad \forall f, g, h \in C([a,b])$, etc.

Rmk: We say that $C([a,b])$ is an algebra over $\mathbb{R}$.

Definition Fore every $f \in C([a,b])$ set $\|f\| := \sup_{x \in [a,b]} |f(x)|$

Theorem $\|\cdot\| : C([a,b]) \to \mathbb{R}$ is a norm, that is

(i) $\|kf\| = |k|\, \|f\| \quad \forall k \in \mathbb{R}, f \in C([a,b])$

(ii) $\|f+g\| \leq \|f\| + \|g\| \quad \forall f, g \in C([a,b])$

(iii) $\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$

Moreover:

(iv) $\|fg\| \leq \|f\|\, \|g\|$

proof of (iv):

$\forall x \in [a,b]$ we have $|f(x) g(x)| = |f(x)|\, |g(x)| \leq \|f\|\, \|g\|$

since $|f(x)| \leq \sup_y |f(y)| = \|f\|$ & $|g(x)| \leq \sup_y |g(y)| = \|g\|$

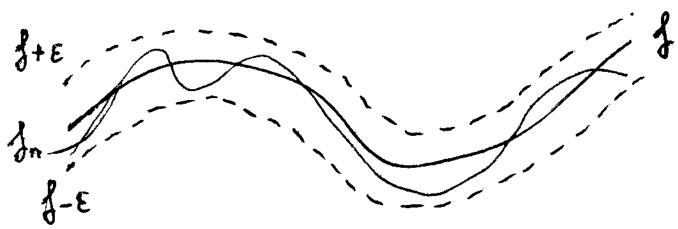
Hence $\|fg\| = \sup_x |f(x) g(x)| \leq \|f\|\, \|g\|$. ∎

§2. Uniform Convergence

Definition Let $\{f_n\}$ be a sequence in $C([a,b])$. We say that f_n converges uniformly to f if $\forall\, \varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t.

$$\|f-f_n\|<\varepsilon \quad \forall\, n\geq N.$$

Rmk: Equivalently, we say that f_n converges uniformly to f if $\forall\, \varepsilon>0$, $\exists\, N\in\mathbb{N}$ s.t. $|f_n(x)-f(x)|<\varepsilon \quad \forall\, n\geq N,\ x\in[a,b]$



Theorem (Uniform convergence & continuity)
Let $\{f_n\}$ be a sequence in $C([a,b])$. If f_n converges uniformly to f then $f\in C([a,b])$.

proof. Fix $x\in[a,b]$. We prove that f is continuous at x.
Fix $\varepsilon>0$.
Since f_n converges uniformly to f, $\exists\, N\in\mathbb{N}$ s.t. $|f_n(y)-f(y)|<\frac{\varepsilon}{3} \quad \forall\, n\geq N, \forall\, y\in[a,b]$
Fix $n\geq N$. Since f_n is continuous on $[a,b]$ (in particular, it is continuous at x)

$\exists\, \delta>0$ s.t. $|f_n(x)-f_n(y)|<\frac{\varepsilon}{3} \quad \forall\, y\in(x-\delta,x+\delta)$.

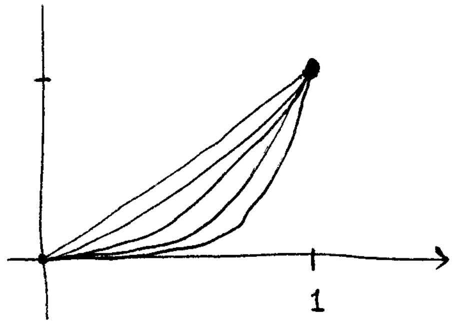
Now, if $y\in(x-\delta,x+\delta)$ we have

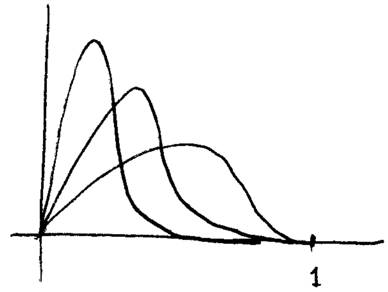
$$\begin{aligned}|f(x)-f(y)| &= |f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)|\\ &\leq |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

∎

Definition A sequence $\{f_n\}$ in $C([a,b])$ is said to converge pointwise to f if $\lim_{n\to\infty} f_n(x)=f(x) \quad \forall\, x\in[a,b]$.

Example: $f_n:[0,1]\longrightarrow\mathbb{R}$, $f_n(x)=x^n$.
Then f_n converges pointwise to $f(x)=\begin{cases}0 & \text{if } 0\leq x<1\\ 1 & \text{if } x=1\end{cases}$, which is not continuous.





<u>Example:</u>

$f_n : [0,1] \longrightarrow \mathbb{R}$, $\quad f_n(x) = n\,x\,(1-x^2)^n$

The pointwise limit of f_n is $f(x) = 0$, but the limit is not uniform.

$$\int_0^1 f_n(x)\,dx = -\frac{n}{2}\,\frac{(1-x^2)^{n+1}}{n+1}\Bigg|_0^1 = \frac{1}{2}\,\frac{n}{n+1} \xrightarrow[n\to\infty]{} \frac{1}{2}$$

Hence $\displaystyle \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

<u>**Theorem**</u> (Uniform ~~continuity~~ convergence & integration)

Let $\{f_n\}$ be a sequence in $C([a,b])$. Assume that f_n converges uniformly to f.

Define ~~all~~ functions $\displaystyle F_n(x) = \int_a^x f_n(t)\,dt$

$$F(x) = \int_a^x f(t)\,dt.$$

Then F_n converges uniformly to F on $[a,b]$.

<u>proof.</u>

Note that F_n, F are well-defined since f_n & f are continuous and therefore integrable.

By uniform convergence of f_n to f, $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall\ n \geqslant N.$$

Hence for $n \geqslant N$:

$$|F_n(x) - F(x)| = \left|\int_a^b f_n(t) - f(t)\,dt\right| \leqslant \int_a^b |f_n(t) - f(t)|\,dt$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dt = \varepsilon \qquad \blacksquare$$

Example / Exercise

$f_n : [-1,1] \longrightarrow \mathbb{R}, \quad f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$

Show that f_n converges uniformly to $|x| = f(x)$

Note that f_n is a differentiable function, but f isn't.

Theorem (Uniform convergence & differentiation)

Let $\{f_n\}$ be a sequence in $C([a,b])$. Assume that f_n is differentiable on $[a,b]$ and that f_n' is continuous. Assume that f_n converges pointwise to a function f and that f_n' converges uniformly to a (continuous) function g.

Then f_n converges uniformly to f & $g = f'$.

proof.

By the theorem on uniform convergence & integration

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt \quad \text{converges uniformly to} \int_a^x g(t)\,dt$$

On the other hand the pointwise limit of $f_n(x) - f_n(a)$ is $f(x) - f(a)$.

Thus f is differentiable & $f' = g$. ∎

§3. More on uniform convergence

Definition A sequence $\{f_n\}$ in $C([a,b])$ is a Cauchy sequence if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|f_n - f_m\| < \varepsilon \quad \forall n, m \geq N$.

Theorem (Cauchy criterion)

A sequence $\{f_n\}$ in $C([a,b])$ is convergent if and only if it is Cauchy.

proof.

- convergent $\Rightarrow$ Cauchy:

Suppose f_n converges uniformly to f. Then $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\|f_n - f\| < \frac{\varepsilon}{2}$, if $n \geq N$.

Thus if $n, m \geq N$

$$\|f_n - f_m\| = \|f_n - f + f - f_m\| \leq \|f - f_n\| + \|f - f_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

- Cauchy $\Rightarrow$ convergent:

We know that $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in [a,b]$.

Hence for each fixed $x \in [a,b]$ the sequence of real numbers $\{f_n(x)\}$ is convergent.

Set $f(x) := \lim_{n \to \infty} f_n(x)$.

Fix $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall x \in [a,b],\ n, m \geq N$.

Fix $n \geq N$ and consider $|f(x) - f_n(x)|$.

Since $\lim_{m\to\infty} f_m(x) = f(x)$, $\exists N_x \in \mathbb{N}$ s.t. $|f(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall m \geq N_x$.

Choose $m \geq \max\{N, N_x\}$. Then we have

$$|f_n(x) - f(x)| = |f_n(x) - f_m(x) + f_m(x) - f(x)| \leq |f(x) - f_m(x)| + |f_n(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

<u>Theorem</u> (Dini's Theorem)

Let $\{f_n\}$ be a sequence of continuous functions $f_n : [a,b] \longrightarrow \mathbb{R}$ with pointwise limit $f : [a,b] \longrightarrow \mathbb{R}$. Assume that f is continuous and $\{f_n(x)\}$ is increasing (decreasing) $\forall x \in [a,b]$.

Then f_n converges uniformly to f.

<u>proof.</u>

If $f_n(x) \leq f_{n+1}(x)$ define $g_n(x) = f(x) - f_n(x)$

If $f_n(x) \geq f_{n+1}(x)$ define $g_n(x) = f_n(x) - f(x)$

Either way, we have a sequence of functions $g_n : [a,b] \longrightarrow \mathbb{R}$ s.t.

- g_n is continuous
- $g_n(x) \geq g_{n+1}(x) \quad \forall x \in [a,b]$
- $\lim_{n\to\infty} g_n(x) = 0$ for every fixed $x \in [a,b]$

We have to show that g_n converges to 0 uniformly.

<u>Observation</u>: $g_n(x) \geq 0 \quad \forall x \in [a,b],\ n \in \mathbb{N}$

If not, $\exists n \in \mathbb{N}$ & $x \in [a,b]$ s.t. $g_n(x) < 0$. But then $\forall m \geq n$

$$g_m(x) \leq g_n(x) < 0$$

and therefore $0 = \lim_{m\to\infty} g_m(x) \leq g_n(x) < 0$, which is a contraddiction.

Now, we want to prove: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $0 \leq g_n(x) < \varepsilon \quad \forall n \geq N,\ x \in [a,b]$.

Assume this is not the case: $\exists \varepsilon_0 > 0$ s.t. $\forall n \in \mathbb{N}$ $\exists x_n \in [a,b]$ with

$$g_n(x_n) \geq \varepsilon_0 > 0$$

The sequence $\{x_n\}$ is a bounded sequence in $\mathbb{R}$. By the Bolzano-Weierstrass Theorem $\exists$ a convergent subsequence $\{x_{n_k}\}$ w/ $\lim_{k\to\infty} x_{n_k} = x$.

Now, since $\lim_{n\to\infty} g_n(x) = 0$, $\exists\ N_x \in \mathbb{N}$ s.t. $0 \le g_n(x) < \frac{\varepsilon_0}{2}\quad \forall\ n \ge N$.

Fix $n \ge N$. Since g_n is continuous, $\exists\ \delta_n > 0$ s.t. $|g_n(x) - g_n(y)| < \frac{\varepsilon_0}{2}$ provided $|x-y| < \delta_n$.

Since $\lim_{k\to\infty} x_{n_k} = x$, $\exists\ K \in \mathbb{N}$ s.t. $|x - x_{n_k}| < \delta_n \quad \forall\ k \ge K$.

Now choose $k \ge K$ s.t. $n_k \ge n$ (this is possible since $\lim_{k\to\infty} n_k = \infty$)

Then

$$\varepsilon_0 \le g_{n_k}(x_{n_k}) \underset{\substack{\uparrow \\ n_k \ge n \\ g_n \downarrow}}{\le} g_n(x_{n_k}) = g_n(x_{n_k}) - g_n(x) + g_n(x) \le |g_n(x_{n_k}) - g_n(x)| + g_n(x)$$

$$< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0 \quad ↯$$

∎

<u>Example</u>

$f_n(x) = x^n$ converges uniformly to 0 on $[0,a]$ for every $0<a<1$ ~~converges~~

§4. <u>Equicontinuous families of functions</u>

Q: The Bolzano-Weierstrass Theorem states that every bounded sequence of real numbers has a convergent subsequence. Is something similar true for sequences of fcts and uniform convergence?

<u>Definition</u> Let $\{f_n\}$ be a sequence of functions on $[a,b]$. We say that $\{f_n\}$ is:

(i) pointwise bounded if $\forall\ x \in [a,b]$, $\exists\ M_x > 0$ s.t. $|f_n(x)| \le M_x \quad \forall\ n \in \mathbb{N}$

(ii) uniformly bounded if $\exists\ M>0$ s.t. $|f_n(x)| \le M$, $\forall\ n \in \mathbb{N}, x \in [a,b]$

<u>Example</u>

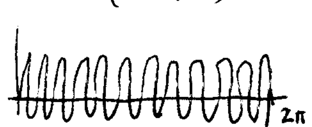
$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}, \quad x \in [0,1]$$

$\{f_n\}$ is uniformly bounded: $|f_n(x)| \le 1 \quad \forall\ n \in \mathbb{N}, x \in [0,1]$

$\lim_{n\to\infty} f_n(x) = 0$ pointwise

$f_n\left(\frac{1}{n}\right) = 1$ so no subsequence can converge to 0 uniformly.

<u>Example</u> $f_n(x) = \sin(nx)$, $x \in [0, 2\pi]$



__Definition__ Let $\{f_n\}$ be a sequence of functions $f_n : [a,b] \to \mathbb{R}$.
We say that $\{f_n\}$ is equicontinuous if $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t.

$$|f_n(x) - f_n(y)| < \varepsilon \quad \forall\ x, y \in [a,b] \text{ w/ } |x-y| < \delta,\ \forall\ n \in \mathbb{N}$$

__Proposition__

Let $\{f_n\}$ be a sequence in $C([a,b])$ that converges uniformly to f. Then $\{f_n\}$ is equicontinuous.

__proof.__ Fix $\varepsilon > 0$:

$f_n \to f$ uniformly: $\exists\ N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon/3\ \ \forall\ n \geq N,\ x \in [a,b]$

f uniformly continuous: $\exists\ \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon/3$ whenever $|x-y| < \delta$

$f_1, \dots, f_{N-1}$ uniformly continuous: $\exists\ \delta_1, \dots, \delta_{N-1} > 0$ s.t.

$$|f_i(x) - f_i(y)| < \varepsilon \quad \text{whenever} \quad |x-y| < \delta_i$$

Now if $|x-y| < \min\{\delta_1, \dots, \delta_{N-1}, \delta\}$ we have:

$$|f_i(x) - f_i(y)| < \varepsilon \qquad \forall\ i = 1, \dots, N-1$$

$$|f_n(x) - f_n(y)| = |f_n(x) - f(x) + f(x) - f(y) + f(y) - f_n(y)|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\forall\ n \geq N$ ∎

__Proposition:__ Let $\{f_n\}$ be a sequence in $C([a,b])$ which is pointwise bounded and equicontinuous. Then $\{f_n\}$ is uniformly bounded.

__proof.__

Suppose not: then $\forall\ n \in \mathbb{N},\ \exists\ x_n \in [a,b]$ s.t. $|f_n(x_n)| > n$.

By Bolzano-Weierstrass $\{x_n\}$ has a convergent subsequence $x_{n_k} \xrightarrow[k\to\infty]{} x \in [a,b]$.

Since $\{f_n\}$ is pointwise bounded $\exists\ M_x > 0$ s.t. $|f_n(x)| \leq M_x\ \ \forall\ n \in \mathbb{N}$.

Now choose $K' \in \mathbb{N}$ s.t. $n_k \geq M_x + 1\ \ \forall\ k \geq K'$.

Since $\{f_n\}$ is equicontinuous, $\exists\ \delta > 0$ s.t. $|x-y| < \delta \implies |f_n(x) - f_n(y)| < \frac{1}{2}\ \ \forall\ n \in \mathbb{N}$

Since $\lim_{k\to\infty} x_{n_k} = x$, $\exists\ K'' \in \mathbb{N}$ s.t. $k \geq K'' \implies |x - x_{n_k}| < \delta$

Then if $k \geq K = \max\{K', K''\}$

$$M_x \geq |f_{n_k}(x)| = |f_{n_k}(x_{n_k}) + f_{n_k}(x) - f_{n_k}(x_{n_k})| \geq |f_{n_k}(x_{n_k})| - |f_{n_k}(x) - f_{n_k}(x_{n_k})| \geq M_x + 1 - \frac{1}{2}$$ ↯

Theorem (Arzelà-Ascoli Theorem)

Let $\{f_n\}$ be a sequence in $C([a,b])$ which is uniformly bounded and equicontinuous. Then there exists a subsequence $\{f_{n_k}\}$ which is uniformly convergent.

Rmk: By the previous proposition we can replace uniformly bounded w/ pointwise bounded.

proof.

Step 1: $\exists$ a subsequence $\{f_{n_k}\}$ s.t. $\{f_{n_k}(x)\}$ converges $\forall\ x \in [a,b] \cap \mathbb{Q}$

Since $[a,b]\cap\mathbb{Q}$ is countable we can enumerate it: $[a,b]\cap\mathbb{Q} = \{x_i\}_{i=1}^{\infty}$

Consider the sequence of numbers $\{f_n(x_1)\}$. This is a bounded sequence, so by the Bolzano-Weierstrass Theorem there exists a subsequence $\{f_{1,k}\}_k$ s.t. $\{f_{1,k}(x_1)\}$ converges.

Consider the bounded sequence $\{f_{1,k}(x_2)\}$. By Bolzano-Weierstrass there exists a subsequence $\{f_{2,k}\}_k$ s.t. $\{f_{2,k}(x_2)\}$ & $\{f_{2,k}(x_1)\}$ converge.

Proceeding inductively like this we construct sequences

$$\{f_n\}_n \supset \{f_{1,k}\}_k \supset \{f_{2,k}\}_k \supset \cdots \supset \{f_{i,k}\}_k \quad \text{s.t.}$$

$$\{f_{i,k}(x_j)\}_k \text{ converges } \forall\ j=1,\dots,i$$

$$\begin{array}{ccccc} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \end{array}$$

We can then choose $f_{n_k} = f_{k,k}$. Since $\{f_{n_k}\}$ is a subsequence of $\{f_{i,k}\}$ for every $i=1,2,3,\dots$ we have $\{f_{n_k}(x_i)\}$ converges $\forall\ i=1,2,3,\dots$

Step 2: $\{f_{n_k}\}$ converges uniformly.

We prove that $\{f_{n_k}\}$ is a Cauchy sequence in $C([a,b])$.

Fix $\varepsilon>0$. $\forall\ x\in[a,b]$ $\exists$ sequence $\{x_i\}\subset[a,b]\cap\mathbb{Q}$ s.t. $\lim_{i\to\infty} x_i = x$

Moreover, since $\{f_n\}$ is equicontinuous $\exists\ \delta>0$ s.t.

$$|f_n(x) - f_n(y)| < \varepsilon/3 \quad \forall\ n\in\mathbb{N},\ x,y\in[a,b] \text{ s.t. } |x-y|<\delta.$$

~~We can then find~~ Since $\lim_{i\to\infty} x_i = x$, we can find i s.t. $|x-x_i|<\delta$.

Since $x_i \in [a,b]\cap\mathbb{Q}$, $\exists\ K\in\mathbb{N}$ s.t. $|f_{n_k}(x_i)-f_{n_h}(x_i)| < \frac{\varepsilon}{3}\quad \forall\ k,h\geq K$.

Then if $k,h\geq K$ we have:

$$\begin{aligned}|f_{n_k}(x)-f_{n_h}(x)| &= |f_{n_k}(x)-f_{n_k}(x_i)+f_{n_k}(x_i)-f_{n_h}(x_i)+f_{n_h}(x_i)-f_{n_h}(x)|\\ &\leq |f_{n_k}(x)-f_{n_k}(x_i)|+|f_{n_k}(x_i)-f_{n_h}(x_i)|+|f_{n_h}(x_i)-f_{n_h}(x)|\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

■

§5. Approximating continuous functions with polynomials, I

Theorem (Weierstrass Approximation Theorem)

Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then there exists a sequence of polynomials $\{P_n\}$ such that $\{P_n\}$ converges uniformly to f.

proof.

First of all we can assume without loss of generality that $f:[0,1]\to\mathbb{R}$ with $f(0)=0=f(1)$. Indeed, given $f:[a,b]\to\mathbb{R}$ define

$$g(x) = f\big(a+x(b-a)\big) - f(a) - x\,[f(b)-f(a)]$$

Then $g:[0,1]\to\mathbb{R}$ w/ $g(0)=0=g(1)$

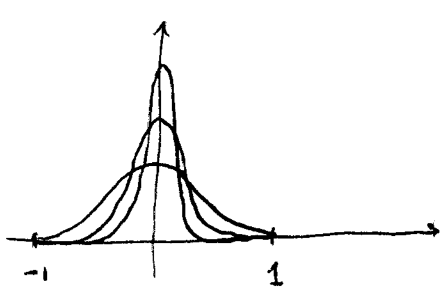
Moreover, if $\{P_n\}$ is a sequence of polynomials that converge uniformly to g then

$P_n\left(\frac{x-a}{b-a}\right)+f(a)+\frac{x-a}{b-a}\,[f(b)-f(a)]$ are polynomials that converge uniformly to f.

Now, since $f(0)=0=f(1)$ we can think of f as a ~~function~~ continuous function $f:\mathbb{R}\to\mathbb{R}$ by setting $f(x)=0$ if $x\geq 1$ or $x\leq 0$.

Consider the sequence of polynomials $Q_n:[-1,1]\to\mathbb{R}$ defined by

$$Q_n(x) = c_n(1-x^2)^n,\ \text{where}\quad \frac{1}{c_n}=\int_{-1}^{1}(1-x^2)^n\,dx$$



Note that $(1-x^2)^n \geqslant 1-nx^2$ $\forall x \in [0,1]$ and therefore

$$\int_{-1}^{1} (1-x^2)^n dx \geqslant 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx \geqslant 2\int_0^{1/\sqrt{n}} 1-nx^2 \, dx = \frac{4}{3}\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}$$

$\Rightarrow \quad c_n < \sqrt{n}.$

Now define

$$P_n(x) = \int_{-1}^{1} f(x+y)\, Q_n(y)\, dy = \int_{-x}^{1-x} f(x+y)\, Q_n(y)\, dy = \int_0^1 f(z)\, Q_n(z-x)\, dz$$

So P_n is a polynomial in x.

Now since f is uniformly continuous on $[0,1]$, $\forall \varepsilon > 0$, $\exists \delta$ s.t.

$$|f(x+y) - f(x)| < \frac{\varepsilon}{2} \qquad \forall y \text{ s.t. } |y| < \delta$$

Hence:

$$|P_n(x) - f(x)| \leq \int_{-1}^{1} |f(x+y) - f(x)|\, Q_n(y)\, dy$$

$$\leq 2\|f\| \int_{-1}^{-\delta} Q_n(y)\, dy + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(y)\, dy + 2\|f\| \int_{\delta}^{1} Q_n(y)\, dy$$

$$\leq 4\|f\| \sqrt{n}\, (1-\delta^2)^n + \frac{\varepsilon}{2} < \varepsilon$$

if $n \geqslant N$ where $4\|f\| \sqrt{N}\, (1-\delta^2)^N < \frac{\varepsilon}{2}$ (note that $\lim_{n\to\infty} 4\|f\|\sqrt{n}\,(1-\delta^2)^n = 0$) ∎

Problem sheet 4 MAT 142, Spring 2017

1. Let $f : [0,1] \to [0,1]$ be an increasing continuous function. Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.
(Hint: start with any point $x_0 \in [0,1]$; if $f(x_0) = x_0$ then you're done; assume then that $f(x_0) \neq x_0$ and consider the sequence $x_1 = f(x_0)$, $x_2 = f(x_1), \ldots, x_n = f(x_{n-1})$ when $f(x_0) > x_0$ and when $f(x_0) < x_0$.)

2. Fix $c \in [0.5, 1]$ and consider the function

$$f_c(x) = c\left(x + \frac{1}{x}\right).$$

(a). Prove that $f_c : [1,\infty) \to [1,\infty)$.

(b). Prove that if $c < 1$ then f_c is a contraction. The theorem proved in class then guarantees that f_c has a unique fixed point in $[1,\infty)$. Can you find it?

(c). Suppose now that $c = 1$. Prove that f_1 satisfies

$$|f_1(x) - f_1(y)| < |x - y|, \quad \text{for all } x, y \in [1,\infty) \text{ with } x \neq y.$$

(d). Using part (c) show that f_1 has at most one fixed point in the interval $[1,\infty)$.

(e). Show that f_1 has no fixed point in $[1,\infty)$.

3. (Exercise 17 in Chapter 3 of [R]) Fix $\alpha > 1$ and $x_0 > \sqrt{\alpha}$. Define a sequence $\{x_n\}$ by

$$x_1 = \frac{\alpha + x_0}{1 + x_0}, \qquad x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a). Prove that $x_1 > x_3 > x_5 > \ldots$ and $x_0 < x_2 < x_4 < \ldots$.

(b). Prove that $\{x_n\}$ converges and that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

4. Let α be any number with $\alpha > 5$ and consider the function $f(x) = \frac{x^3+x^2+1}{\alpha}$.

(a). Show that $f : [-1,1] \to [-1,1]$.

(b). Show that $f : [-1,1] \to [-1,1]$ is a contraction.

(c). Show that the equation $x^3 + x^2 + 1 = \alpha x$ has a unique solution in $[-1,1]$.

5. Use Newton's Method to approximate a zero of the function $f(x) = \cos x - x^2$ near 0. Find the best approximation within the accuracy of your calculator (that is, stop the iteration whenever you start getting the same result over and over again).

6. For the following sequences of functions determine the pointwise limit on the interval indicated and whether the convergence is uniform.

(a). $f_n(x) = e^{-nx^2}$, $x \in [-1, 1]$

(b). $f_n(x) = \frac{e^{-x^2}}{n^2}$, $x \in \mathbb{R}$

(c). $f_n(x) = x^n - x^{2n}$, $x \in [0, 1]$

(d). $f_n(x) = \sqrt{x + \frac{1}{n}}$, $x \in [0, \infty)$

(e). $f_n(x) = n\left(\sqrt{x + \frac{1}{n}} - \sqrt{x}\right)$, $x \in [a, \infty)$ for some $a > 0$

Problem sheet 5 MAT 142, Spring 2017

1. Let $\{f_n\}$ be a sequence of continuous functions on a closed bounded interval $[a,b]$ and assume that f_n converges uniformly to f.

(a). Let $\{x_n\}$ be a sequence of points in $[a,b]$ such that $\lim_{n\to\infty} x_n = x$. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

(b). Prove the converse to part (a): Let f be a continuous functions defined on $[a,b]$ and let f_n be a sequence of functions such that $\lim_{n\to\infty} f_n(x_n) = f(x)$ whenever $\lim_{n\to\infty} x_n = x$. Then f_n converges to f uniformly.

2. Suppose that $f_n, g : [0,\infty) \to \mathbb{R}$ are continuous functions such that $\int_0^\infty g(x)\,dx$ exists, $|f_n(x)| \le g(x)$ for all $x \in [0,\infty)$ and f_n converges uniformly to a function f on $[0,T]$ for every $T > 0$. Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx.$$

3. For every function $g : [0,1] \to \mathbb{R}$ with continuous derivative let $\ell(g)$ denote the length of the parametrized curve $\mathbf{c}(t) = \Big(t, g(t)\Big)$, $t \in [0,1]$ (this is the most obvious parametrization of the graph of g). Find a sequence of functions $f_n : [0,1] \to \mathbb{R}$ that converge uniformly to a function f with $\ell(f) \neq \lim_{n\to\infty} \ell(f_n)$.

4. Dini's Theorem states that if $\{f_n\}$ is a sequence of functions $f_n : I \to \mathbb{R}$ (where I is an interval in $\mathbb{R}$) such that:

(i) f_n is continuous for all n;

(ii) $I = [a,b]$ is a closed bounded interval;

(iii) $f_n(x) \le f_{n+1}(x)$ for all $x \in I$ (or $f_n(x) \ge f_{n+1}(x)$ for all $x \in I$);

(iv) f_n converges pointwise to a continuous function f;

then f_n converges uniformly to f.

(a). Assume hypotheses (i), (ii), (iii) are satisfied. Show that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in I$. The content of the hypothesis (iv) is to assume that this pointwise limit is a continuous function.

(b). Consider the functions

$$f_n : [0,1] \to \mathbb{R}, \qquad g_n : [0,\infty) \to \mathbb{R}, \qquad h_n : [0,1] \to \mathbb{R}, \qquad k_n : [0,1] \to \mathbb{R}$$

defined by:

$$f_n(x) = x^n \qquad g_n(x) = \begin{cases} 0 & \text{if } 0 \le x \le n \\ x - n & \text{if } n \le x \le n+1 \\ 1 & \text{if } x > n+1 \end{cases} \qquad h_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 - \frac{1}{n} \\ 0 & \text{if } 1 - \frac{1}{n} < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

and k_n is the function whose graph is:

- the straight line segment from $(0,0)$ to $(\frac{1}{2n}, 1)$,
- the straight line segment from $(\frac{1}{2n}, 1)$ to $(\frac{1}{n}, 0)$,
- the straight line segment from $(\frac{1}{n}, 0)$ to $(1, 0)$.

Use these functions to show that all hypotheses in Dini's Theorem are necessary. In other words, for each of these sequences of functions decide whether hypotheses (i)–(iv) hold, calculate the pointwise limit and decide whether the convergence is uniform.

5. Let $\{f_n\}$ be a sequence of continuous functions on the closed bounded interval $[a, b]$. Assume that $\{f_n\}$ is equicontinuous and that f_n converges pointwise to f.

(a). Show that f is continuous.

(b). Show that f_n converges uniformly to f.

6. Let $\{f_n\}$ be a sequence of continuous functions on the closed bounded interval $[a, b]$. Assume that f_n converges pointwise to f and let $\{f_{n_k}\}$ be a subsequence.

(a). Prove that if f_{n_k} converges pointwise to g then $f = g$.

(b). Prove that if f_{n_k} converges uniformly to f then f_n converges uniformly to f.

(c). Use part (b) and the Arzelà–Ascoli Theorem to give a different proof of part (b) in Problem 5.

Problem sheet 6 MAT 142, Spring 2017

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $f_n(x) = f(nx)$ for $x \in [0, 1]$. Assume that $\{f_n\}$ is equicontinuous. Prove that f is constant.

2. You are going to prove that the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = |x|$ can be approximated uniformly by polynomials without using Weierstrass Approximation Theorem.

(a). Given $x \in [0, 1]$, show that the sequence

$$y_1 = 1, \qquad y_{n+1} = \frac{1}{2}\left(x + 2y_n - y_n^2\right)$$

defines a decreasing sequence in $[0, 1]$ converging to $\sqrt{x}$.

(b). Deduce from part (a) that there exists polynomials $P_n : [-1, 1] \to \mathbb{R}$ such that the sequence $\{P_n\}$ converges pointwise to $f(x) = |x|$. (Hint: define $P_1(x) = 1$ and $P_{n+1}(x) = \frac{1}{2}\left(x^2 + 2P_n(x) - P_n(x)^2\right)$ for all $n \geq 1$.)

(c). Use Dini's Theorem to show that $\{P_n\}$ converges uniformly to $f(x) = |x|$ in $[-1, 1]$. (Hint: deduce from part (a) that $P_n(x) \geq P_{n+1}(x) \geq 0$ for all $x \in [-1, 1]$.)

3. Prove that every continuous function on a closed bounded interval $[a, b]$ can be approximated uniformly by piece-wise linear functions, that is, functions whose graph is a polygonal curve.

4. Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 f(x)\, x^n\, dx = 0$$

for every $n \in \mathbb{N}$. Prove that $f(x) = 0$ for all $x \in [0, 1]$. (Hint: use the Weierstrass Approximation Theorem to show that $\int_0^1 f^2\, dx = 0$.)

5. Exercise 10 in §7.4 of [A].

6. Exercise 9 in §7.8 of [A].

Note: [A] indicates Apostol, Calculus I.

Problem sheet 7 MAT 142, Spring 2017

1. Let $\{a_n\}$ be a sequence of real numbers and suppose that there exists $x_0 \neq 0$ such that $\sum_{n=0}^{\infty} a_n x_0^n$ converges. Fix $0 < r < |x_0|$. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly to a continuous function f on $[-r, r]$.

2. Let f be the function obtained in Problem 1. Show that f is integrable on $[-r, r]$ and

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

3. Let f be the function obtained in Problem 1. Show that f is differentiable on $[-r, r]$ and

$$\int_0^x f(t)\,dt = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

4. Let $\{a_n\}$ be a sequence such that $\sum_{n=1}^{\infty} a_n$ converges. By Problem 1, $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-a, a]$ for every $0 < a < 1$.

(i). Prove Abel's Theorem: $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent in $[0, 1]$. (Hint: You can use the following fact without proof:

$$|a_m + a_{m+1} + \cdots + a_{m+k}| < \epsilon \qquad \Longrightarrow \qquad |a_m x^m + a_{m+1} x^{m+1} + \cdots + a_{m+k} x^{m+k}| < \epsilon$$

for every $x \in [0, 1]$.)

(ii). Find a sequence $\{a_n\}$ such that $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} a_n x^n$ does not converge for $x = -1$.

5. You are going to prove Bernstein's Theorem: Assume that f is a function $f : [0, r] \to \mathbb{R}$ such that $f^{(n)}(x) \geq 0$ for all $n \geq 0$ and $x \in [0, r]$; then the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converges to $f(x)$ for every $x \in [0, r)$.

(i). Set $E_n = f - T_n(f)$, where $T_n(f)$ is the Taylor polynomial of f of degree n centered at 0. Show that $0 \leq E_n(x) \leq f(x)$ for all $x \in [0, r]$.

(ii). Show that

$$\frac{E_n(x)}{x^{n+1}} = \frac{1}{n!} \int_0^1 (1-s)^n f^{(n+1)}(sx)\,ds.$$

(iii). Deduce from the formula in part (ii) that

$$\frac{E_n(x)}{x^{n+1}}$$

is a decreasing function of $x \in (0, r]$. In particular, deduce that

$$E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} E_n(r).$$

(iv). Use part (i) and (iv) to deduce that

$$E_n(x) \leq f(r) \left(\frac{x}{r}\right)^{n+1}.$$

(v). Deduce from part (iv) that $\lim_{n\to\infty} E_n(x) = 0$ for all $x \in [0, r)$.

(vi). Is the convergence of $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ to f uniform on $[0, a]$ for any $a < r$?

Notes 5: Differential Equations MAT 142, Spring 2017

1. [**Warm-up**] Assume that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is not always zero and satisfies $f'' + f = 0$.

(a). Prove that $f^2 + (f')^2$ is constant and deduce that either $f(0) \neq 0$ or $f'(0) \neq 0$.

(b). Prove that there exists a function s such that $s'' + s = 0$, $s(0) = 0$ and $s'(0) = 1$. (We will show later in the course that s is the unique such function.) (Hint: look for s of the form $s = af + bf'$ fo constants $a, b \in \mathbb{R}$.)

We can now *define* the trigonometric functions sine and cosine by $\sin x = s(x)$ and $\cos x = s'(x)$. Many of the properties of the trigonometric functions follow easily from the differential equation satisfied by s.

(c). Prove that $(\cos x)' = -\sin x$.

(d). Prove that $\sin(x + a) = \sin x \cos a + \sin a \cos x$ and $\cos(x + a) = \cos x \cos a - \sin x \sin a$. (Hint: show that $f(x) = \sin(x + a)$ and $g(x) = \sin x \cos a + \sin a \cos x$ both satisfy the IVP

$$\begin{cases} y'' + y = 0 \\ y(0) = \sin a \\ y'(0) = \cos a \end{cases}$$

Assume that this IVP has a unique solution and deduce that $f = g$. Taking derivatives now derive the identity involving $\cos(x + a)$.)

More work is required to define π and show that sin and cos are periodic functions of period 2π, that is, $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$ for all $x \in \mathbb{R}$.

(e). Show that $\cos x$ cannot be positive for all $x > 0$. (Hint: you can use the following Theorem, that you proved in the quiz at the very beginning of the course: Let f be a twice-differentiable function $f : [0, \infty) \to \mathbb{R}$ such that $f(x) > 0$ for all $x \geq 0$, f is decreasing and $f'(0) = 0$. Then there exists $x_* > 0$ such that $f''(x_*) = 0$.)

By part (e) there exists a positive number π such that $\frac{\pi}{2}$ is the smallest positive number such that $\cos x = 0$.

(f). Show that $\sin \frac{\pi}{2} = 1$.

(g). Show that $\sin x$ and $\cos x$ are periodic functions of period 2π. (Hint: Use repeatedly part (d), first to calculate $\sin 2\pi$ and $\cos 2\pi$ and then to calculate $\sin(x + 2\pi)$ and $\cos(x + 2\pi)$.)

PART I: FIRST-ORDER EQUATIONS

Reading: §§8.1–8.7 and 8.20–8.27 in [A].

2. [**Exponential growth**] Fix $k \in \mathbb{R}$. You are going to solve the differential equation $y' = ky$ and study some physical phenomena modelled by this equation.

(a). Prove that $y(x) = y_0 e^{kx}$ for some constant $y_0 \in \mathbb{R}$. (Hint: Consider the quantity $y(x)e^{-kx}$.)

(b). The decay of a radioactive substance is modelled by the differential equation $A' = kA$ where $A(t)$ is the amount of substance at time t and k is a constant that depends on the radioactive substance.

- **i.** Assuming k given, find a formula for $A(t)$ in terms of $A_0 = A(0)$.
- **ii.** Exercise 1 in §8.7 on [A].
- **iii.** Show that there exists $\tau > 0$ (called the half-life of the substance) such that $A(t+\tau) = \frac{1}{2}A(\tau)$ for all $t \in \mathbb{R}$.
- **iv.** Exercise 3 in §8.7 of [A].

(c). Newton's Law of Cooling states that the temperature of an object decreases at a rate proportional to the difference of its temperature and the ambient temperature.

- **i.** Find a formula for the temperature $T(t)$ of the object at time t in terms of the temperature T_0 at time $t = 0$ assuming that the ambient temperature T_a is kept at a fixed constant. (Hint: Note that since T_a is a constant $T' = (T - T_a)'$.)
- **ii.** Exercise 7 in §8.7 of [A].

3. [**Linear first-order equations**] A linear first-order differential equation is a differential equation of the form

$$y' + p(x)\, y = q(x)$$

where p, q are given functions. We usually try to solve the IVP

$$y' + p(x)\, y = q(x), \qquad y(x_0) = y_0. \tag{1}$$

(a). Consider first the case $q = 0$. Assume that p is a continuous function on an open interval I such that $x_0 \in I$ and fix a constant $y_0 \in \mathbb{R}$. Prove that the solution y of the IVP (1) is $y(x) = y_0\, e^{-P(x)}$, where $P(x) = \int_{x_0}^{x} p(t)\, dt$ is the (unique) antiderivative of p that vanishes at x_0. (Hint: Consider the quantity $y(x)\, e^{P(x)}$.)

(b). More in general, assume that p, q are continuous functions on an open interval I that contains x_0. Show that the unique solution of the IVP (1) is

$$y(x) = e^{-P(x)} \left(y_0 + \int_{x_0}^{x} e^{-P(t)} q(t)\, dt \right),$$

where P is defined in part (a). (Hint: Consider the quantity $y(x)\, e^{P(x)}$.)

(c). Exercises 1–12 in §8.5 of [A].

4. [Separation of variables and other tricks] There is no general formula to solve non-linear first-order equations. However, in some special cases there exists tricks to reduce the solution of the equation to the Fundamental Theorem of Calculus or to a linear equation.

(a). (Separation of variables, see §8.23 of [A].) Let a be a continuous function defined on an open interval containing y_0 and q a continuous function defined on an open interval containing the point x_0. Assume that the IVP

$$a(y)y' = q(x), \qquad y(x_0) = y_0$$

has a unique solution y. Show that y is defined implicitly by

$$\int_{y_0}^{y(x)} a(s)\,ds = \int_{x_0}^{x} q(t)\,dt.$$

(b). Exercises 1–11 in §8.24 of [A]. Write the solution with arbitrary initial condition $y(x_0) = y_0$, for constants x_0, y_0 such that the hypotheses of part (a) are satisfied.

(c). The Bernoulli Equation: exercises 13–18 in §8.5 of [A].

(d). The Riccati Equation: exercises 19-20 in §8.5 of [A].

5. [Application: population growth] Exercises 13–18 in §8.7 of [A].

6. [Existence and Uniqueness of solutions to first-order differential equations]

(a). Consider the IVP

$$y' = y^2, \qquad y(0) = 1.$$

Find a solution y by separation of variables and show that $\lim_{x\to 1} y(x) = \infty$.

(b). Consider the IVP

$$y' = y^{\frac{2}{3}}, \qquad y(0) = 0.$$

Show that $y(x) = 0$ and $y(x) = \frac{x^3}{27}$ are two distinct solutions of this IVP.

(c). Uniqueness: exercises 26 and 27 in Chapter 5 of [R] (see Review Sheet 2).

(d). Existence: exercise 25 in Chapter 7 of [R] (see Review Sheet 2).

(e). Here's an alternative proof of Existence and Uniqueness of solutions to first-order differential equations. The procedure of proof is called Picard Iteration. Let $\phi : R \to \mathbb{R}$ be a function defined on a rectangle $R = [a, b] \times [\alpha, \beta] \subset \mathbb{R}^2$. Assume that ϕ is continuous on R and moreover there exists $A > 0$ such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

for all $(x, y_1), (x, y_2) \in R$. Fix $x_0 \in (a, b)$ and $y_0 \in (\alpha, \beta)$ consider the IVP

$$y' = \phi(x, y), \qquad y(x_0) = y_0. \tag{2}$$

We are going to prove that this IVP has a unique solution provided $b - a$ is sufficiently small using ideas related to the Contraction Mapping Theorem, which was our main tool to prove the convergence of Newton's Method.

i. Show that y is a solution of the IVP if and only if $y = y_0 + \int_{x_0}^{x} \phi\big(t, y(t)\big)\,dt$.

Let $C([a, b])$ be the space of continuous real-valued functions on the interval $[a, b]$. Recall that we can define a norm on $C([a, b])$ by

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Define a "function" $T : C([a, b]) \to C([a, b])$ by

$$T(f) = y_0 + \int_{x_0}^{x} \phi\big(t, f(t)\big)\,dt.$$

By part i we have to show that T has a unique fixed point.

ii. Prove that that

$$\|T(f) - T(g)\| \le A(b - a)\|f - g\|$$

for every $f, g \in C([a, b])$. In particular, by considering a smaller interval we can assume that $b-a$ is small enough so that T is a contraction: there exists $0 < c < 1$ such that

$$\|T(f) - T(g)\| \le c\|f - g\|$$

for every $f, g \in C([a, b])$.

iii. Deduce that the IVP (2) has at most one solution.

iv. Consider the sequence of functions $y_n \in C([a, b])$ defined by

$$y_1 = 0, \qquad y_{n+1} = T(y_n).$$

Prove that $\{y_n\}$ is a Cauchy sequence and that y_n converges uniformly to a solution of the IVP (2).

7. **[Integral curves]** Let $y : [a, b] \to \mathbb{R}$ be a solution of the differential equation $y' = \phi(x, y)$. We can consider the graph of y as a curve in $\mathbb{R}^2$. In fact we can think of the differential equation as describing a family of curves, called *integral curves* of the differential equation, by prescribing their slopes. If ϕ satisfies the conditions in our Existence and Uniqueness Theorems, then for every point in $\mathbb{R}^2$ there exists a unique integral curve of the differential equation passing through that point.

(a). Exercises 1–12 in §8.22 of [A].

(b). Exercises 1–11 in §8.26 of [A].

(c). For the examples of part (a) try to study the orthogonal trajectories to the given family of curves.

PART II: SECOND-ORDER EQUATIONS

Reading: §§8.8–8.19 in [A].

8. [**Uniqueness of solutions to** $y'' + b\,y = 0$] Fix $b \in \mathbb{R}$ and consider the IVP

$$\begin{cases} y'' + b\,y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = z_0 \end{cases} \tag{3}$$

You are going to prove that this IVP has a unique solution, in two different ways.

(a). Show that the IVP (3) has a unique solution if the IVP

$$\begin{cases} y'' + b\,y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \tag{4}$$

has the unique solution $y = 0$.

(b). The first way of proving uniqueness uses Taylor polynomials.

i. Let y be a solution to the IVP (4). Prove that

$$y^{(2n)}(x) = (-1)^n b^n y(x), \qquad y^{(2n+1)}(x) = (-1)^n b^n y'(x).$$

ii. Deduce from i. that the Taylor polynomial of y at 0 of degree $2n-1$ is 0 and therefore $y(x) = E_{2n-1}(x)$.

iii. Show that for every $c > 0$ there exists a constant $M \geq 0$ such that

$$|y^{(2n)}(x)| \leq M|b|^n$$

for all $x \in [-c, c]$. (Hint: set $M = \max_{[-c,c]} |y(x)|$, which exists since y is continuous.)

iv. By choosing n sufficiently large, show that $|y(x)| < \varepsilon$ on $[-c, c]$ for every $\varepsilon > 0$ and deduce that $y = 0$.

(c). This proof is more elementary but more "clever".

i. Prove that $b\,y^2 + (y')^2 = 0$.

ii. If $b \geq 0$ deduce immediately from part i. that $y = 0$.

iii. Assume now that $b = -k^2 < 0$. Suppose that $y(x) \neq 0$ for all $x \in [\alpha, \beta]$. Use part i. to prove that there exists $C \in \mathbb{R}$ such that either $y(x) = Ce^{kx}$ or $y(x) = Ce^{-kx}$ for all $x \in [\alpha, \beta]$.

iv. Assume that $y(x_*) \neq 0$ for some $x_* \neq 0$. Use the continuity of y to show that there exists a point a with $0 \leq |a| < |x_*|$ such that $y(a) = 0$ but $y(x) \neq 0$ on the whole open interval joining a and x_*. Use this fact and part iii. to prove that $y = 0$.

9. [Homogeneous linear second-order equations with constant coefficients] We can now study existence of solutions to *homogeneous linear second-order constant-coefficients* equations, that is, differential equations of the form

$$y'' + a\,y' + b\,y = 0$$

for constants $a, b \in \mathbb{R}$.

(a). Fix $b \in \mathbb{R}$. Find *all* solutions to the equation

$$y'' + b\,y = 0$$

(Hint: Treat separately the case $b = 0$, $b = k^2 > 0$ and $b = -k^2 < 0$.)

(b). Fix $a, b \in \mathbb{R}$ and consider the equation

$$y'' + a\,y' + b\,y = 0$$

i. Show that y satisfies $y'' + a\,y' + b\,y = 0$ if and only if $u(x) = e^{\frac{a}{2}x}y(x)$ satisfies

$$u'' + \frac{4b - a^2}{4}u = 0$$

ii. Combine parts i. and (a) to find all solutions to $y'' + a\,y' + b\,y = 0$.

iii. Deduce that the IVP

$$\begin{cases} y'' + a\,y' + b\,y = 0 \\ y(x_0) = y_0 \\ y'(x_0) = z_0 \end{cases}$$

has a unique solution.

(c). Exercises 1–16, 18, 20 in §8.14 of [A].

10. [Inhomogeneous linear second-order constant-coefficients equations] Fix $a, b \in \mathbb{R}$. For a twice-differentiable function y write $L(y) = y'' + a\,y' + b\,y$. L is called a *differential operator*.

(a). Show that $L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2)$ for every pair of twice-differentiable functions y_1, y_2 and constants c_1, c_2. This is why we say that L is a *linear* differential operator.

(b). Let $R : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that y_* is a solution of the differential equation $L(y_*) = R$. Show that all solutions of the differential equation $L(y) = R$ are of the form $y = y_* + y_h$, where y_h is a solution of the homogeneous equation $L(y_h) = 0$.

(c). By Problem 9, $y_h = c_1y_1 + c_2y_2$, where $c_1, c_2 \in \mathbb{R}$ and y_1, y_2 are two solutions of $L(y) = 0$ such that y_1/y_2 is not constant. The *Wronskian* of y_1 and y_2 is the function

$$W(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x).$$

Exercises 21–23 in §8.14 of [A] establish properties of W.

(d). Fix $x_0 \in \mathbb{R}$ and define

$$c_1(x) = -\int_{x_0}^{x} y_2(t)\,\frac{R(t)}{W(t)}dt, \qquad c_2(x) = \int_{x_0}^{x} y_1(t)\,\frac{R(t)}{W(t)}dt.$$

Set $y_*(x) = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$ and show that $L(y_*) = R$. (This method to obtain a particular solution of the equation $L(y) = R$ is called *variation of parameters*.)

(e). Exercises 1–25 in §8.17 of [A]. (Hints: 1. If $b = 0$ then the solution is found more quickly by applying the Fundamental Theorem of Calculus twice since $y'' + a\,y' = (y' + a\,y)'$. 2. When $R(x)$ is a polynomial of degree d and $b \neq 0$ then one can quickly find a particular solution y_* of the equation $L(y) = R$ guessing that y_* must be another polynomial of degree d. 3. If $e^{-mx}R(x)$ is a polynomial of degree d the we can look for a particular solution of the form $y_* = e^{mx}\times$ a polynomial of degree d.)

11. [Application: simple harmonic motion (Example 1 in §8.18 of [A])] Exercises 1–7 in §8.19 of [A].

12. [BVP vs. IVP] We saw above that the IVP problem for second-order linear constant coefficient equations always has a unique solution. One is also interested in studying *boundary value problems* (BVP) instead of IVPs: fix an interval $[x_1, x_2] \subset \mathbb{R}$, constants $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and try to find all twice-differentiable functions y such that

$$\begin{cases} y'' + a\,y' + b\,y = 0 \\ \alpha_1 y(x_1) + \beta_1 y'(x_1) = 0 \\ \alpha_2 y(x_2) + \beta_2 y'(x_2) = 0 \end{cases}$$

In contrast to the IVP, in general a solution to the BVP does not exists. Do exercises 17 and 19 in §8.14 of [A] for some examples of this.

13. [Linear constant-coefficients homogeneous equations] Fix constants $a_0, \ldots, a_{n-1}$ and consider a degree–n linear constant-coefficients equation

$$y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_1\,y' + a_0\,y = 0. \tag{5}$$

In this problem you will get a glimpse of how the theory we have developed for second-order equations generalises to higher-order equations.

(a). Suppose that $\alpha \in \mathbb{C}$ is a root of the polynomial

$$x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0 = 0. \tag{6}$$

Show that $y(x) = e^{\alpha x}$ is a (complex-valued) solution of the differential equation (5).

(b). Write $\alpha = a + ib$ and deduce that $y(x) = e^{ax}\cos(bx)$ and $y(x) = e^{ax}\sin(bx)$ are two (real-valued) solutions of (5).

(c). Suppose that α is a double root of the polynomial (6). Show that $y(x) = xe^{\alpha x}$ is a second (complex-valued) solution of (5).

(d). Suppose that α is a root of the polynomial (6) of order r. Show that $y(x) = x^k e^{\alpha x}$ is a (complex-valued) solution of (5) for all $0 \le k \le r$.

(In this way one can always find n (real-valued) solutions $y_1, \ldots, y_n$ of (5) (why?) and in fact every solution of the differential equation can be written as $y = c_1\, y_1 + \cdots + c_n\, y_n$.)

14. [**Linear second-order equations**] A homogeneous linear second-order equation is an equation of the form

$$y'' + a(x)\, y'(x) + b(x)\, y = R(x),$$

where a, b, R are given continuous functions. One can prove existence and uniqueness for the IVP (see Exercises 28–29 in Chapter 5 and 26 in Chapter 7 of [R]), but there exists no general formula for writing the solution.

(a). In this problem we show that we can always reduce to the case

$$y'' + g(x)\, y = f(x).$$

Unfortunately there is no general formula for the solution to such an equation.

i. Show that every linear second-order equation can be re-written in the form

$$\big(p(x)\, y'\big)' + q(x)\, y = r(x)$$

for continuous functions p, q, r with $p(x) > 0$ for all x. (Hint: calculate the derivative of $e^{A(x)}y$ and then choose the function A appropriately; this is similar to how we dealt with linear first-order equations.)

ii. By making a change of variable $s = s(x)$ such that $s'(x) = \frac{1}{p(x)}$, reduce the previous equation further to an equation of the form

$$u'' + g(s)\, u = f(s)$$

where $y(x) = u\big(s(x)\big)$.

(b). You are going to prove a version of the Sturm Comparison Theorem: suppose that y_1 and y_2 are solutions to

$$y_1'' + g_1(x)\, y_1 = 0, \qquad y_2'' + g_2(x)\, y_2 = 0,$$

for continuous functions g_1, g_2 such that $g_2(x) > g_1(x)$. If a and b are consecutive zeroes of y_1 then y_2 must have a zero in the interval (a, b).

i. Show that $y_1''\, y_2 - y_2''\, y_1 = (g_2 - g_1)\, y_1\, y_2$.

ii. Assume that $y_1(x), y_2(x) > 0$ for all $x \in (a, b)$ and show that

$$\int_a^b y_1''(x)\, y_2(x) - y_2''(x)\, y_1(x)\, dx > 0.$$

iii. Deduce that

$$\Big(y_1'(b)\,y_2(b) - y_1'(a)\,y_2(a)\Big) - \Big(y_1(b)\,y_2'(b) - y_1(a)\,y_2'(a)\Big) > 0.$$

iv. Deduce that it is impossible that $y_1(a) = 0 = y_1(b)$. (Hint: consider the sign of $y_1'(a)$ and $y_1'(b)$.)

v. Similarly, prove that we cannot have $y_1(a) = 0 = y_1(b)$ if we assume that $y_1(x) > 0$ and $y_2(x) < 0$ for all $x \in (a,b)$, or $y_1(x) < 0$ and $y_2(x) > 0$ for all $x \in (a,b)$, or $y_1(x) < 0$ and $y_2(x) < 0$ for all $x \in (a,b)$.

(c). Let y be a solution of the equation

$$y'' + (x^2 + k^2)\,y = 0.$$

Use part (b) to show that y has infinitely many zeroes which are within $\frac{\pi}{k}$ of each other.

Review Sheet 1 MAT 142, Spring 2017

1. For vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$ we have defined:

- The dot product: $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$
- The cross product or determinant: $\mathbf{x} \times \mathbf{y} = x_1 y_2 - x_2 y_1$
- The norm: $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2}$
- The distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

We showed that

- $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta$
- $\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \sin\theta$

where θ is the angle between the two vectors.

2. We defined polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

- We studied polar equations $r = f(\theta)$, $\theta \in [a, b]$
- We showed that if f^2 is integrable then the area of the polar region R bounded by the polar equation $r = f(\theta)$, $\theta \in [a, b]$ is

$$\text{area}(R) = \frac{1}{2}\int_a^b f^2(\theta)\, d\theta$$

3. We have studied regular parametrized curves $\mathbf{c} : [a, b] \to \mathbb{R}^2$.

- We have defined the tangent vector $\mathbf{c}'(t)$ and the unit normal vector $\mathbf{n}(t)$ to $\mathbf{c}$
- We have defined the length $\ell(\mathbf{c})$ of $\mathbf{c}$ and proved that if $\mathbf{c}'$ is continuous then

$$\ell(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\|\, dt$$

- We have defined the arc length of $\mathbf{c}$:

$$s(t) = \int_{t_0}^{t} \|\mathbf{c}'(u)\|\, du,$$

where $t_0 \in [a, b]$. We said that $\mathbf{c}$ is parametrized by arc length if $\|\mathbf{c}'(t)\| = 1$ for all $t \in [a, b]$

- We have defined the curvature κ of $\mathbf{c}$: if $\mathbf{c}$ is parametrized by arc length then

$$\mathbf{c}''(t) = \kappa(t)\,\mathbf{n}(t)$$

If $\mathbf{c}$ is not necessarily parametrized by arc length, we found the formula

$$\kappa(t) = \frac{\mathbf{c}'(t) \times \mathbf{c}''(t)}{\|\mathbf{c}'(t)\|^3}$$

- We have proved that for every differentiable function $\kappa : [a, b] \to \mathbb{R}$ there exists a unique curve up to rigid motions with curvature κ

- We have defined simple closed curves and stated the Jordan Closed Theorem: every such curve encloses a bounded connected region $\mathrm{int}(\mathbf{c})$ of the plane

- We have shown the Green's Formula for area for convex simple closed curves with period T and with continuous derivative:

$$\mathrm{area}\big(\mathrm{int}(\mathbf{c})\big) = \frac{1}{2}\int_0^T \mathbf{c}(t) \times \mathbf{c}'(t)\,dt$$

- We have proved the Isoperimetric Inequality: for every simple closed curve with continuous $\mathbf{c}'$ we have

$$\mathrm{area}\big(\mathrm{int}(\mathbf{c})\big) \leq \frac{1}{4\pi^2}\ell(\mathbf{c})^2$$

Moreover, equality holds if and only if $\mathbf{c}$ is a circle.

4. We have studied sequences of real numbers.

- We proved that every monotonic sequence is convergent

- We defined the notion of a subsequence and proved the Bolzano–Weiestrass Theorem: every bounded sequence has a convergent subsequence

- We proved the Cauchy Criterion: a sequence converges if and only if it is a Cauchy sequence