MAT 132: Calculus II

Fall 2008

Department of Mathematics
SUNY at Stony Brook

Welcome to Calculus II!

Calculus is the mathematics of change. It is not a mere abstraction, but represents how the world actually works. Applications are found wherever change and continuity are studied in a precise way, and is vital to most areas of engineering and the technical sciences, but is also important in many areas of economics, business, architecture, and even occasionaly in art and music. Seemingly more remote sciences as anthropology or botany use calculus and other forms of mathematics (statistics, etc) in subsidiary but important ways.

Course Content

We will study three main topics: integration (Ch 5, Ch 6), differential equations (Ch 7), and infinite series (Ch 8). The ability to actually apply the techniques of calculus is vitally important, so we will emphasize examples and applications.

Warning! This is a 4 credit course, and we will move quickly through a large amount of material. There is a lot of homework, but because the course is fast-paced and difficult, it is absolutely vital to keep up with it.

Here is a link to the syllabus.
Here is a link to the homework assignments.
Here is a link to Brian's lecture notes.

Here is a link to Test I information.
Here is a link to Test II information.

Here is a link to an Extra Credit assignment, due week of Dec 8 (more typos fixed)
Here is a link to another Extra Credit Assignment, due any time before the Final

Here is a link to the Department Notes on second order differential equations

**Brian's Review Session: Javitz 103, at 1pm Tuesday Dec 16th
**Some Extra Office Hours this week:
Brian will be in the MLC from 1-2 on Monday
Brian's Extra office hours: Monday 2-4, Tuesday 11-1, Wednesday 3-5
***Final Exam Rooms:

Brian's Lecture (Rec 06-10) Old Chem 116
Thomas' Lecture (Rec 01-05) Harriman 137

The important stuff:

Textbook: *Single Variable Calculus* by James Stewart, Stony Brook Edition, 3ed

One homework assugnment will be due each week (exceptions being test weeks). Assignments are due at the beginning of your second recitation of the week.

Homeworks: 20% of total grade
Exam I: 20% of total grade
Exam II: 20% of total grade
Final Exam: 40% of total grade

Instructors: (click on the name for more information)

Brian Weber, MWF 9:35am Lecture
Jan Gutt, TuTh 2:20 Recitation
Evan Wright, MW 11:45,3:50 Recitations
Ye Sle Cha, TuTh 8:20 Recitation
Michael Williams, TuTh 11:20 Recitation

Thomas Poole, MW 5:20pm Lecture
Jiansong Chen, MW 11:45, 6:50 Recitations
Joseph Walsh, TuTh 5:20 Recitation
Prachi Bemalkhedkar, MF 12:50 Recitation
Frank Palladino, MW 3:50 Recitation

Americans with Disabilities Act

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. **They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students who requiring assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.www.ehs.stonybrook.edu/fire/disabilities.asp**

Syllabus for Math 132, Calculus II

Fall 2008

Instructors (all email addresses are @math.sunysb.edu)

Brian Weber (Lecturer) brweber	Thomas Poole (Lecturer) tpoole	Yi Zhu yzhu	Joseph Walsh jwalsh
Jiansong Chen jschen	Frank Palladino fpalladino	Inyoung Kim inkim	Brandon Williams mbw

Course Text

Single Variable Calculus, 3ed Stony Brook University Edition, by James Stewart

Prerequisites

Official prerequisites are a grade of C or higher in one of MAT 131, MAT 141, or AMS 151, or else a level 7 or higher on the mathematics placement exam. Unofficially, what you need is a firm knowledge of derivatives and of basic algebra.

Exams

Any necessary special formulas will be provided on the exam, and the problems will be designed so that calculators won't be necessary. Thus all you'll need is your brain and a pencil. No notes, books, cheatsheets or calculators will be allowed.

Midterm 1: Tuesday Oct 14, 8:30PM (20% of grade)
Midterm 2: Thursday Nov 6, 8:30PM (20% of grade)
Final: Thursday Dec 18, 2-4:30PM (40% of grade)

Homework (20% of grade)

One problem set will be due each week. The problems will be turned in at the *beginning* of class of your second weekly recitation section. Exam questions will be modelled on homework questions, so doing and understanding the homework is the best way to prepare for the tests.

This is a 4 credit course, and as a fair warning, you will have to work hard to be successful. If you fall seriously behind on the homework, you will not be able to keep up in class and will not be prepared for the exams. You are encouraged to work in groups, but you must write up your own solutions.

You must always show your work. No credit will be given for correct answers without correct work, on either exams or homeworks. No exceptions.

Makeup policy

All of your responsibilities for this class have been announced well ahead of time, namely in the first week of classes. Thus almost no requests for makeup homeworks or exams will be granted. The only exceptions, assuming evidence is provided, will be for serious illness, family emergency, or an unforeseeable catastrophe (tornado, car wreck, etc).

Grading policy

The grading will be curved. This means your letter grade will be influenced by your performance relative to the rest of the class. An approximate curve will be made after each exam. The final curve, by which your course grade will be determined, will be set using the same process used for the individual exams, so the individual exam curves should be a good measure of how well you are doing.

Academic Integrity Each student must pursue his or her academic goals honestly and be personally accountable for all submitted work. Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/uaa/academicjudiciary/.
Course Withdrawals The academic calendar, published in the Undergraduate Class Schedule, lists various dates that students must follow. Permission for a student to withdraw from a course after the deadline may be granted only by the Arts and Sciences Committee on Academic Standing and Appeals or the Engineering and Applied Sciences Committee on Academic Standing. The same is true of withdrawals that will result in an underload. A note from the instructor is not sufficient to secure a withdrawal from a course without regard to deadlines and underloads.

Homework assignments for Math 132, Calculus II

Fall 2008

Material Presented Week of	Chapter	Problems	Problems Due Week of
9/1	5.3 5.4	1, 2, 4, 6, 8, 9, 12, 18, 28, 47, 49 2, 6, 7, 8, 11, 12, 13	9/8
9/8	5.5 5.6	1, 7, 8, 10, 22, 30, 41, 42, 46, 50, 64 1, 3, 4, 5, 6, 10, 16, 18, 19, 25, 26	9/15
9/15	5.7	1, 2, 3, 6, 10, 13, 14, 17, 18, 20, 22, 25, 26, 28	9/22
9/22	5.9 5.10	1, 8, 15 3, 5, 6, 8, 10, 14, 32, 41, 42, 44	9/29
9/29	6.1 6.2 6.3	1, 3, 5, 7, 8, 14, 25 1, 2, 3, 4, 5, 13, 34, 35 3, 4, 5, 6, 7, 9	10/6
10/6	6.4 6.5	1, 2, 4, 5, 10, 18 1, 2, 3, 6, 9, 10, 12, 14, 17, 18	10/20
10/13 *Exam I on 10/14*	6.6 7.1	1, 2, 4, 10 1, 4, 7, 10, 13	10/20
10/20	7.2 7.3	3, 4, 5, 6, 9, 10 1, 2, 4, 6, 8, 11, 12	10/27
10/27	7.4 7.5	2, 3, 8, 10, 13 1, 3, 8	11/10
11/3 *Exam II on 11/6*	Dept Notes	1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14	11/10
11/10	8.1 8.2	3, 4, 10, 11, 14, 23, 24, 28, 29, 33 9, 13, 14, 19, 20, 22, 23, 55(b)	11/17
11/17	8.3 8.4	1, 3, 4, 6, 7, 14, 16, 22, 24 3, 4, 5, 6, 7, 8, 20, 21, 22	11/24
11/24	8.5	3, 4, 7, 9, 11, 12, 18, 21	12/1
12/1	8.6 8.7	4, 5, 12, 16, 22, 24 4, 5, 8, 9, 11, 12, 16, 20, 23, 24, 34, 36	12/8
12/8		*Review for the Final*	

Lecture notes are intended as a suppliment to the lectures, recitations, and the book.
These notes not necessarily complete and are not a substitute for attending lecture.

1 Lecture 1 - FTC II

Calculus is the mathematics of change. It is divided into two branches, differential calculus and integral calculus, which interact strongly with each other.

1.1 Derivatives

Derivatives measure instantaneous rates of change.

Given a function $y = f(x)$, we can measure the discrete change in the y-value, denoted $\triangle y$, when a discrete change, $\triangle x$, occurs in the x-value:

$$\text{Average rate of change of } f(x) \text{ when } x \text{ changes by the amount } \triangle x \quad = \quad \frac{\triangle y}{\triangle x}.$$

If we make the discrete change $\triangle x$ smaller and smaller, thereby measuring the change of the function $f(x)$ over smaller and smaller intervals, in the limit we get the instantaneous rate of change

$$\text{Instantaneous rate of change} \quad = \quad \frac{dy}{dx} \quad = \quad \lim_{\triangle x \to 0}.$$

Here "dx" and "dy" indicate *infinitesimal*, as opposed to discrete, changes in the variables x and y [1]. If $y = f(x)$ is a function, the symbols

$$f'(x) \qquad \frac{df}{dx} \qquad \frac{d}{dx}(f(x)) \qquad \frac{dy}{dx}$$

all mean precisely the same thing: the derivative of f with respect to x. In class, on tests, and in homeworks, all of these notations will be used.

You will be required to know the following basic differentiation rules:

[1] the term "infinitesimal" is not mathematically precise, but we shall not deal with this in this here.

Power Rule:

$$\frac{d}{dx}\left(x^n\right) \;=\; n\,x^{n-1}$$

Exponential Rule:

$$\frac{d}{dx}\left(e^{kx}\right) \;=\; k\,e^{kx}$$

Logarithm Rule:

$$\frac{d}{dx}\left(\ln(x)\right) \;=\; \frac{1}{x}$$

Trig rules:

$$\tfrac{d}{dx}\left(\sin(x)\right) \;=\; \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) \;=\; -\sin(x)$$

$$\tfrac{d}{dx}\left(\tan(x)\right) \;=\; \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) \;=\; -\csc^2(x)$$

$$\tfrac{d}{dx}\left(\sec(x)\right) \;=\; \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) \;=\; -\csc(x)\cot(x)$$

Constant multiple rule:

$$\frac{d}{dx}\left(a\,f(x)\right) \;=\; a\,\frac{d}{dx}\left(f(x)\right)$$

Sum/difference rule:

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) \;=\; \frac{d}{dx}\left(f(x)\right) \,\pm\, \frac{d}{dx}\left(g(x)\right)$$

Product rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) \;=\; \frac{d}{dx}\left(f(x)\right)\cdot g(x) \,+\, f(x)\cdot\frac{d}{dx}\left(g(x)\right)$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \;=\; \frac{\tfrac{d}{dx}\left(f(x)\right)\cdot g(x) \,-\, f(x)\tfrac{d}{dx}\left(g(x)\right)}{\left(g(x)\right)^2}$$

Chain Rule:

$$\frac{d}{dx}\left(f(g(x))\right) \;=\; f'(g(x))\cdot g'(x)$$

1.2 Integrals

Integrals measure total accumulated change.
In terms of graphs of functions, this equates to the (signed) area under a curve.

Given a function $y = f(x)$, one can *approximate* the area under its graph, say between $x = a$ and $x = b$, by breaking the graph into uniformly-spaces rectangles of width $\triangle x$. If you use n many rectangles and x_i is a point in the i^{th} rectangle, then the height of the rectangle should be $f(x_i)$, and

$$\begin{aligned}\text{Area of the } i^{th} \text{ rectangle} &= \textit{height} \times \textit{length} = f(x_i) \cdot \triangle x \\ \text{Sum of the rectangles}' \text{ areas} &= \sum_{i=1}^{n} f(x_i) \cdot \triangle x\end{aligned}$$

But using rectangles with discrete length $\triangle x$ will give just an approximation. To make the approximation better, you make $\triangle x$ smaller and smaller: in the limit, the discrete width $\triangle x$ become the infinitesimal width dx, and the discrete sum $\sum_{i=1}^{n}$ becomes the continuous sum $\int_a^b$.

$$\sum_{i=1}^{n} f(x_i)\, \triangle x \qquad \text{becomes} \qquad \int_a^b f(x)\, dx.$$

The symbol $\int_a^b f(x)dx$ literally means "the (signed) area under the graph of $y = f(x)$ between $x = a$ and $x = b$". Theoretically, it is obtained by summing up the infinitesimal areas of the infinitely many rectangles that live under the graph.[2]

[2]Again, this is mathematically imprecise. For the purposes of this class however, we consider this to be a minor issue.

1.3 The Fundamental Theorem of Calculus

If $y = f(x)$ is a function, we usually denote the derivative by

$$f'(x) \qquad\qquad \frac{df}{dx} \qquad\qquad \text{etc.}$$

and we denote an *antiderivative* of $f(x)$ by using capitals:

$$F(x).$$

Remarkably, derivatives (rates of change) and integrals (areas under graphs, or total accumulated change) are actually related to each other. This is the **Fundamental Theorem of Calculus**, version II of which we state here:

Theorem 1.1 (Fundamental Theorem of Calculus, version II) *If the antiderivative of $f(x)$ is $F(x)$, then*

$$\int_a^b f(x)\,dx \;=\; F(b) \;-\; F(a).$$

□

Because of the FTC, it is as important to know the rules for antiderivatives as it is for derivatives:

$$\begin{array}{rlcl}
\text{Power Rule for } n \neq 1: & f(x) \;=\; x^n & \Longrightarrow & F(x) \;=\; \frac{1}{n+1}\,x^{n+1} \\
\text{Power Rule for } n = 1: & f(x) \;=\; x^{-1} & \Longrightarrow & F(x) \;=\; \ln(x) \\
\text{Exponential Rule:} & f(x) \;=\; e^{kx} & \Longrightarrow & F(x) \;=\; \frac{1}{k}\,e^{kx} \\
\text{Trig rules:} & f(x) \;=\; \sin(x) & \Longrightarrow & F(x) \;=\; -\cos(x) \\
 & f(x) = \cos(x) & \Longrightarrow & F(x) \;=\; \sin(x)
\end{array}$$

There is no product rule or quotient rule for integrals!

1 Lecture 2 - FTC I

1.1 The Chain Rule

We begin with a review of old material. Everyone should be familiar with the chain rule:

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

It is equally important to understand the chain rule in differential notation:

substitution : $u = g(x)$

$$\begin{aligned} \frac{d}{dx}(f(g(x))) &= \frac{d}{dx}(f(u)) \\ &= \frac{du}{dx}\frac{d}{du}(f(u)). \end{aligned}$$

Often, this version of the chain rule is written in abbreviated form $\frac{d}{dx} = \frac{du}{dx}\frac{d}{du}$.

Example 1: Evaluate $\frac{d}{dx}\sqrt{x^2+1}$.
Solution: Use the substitution $u = x^2 + 1$ to write

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}\sqrt{u}$$

But there is a problem! The function is $\sqrt{u}$, with variable u, but the derivative $\frac{d}{dx}$ is with respect to x, not u!
To rectify this, use the **chain rule** $\frac{d}{dx} = \frac{du}{dx}\frac{d}{du}$, to get

$$\begin{aligned} \frac{d}{dx}\sqrt{x^2+1} &= \frac{d}{dx}\sqrt{u} \\ &= \frac{du}{dx} \cdot \frac{d}{du}\sqrt{u} \\ &= 2x \cdot \frac{1}{2}u^{-1/2} \\ &= x \cdot \left(x^2+1\right)^{-1/2}. \end{aligned}$$

Example 2: Evaluate $\frac{d}{dx} \sin\left(x^4 + x\right)$.
Solution: use $u = x^4 + x$ and the chain rule $\frac{d}{dx} = \frac{du}{dx}\frac{d}{dx}$ to get

$$\begin{aligned}
\frac{d}{dx} \sin\left(x^4 + x\right) &= \frac{d}{dx} \sin(u) \\
&= \frac{du}{dx} \cdot \frac{d}{du} (\sin(u)) \\
&= \frac{d}{dx}\left(x^4 + x\right) \cdot \frac{d}{du} (\sin(u)) \\
&= \left(3x^3 + 1\right) \cdot \cos(u) \\
&= \left(3x^2 + 1\right) \cdot \cos\left(x^4 + x\right).
\end{aligned}$$

1.2 FTC I

The first version of the fundamental theorem of calculus states explicitly that the derivative is the inverse of the integral

Theorem 1.1 (Fundamental Theorem of Calculus, version I)

$$\frac{d}{dx}\int_a^x f(u)\,du \quad = \quad f(x).$$

□

Example 3. Evaluate

$$\frac{d}{dx}\int_{-1}^x u^4\,du$$

in two ways: a) by directly evaluating and b) by using the fundamental theorem.
a) Direct evaluation:

$$\begin{aligned}
\frac{d}{dx}\left(\int_{-1}^x u^4\,du\right) &= \frac{d}{dx}\left(\frac{1}{5}u^5\Big|_{-1}^x\right) \\
&= \frac{d}{dx}\left(\frac{1}{5}x^5 - \frac{1}{5}(-1)^5\right) \\
&= \frac{d}{dx}\left(\frac{1}{5}x^5\right) \\
&= x^4
\end{aligned}$$

b) Fundamental theorem: there is no work involved! This problem fits the pattern of the fundamental theorem exactly:

$$\frac{d}{dx}\int_{-1}^x u^4\,du \quad = \quad x^4.$$

Example 4. Use the fundamental theorem to evaluate

$$\frac{d}{dx}\int_2^{x^2} (u^2+1)\,du$$

Solution: This problem DOES NOT directly fit the pattern of the fundamental theorem. We have to use a substitution

$$v = x^2$$
$$\frac{d}{dx}\int_2^v (u^2+1)\,du.$$

Now our variable is v, but the derivative is with respect to x. Thus we use the chain rule: $\frac{d}{dx} = \frac{dv}{dx}\frac{d}{dv}$ to get

$$\begin{aligned}
\frac{d}{dx}\int_2^v (u^2+1)\,du &= \frac{dv}{dx}\cdot\frac{d}{dv}\int_2^v (u^2+1)\,du \\
&= 2x\cdot\big(v^2+1\big) \\
&= 2x\cdot\big(x^4+1\big) \;=\; 2x^5 \;+\; 2x.
\end{aligned}$$

1 Lecture 3 - Substitution in Integrals

1.1 Odds and ends

Given a function $f(x)$, its antiderivative can be denoted as either

$$F(x) \qquad \text{or} \qquad \int f(x)\,dx.$$

The antiderivative is also known as the *indefinite integral*.

An expression like

$$\int_1^2 x^3\,dx$$

involves no uncertainty whatsoever. It has a definite value: it comes to 15/16 if you work it out. However there *appears* to be a variable, namely x. But actual variables may freely take on any value, while x is constrained to move between 1 and 2. The x in this case is called an *apparent variable*; it's only there to serve as a placeholder until the integral has been evaluated.

Example of an FTC I problem Given $h(x) = \int_{e^x}^0 \sin^3(t)\,dt$, find $h'(x)$. Solution: use the substitution $u = e^x$ to get

$$\begin{aligned}
\frac{dh}{dx} &= -\frac{d}{dx}\int_0^{e^x} \sin^3(t)\,dt \\
&= -\frac{du}{dx}\frac{d}{du}\int_0^u \sin^3(t)\,dt \\
&= -e^x \sin^3(u) \\
&= -e^x \sin^3\left(e^x\right).
\end{aligned}$$

1.2 Substitution in Integrals

- The rule of thumb is to pick the substitution u to be either 1) a function inside a function, or 2) the denominator.
- You must convert all x's and dx's to u's and du's
- For indefinite integrals, switch back to x's at the end
- For definite integrals, you don't need to switch back to x's, but you do need to convert the limits.

<u>Example 1</u> (Indefinite Integral) Find the antiderivative of $f(x) = \sqrt{2x+1}$.

<u>Solution</u>

$$
\begin{aligned}
F(x) &= \int \sqrt{2x+1}\, dx \\
& \quad \text{substitute} \quad u \;=\; 2x+1 \\
& \qquad\qquad\qquad \frac{du}{dx} = 2 \qquad \frac{1}{2} du \;=\; dx \\
F(x) &= \int \sqrt{u}\, \frac{1}{2}\, du \\
&= \frac{1}{2} \int u^{\frac{1}{2}}\, du \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + C \\
&= 3\,(2x+1)^{\frac{3}{2}} + C
\end{aligned}
$$

Example 2 (Indefinite Integral) Find the antiderivative of $g(x) = x^2 \sqrt[3]{x^3+2}$.

Solution

$$
\begin{aligned}
G(x) &= \int x^2\,(x^3+2)^{\frac{1}{3}}\,dx \\
& \quad \text{substitute} \quad u \;=\; x^3+2 \\
& \qquad\qquad\quad \frac{du}{dx} \;=\; 3x^2 \qquad \frac{1}{3}du \;=\; x^2\,dx \\
G(x) &= \int u^{\frac{1}{3}}\,\frac{1}{3}\,du \\
&= \frac{1}{3}\cdot\frac{3}{4}\cdot u^{\frac{4}{3}} + C \\
&= \frac{1}{4}\left(x^3 + 2\right)^{\frac{4}{3}} + C.
\end{aligned}
$$

Example 3 (Definite Integral) Find $\int_0^1 \frac{e^x}{e^x+1}\,dx$.

Solution Use $u = e^x + 1$, so that $du = e^x\,dx$.

$$
\begin{aligned}
\int_1^2 \frac{e^x}{e^x+1}\,dx &= \int_{e+1}^{e^2+1} \frac{1}{u}\,du \\
&= \ln|u|\Big|_{e+1}^{e^2+1} = \ln(e^2+1) - \ln(e+1) \\
&= \ln\left(\frac{e^2+1}{e+1}\right).
\end{aligned}
$$

Example 4 (Definite Integral) Find $\int_0^{\frac{\pi}{4}} \frac{\sin(x)\cos(x)}{1+\cos^2(x)}dx$.

Solution Use $u = 1 + \cos^2(x)$, so that $du = -2\cos(x)\sin(x)$.

We will also have to change the limits:

- Lower limit: $x = 0$ implies $u = 2$
- Upper limit: $x = \frac{\pi}{4}$ implies $u = \frac{3}{2}$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin(x)\cos(x)}{1+\cos^2(x)}dx &= -\frac{1}{2}\int_2^{\frac{3}{2}} \frac{du}{u} \\
&= -\frac{1}{2}\int_2^{\frac{3}{2}} u^{-1}\,du \\
&= -\frac{1}{2}\,\ln|u|\Big|_2^{\frac{3}{2}} \\
&= \left(-\frac{1}{2}\,\ln\left(\frac{3}{2}\right)\right) - \left(-\frac{1}{2}\,\ln(2)\right) \\
&= -\frac{1}{2}\ln\left(\frac{4}{3}\right).
\end{aligned}
$$

1 Lecture 4 - Integration by parts

1.1 Odds and ends: the "d" operator

The notation

$$\triangle x$$

indicates a small but finite change in the variable x. The notation

$$dx$$

indicates an infinitesimal[1] change in the variable x. If $f(x)$ is a function, then the derivative of f w.r.t x, written

$$\frac{df}{dx}$$

literally indicates the infinitesimal change in f compared to the infinitesimal change in x.

But the operator "d" actually behaves like a derivative:

$$\begin{array}{lrcl} \text{given} & f(x) & = & x^2 \\ \text{then} & df(x) & = & 2x\,dx. \end{array}$$

If we divide both sides by dx, we get

$$\frac{df}{dx} = 2x,$$

which is the correct derivative. The d-operator (the infinitesimal change operator) is used very frequently in calculus.

[1]Mathematically speaking, this is imprecise. After all, what, exactly, is the mathematical meaning of "infinitesimal"? Maybe a number that is really really really really really small?? Maybe some kind of number smaller than all positive numbers but bigger that 0?? Or is and 'infinitesimal' really a number at all?? We shall sweep such questions under the rug.

1.2 Integration by parts for Indefinite integrals

The theory: Integration by parts is, roughly speaking, the product rule for integrals.

To develop integration by parts, we start with the product rule for derivatives: given arbitrary functions $v(x)$ and $u(x)$:

$$u(x), \qquad v(x)$$

Take "d" of the product $u(x)v(x)$:

$$d\ (u(x) \cdot v(x)) \ = \ du(x)\ \cdot\ v(x)\ +\ u(x) \cdot dv(x) \qquad \text{(product rule)}$$

Now take the integral, $\int$, of both sides:

$$\begin{aligned} \int d\,(u(x)\,v(x)) \ &= \ \int du(x)\ \cdot\ v(x)\ +\ \int u(x) \cdot dv(x) \\ \text{rearrange}: \int u(x)\,dv(x) \ &= \ \int d\,(u(x)\,v(x))\ -\ \int v(x)\,du(x). \end{aligned}$$

The **Fundamental Theorem of Calculus** states that the *integral of the derivative* gives the function back again: we can interpret this to mean $\int d(f(x)) = f(x)$, or in our case, that $\int d(u\,v) = u\,v$.

This gives us the **Formula for Integration by Parts:**

$$\int u\,dv \ = \ u \cdot v - \int v\,du.$$

Example 1 Find the antiderivative of $f(x) = x\sin(x)$.

Solution

We must evaluate
$$\int x \sin(x)\, dx.$$
This appears to be a product of two unlike functions, so we should use Integration by Parts:

Choose
$$\begin{aligned} u &= x \\ dv &= \sin(x). \end{aligned}$$
Then calculate
$$\begin{aligned} du &= dx \\ v &= -\cos(x). \end{aligned}$$
By the formula for integration by parts we get
$$\begin{aligned} \int u\, dv &= u \cdot v - \int v\, du \\ \int x \sin(x)\, dx &= -x\cos(x) + \int \cos(x)\, dx \\ &= -x\cos(x) + \sin(x). \end{aligned}$$

1.3 Integration by Parts for Definite integrals

The formula is nearly the same:

$$\int_a^b u\,dv \;=\; u\cdot v\Big|_a^b - \int_a^b v\,du.$$

<u>Example 2</u> Evaluate $\int_1^2 x\,e^x\,dx$.

<u>Solution</u>
This is a product of unlike functions, so we should use integration by parts. Choose

$$\begin{aligned} u &= x \\ dv &= e^x\,dx. \end{aligned}$$

The calculate

$$\begin{aligned} du &= dx \\ v &= e^x \end{aligned}$$

Thus we get

$$\begin{aligned} \int u\,dv &= u\cdot v - \int v\,du \\ \int_1^2 x\,e^x\,dx &= x\,e^x\Big|_1^2 - \int_1^2 e^x\,dx \\ &= x\,e^x\Big|_1^2 - \int_1^2 e^x\,dx \\ &= x\,e^x\Big|_1^2 - e^x\Big|_1^2 \\ &= 2\cdot e^2 - e^1 - e^2 + e^1 \\ &= e^2. \end{aligned}$$

1 Lecture 5 - Examples of Integration by parts and substitution

Many techniques for integration exist, some harder than others.

When figuring out which techniques to use against a particular problem, you usually try the simplest technique first, and try harder and harder techniques if they seem necessary.

For example, substitution is easier that integration by parts, so you should usually try substitution first. If substitution doesn't work, consider using integration by parts.

Example 1 Find $\int v\sqrt{1+v^2}\,dv$.

Solution
There is a product inside the integral, so integration-by-parts is tempting. However, you should try a substitution first, just because it is easier:
Use $u = 1+v^2$, $du = 2vdv$ to get

$$\begin{aligned}\int v\sqrt{1+v^2}\,dv &= \int v\left(1+v^2\right)^{\frac{1}{2}}\,dv \\ &= \frac{1}{2}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}}\cdot u^{-\frac{1}{2}} + C \\ &= -\left(1+v^2\right)^{-\frac{1}{2}} + C\end{aligned}$$

Example 2 Find $\int y^5 \ln(y)\,dy$.

Solution
Substitution will not work. For integration by parts, let u be something that gets *simpler* when you differentiation it. Thus we choose

$$\begin{aligned} u &= \ln(y) \\ dv &= y^5\,dy \end{aligned}$$

and we calculate

$$\begin{aligned} du &= y^{-1}\,dy \\ v &= \frac{1}{6}y^6 \end{aligned}$$

To get

$$\begin{aligned} \int u\,dv &= u\,v - \int v\,du \\ \int \ln(y)\,y^5\,dy &= \frac{1}{6}\,y^6\,\ln(y) - \frac{1}{6}\int y^6\,y^{-1}\,dy \\ &= \frac{1}{6}\,y^6\,\ln(y) - \frac{1}{6}\int y^5\,dy \\ &= \frac{1}{6}\,y^6\,\ln(y) - \frac{1}{36}y^6 + C. \end{aligned}$$

Example 3 Find $\int x^2\, e^{-x}\, dx$.

Solution

This is another integration by parts problem. The function e^{-x} does not get simpler when you differentiate, however, the function x^2 *does*, so we should let

$$\begin{aligned} u &= x^2 \\ dv &= e^{-x}\, dx \end{aligned}$$

and compute

$$\begin{aligned} du &= 2x\, dx \\ v &= -e^{-x}. \end{aligned}$$

Thus

$$\begin{aligned} \int u\, dv &= u\, v - \int v\, du \\ \int x^2\, e^{-x}\, dx &= -x^2\, e^{-x} + 2 \int x\, e^{-x}\, dx \end{aligned}$$

Despite using integration by parts, we *still* have a difficult integral to evaluate. We have to use integration by parts again: choose

$$\begin{aligned} u &= x \\ dv &= e^{-x}\, dx \end{aligned}$$

and compute

$$\begin{aligned} du &= dx \\ v &= -e^{-x}. \end{aligned}$$

Thus

$$\begin{aligned} \int x^2\, e^{-x}\, dx &= -x^2\, e^{-x} + 2 \int x\, e^{-x}\, dx \\ &= -x^2\, e^{-x} + 2 \left(-x\, e^{-x} + \int e^{-x}\, dx \right) \\ &= -x^2\, e^{-x} - 2x\, e^{-x} - 2e^{-x} + C \end{aligned}$$

Example 4 Find $\int r^2\sqrt{1\,+\,r^3}\,dr$.

Solution This is a substitution:

$$
\begin{aligned}
u &= 1+r^3 \qquad\qquad du = 3r^2dr \\
\int r^2\left(1+r^3\right)^{\frac{1}{2}}\,dr &= \frac{1}{3}\int u^{\frac{1}{2}}\,du \\
&= \frac{1}{3}\cdot\frac{1}{-\frac{1}{2}}\cdot u^{-\frac{1}{2}} \\
&= -\frac{2}{3}\cdot\left(1+r^3\right)^{-\frac{1}{2}}.
\end{aligned}
$$

Example 4 Find $\int r^5 \sqrt{1\,+\,r^3}\,dr$.

Solution We use integration by parts: select

$$\begin{aligned} u &= r^3 \\ dv &= r^2\sqrt{1+r^3}. \end{aligned}$$

and compute

$$\begin{aligned} du &= 4r^2 \\ v &= -\frac{2}{3}\cdot\left(1+r^3\right)^{-\frac{1}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ \int r^5\left(1+r^3\right)^{\frac{1}{2}}\,dr &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{2}{3}\int r^2\left(1+r^3\right)^{-\frac{1}{2}}\,dr. \end{aligned}$$

The last integral on the right is *still* very difficult! But we can use the substitution

$$u = 1+r^3 \qquad\qquad du = 3r^2\,dr$$

to get

$$\begin{aligned} \int r^5\left(1+r^3\right)^{\frac{1}{2}}\,dr &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{2}{3}\left(\frac{1}{3}\int u^{-\frac{1}{2}}\,du\right) \\ &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{4}{27}u^{-\frac{3}{2}} \\ &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{4}{27}\left(1+r^3\right)^{-\frac{3}{2}} + C \end{aligned}$$

1 Lecture 6 - More Integration Techniques

In total, we will study six integration techniques:

- Substitution
- Integration by parts
- Trigonometric integration (integrating powers of sin and cos
- Trigonometric substitution (substituting a trig function into an integral that didn't ave trig functions to begin with)
- Partial fraction expansion
- Long division

We have already studied substitution and integration by parts. In this lecture we study trigonometric integration and trigonometric substitution.

Something many of you have noticed is that integration is hard, much harder than differentiation, and the techniques needed to evaluate integrals are commensurately harder.

Even still, many relatively simple integrals just cannot be evaluated, no matter what techniques are used: for instance $\int \sqrt{x^3+1}\,dx$ or $\int e^{x^2}\,dx$ can not be explicitly evaluated.

1.1 Trigonometric integration

Here we study integrals of the type

$$\int \sin^m(x)\,\cos^n(x)\,dx.$$

1.1.1 Case where either m or n (or both) is an odd number

In this case, one uses the identity

$$\sin^2(x) + \cos^2(x) = 1$$

to reduce to the case where there is either a *single* cosine or else a *single* sine. Here are some examples:

Example 1 Evaluate $\int \sin^3(x)\,dx$

Solution
Use $\sin^2(x) = 1 - \cos^2(x)$ to get

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \sin^2(x)\,\sin(x)\,dx \\ &= \int (1-\cos^2(x))\,\sin(x)\,dx\end{aligned}$$

Then use the substitution $u = \cos(x)$, $du = -\sin(x)\,dx$ to get

$$\begin{aligned}\int \sin^3(x)\,dx &= \int (1-\cos^2(x))\,\sin(x)\,dx \\ &= -\int (1-u^2)\,du \\ &= -u + \frac{1}{3}u^3 + C \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + C.\end{aligned}$$

Example 2 Evaluate $\int \cos^2(x)\,\sin^5(x)\,dx$.

Solution Since the power on the sin function is odd, we can reduce to the case where there is one single sin function, and then use a substitution:

$$\begin{aligned}
\int \cos^2(x)\,\sin^5(x)\,dx &= \int \cos^2(x)\,\left(\sin^2(x)\right)^2\,\sin(x)\,dx \\
&= \int \cos^2(x)\,\left(1-\cos^2(x)\right)^2\,\sin(x)\,dx \\
&= -\int u^2(1-u^2)^2\,du \qquad\qquad u = \cos(x) \quad du = -\sin(x) \\
&= -\int \left(u^2 - 2u^4 + u^6\right)\,du \\
&= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{6}u^6 + C \\
&= -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{6}\cos^6(x) + C.
\end{aligned}$$

Example 3 Evaluate $\int \cos^3(x)\,\sin^5\,dx$.

Solution Here both powers are odd, so we can decide whether we want to get rid of all but a single cos or all but a single sin. Let's go with getting rid of the cosines:

$$\begin{aligned}
\int \cos^3(x)\,\sin^5(x)\,dx &= \int \left(1-\sin^2(x)\right)\,\cos(x)\,\sin^5(x)\,dx \\
&= \int (1-u^2)\,u^5\,du \qquad\qquad u = \sin(x) \quad du = \cos(x)\,dx \\
&= \int (u^5 - u^7)\,du \\
&= \frac{1}{6}u^6 - \frac{1}{8}u^8 + C \\
&= \frac{1}{6}\sin^6(x) - \frac{1}{8}\sin^8(x) + C.
\end{aligned}$$

1.1.2 Case where both powers are even

Here we study what can be done with, for example, $\int \sin^4(x)\,dx$ or $\int \cos^2(x)\,\sin^6(x)\,dx$.

One must use one of two *reduction formulae*:

$$\begin{aligned}\int \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx\\ \int \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.\end{aligned}$$

Example 4 Evaluate $\int \sin^2(x)\,dx$.

Solution Use the reduction formula for sin to get

$$\begin{aligned}\int \sin^2(x)\,dx &= -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int 1\,dx\\ &= -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C.\end{aligned}$$

This can also be solved, for example, using a half-angle formula for sin.

Example 5 Evaluate $\int \sin^4(x)\,dx$.

Solution We have to use the reduction formula *twice*:

$$\begin{aligned}\int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx\\ &= -\frac{1}{4}\sin^3\cos(x) + \frac{3}{4}\left(-\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int 1\,dx\right)\\ &= -\frac{1}{4}\sin^3\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + C\end{aligned}$$

Example 6 Evaluate $\int \cos^4(x)\sin^2(x)\,dx$.

Solution First use $\sin^2(x) = 1-\cos^2(x)$ to convert entirely to cosines:

$$\begin{aligned}\int \cos^4(x)\sin^2(x)\,dx &= \int \cos^4(x)\left(1-\cos^2(x)\right)\,dx \\ &= \int \cos^4(x)\,dx - \int \cos^6(x)\,dx.\end{aligned}$$

Now we have two problems: evaluate $\int \cos^4(x)\,dx$ and evaluate $\int \cos^6(x)\,dx$. First things first: to evaluate $\int \cos^4(x)\,dx$, use the cosine reduction formula twice:

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx \\ &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left(\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int 1\,dx\right) + C \\ &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\cos(x)\sin(x) + \frac{3}{8}x + C.\end{aligned}$$

Now we evaluate $\int \cos^6(x)\,dx$ by using using the reduction formula three times:

$$\begin{aligned}\int \cos^6(x)\,dx &= \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{6}\int \cos^4(x)\,dx \\ &= \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{6}\left(\frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx\right) \\ &= \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{24}\cos^3(x)\sin(x) + \frac{15}{24}\int \cos^2(x)\,dx \\ &= \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{24}\cos^3(x)\sin(x) + \frac{15}{24}\left(\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int 1\,dx\right) \\ &= \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{24}\cos^3(x)\sin(x) + \frac{15}{48}\cos(x)\sin(x) + \frac{15}{48}x + C.\end{aligned}$$

Altogether, we get

$$\begin{aligned}&\int \cos^4(x)\sin^2(x)\,dx = \int \cos^4(x)\,dx - \int \cos^6(x)\,dx \\ &= \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\cos(x)\sin(x) + \frac{3}{8}x \\ &\quad + \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{24}\cos^3(x)\sin(x) + \frac{15}{48}\cos(x)\sin(x) + \frac{15}{48}x + C. \\ &= \frac{1}{6}\cos^5(x)\sin(x) + \frac{11}{24}\cos^3(x)\sin(x) + \frac{33}{48}\cos(x)\sin(x) + \frac{33}{48}x + C.\end{aligned}$$

1.2 Trigonometric substitution

One way of writing the main trigonometric identity is

$$1 - \sin^2(\theta) = \cos^2(\theta).$$

Some problems do not have trig functions, but have some pattern resembling the main trig identity. In these problems, we can substitute a trig function into the integral, and then use the trig identity to simplify the problem.

Example 7 Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution The $1-x^2$ resembles the trig identity $1-\sin^2(\theta) = \cos^2(\theta)$. Therefore use the substitution $x = \sin(\theta)$, $dx = \cos(\theta)\,d\theta$ to get

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta \\
&= \int \frac{1}{\sqrt{\cos^2(\theta)}}\cos(\theta)\,d\theta \\
&= \int d\theta \\
&= \theta + C \\
&= \sin^{-1}(\theta) + C.
\end{aligned}$$

Example 8 Evaluate $\int \frac{x}{1+x^2}\,dx$.

Solution We have to use the trig identity $1 + \tan^2(\theta) = \sec^2(\theta)$. Substitute $x = \tan(\theta)$, $dx = \sec^2(\theta)$ to get

$$\begin{aligned}
\int \frac{x}{1+x^2}\,dx &= \int \frac{\tan(\theta)}{1+\tan^2(\theta)}\sec^2(\theta)\,d\theta \\
&= \int \frac{\tan(\theta)}{\sec^2(\theta)}\sec^2(\theta)\,d\theta \\
&= \int \tan(\theta)\,d\theta \\
&= \ln|\cos(\theta)| + C \\
&= \ln\left|\cos\left(\tan^{-1}(x)\right)\right| + C.
\end{aligned}$$

1 Lecture 7 - More Integration Techniques: Trigonometric substitution

1.1 Odds & Ends: Derivation of the reduction formulae

The two reduction formulae for *indefinite* integrals are

$$\begin{aligned}
\int \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\,\cos(x) \,+\, \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \\
\int \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\,\sin(x) \,+\, \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.
\end{aligned}$$

The two reduction formulae for *definite* integrals are

$$\begin{aligned}
\int_a^b \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\,\cos(x)\Big|_a^b \,+\, \frac{n-1}{n}\int_a^b \sin^{n-2}(x)\,dx \\
\int_a^b \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\,\sin(x)\Big|_a^b \,+\, \frac{n-1}{n}\int_a^b \cos^{n-2}(x)\,dx.
\end{aligned}$$

We show how to derive these. This is not knowledge you need for the test, but it is good to see how it is done.

1.1.1 Derivation of the sin reduction formula

We use a clever integration by parts argument:

$$\begin{aligned}
\int \sin^n(x)\,dx &= \int \sin^{n-1}(x)\,\sin(x)\,dx \\
& \text{use} u = \sin^{n-1}(x) \qquad du = (n-1)\sin^{n-2}(x)\,\cos(x)\,dx \\
& \quad dv = \sin(x)\,dx \qquad v = -\cos(x)\,dx \\
\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) \,+\, (n-1)\int \sin^{n-2}(x)\,\cos^2(x)\,dx.
\end{aligned}$$

Now integration by parts is supposed to make things simpler, but the expression on the right does not look simpler. However, we can use some algebra to manipulate the right side of the equation. Use $\cos^2(x) = 1 - \sin^2(x)$ to get rid of the cos on the right:

$$\begin{aligned}
\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\left(1 - \sin^2(x)\right)\,dx \\
\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx.
\end{aligned}$$

On both the left side and the right side, we have a "$\int \sin^n(x)\,dx$" expression, so we can move both of them to the left side, then simplify:

$$\begin{aligned}
(n-1)\int \sin^n(x)\,dx - \int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx \\
n\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx.
\end{aligned}$$

Now divide both sides by n to get

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\,\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.$$

Thus, after our algebra tricks, the integral on the right is indeed simpler than the integral on the left, which is the integral we started with. We have arrived at the reduction formula for sin.

1.1.2 Derivation of the cos reduction formula

We use the same procedure we used for sin. First use integration by parts:

$$\begin{aligned}
\int \cos^n(x)\,dx &= \int \cos^{n-1}(x)\,\cos(x)\,dx \\
\text{use} u &= \cos^{n-1}(x) \qquad du = -(n-1)\cos^{n-2}(x)\,\sin(x)\,dx \\
dv &= \cos(x)\,dx \qquad v = \sin(x)\,dx \\
\int \sin^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x)\,\sin^2(x)\,dx.
\end{aligned}$$

Then get rid of the "$\sin^2(x)$" on the right by using $\sin^2(x) = 1 - \cos^2(x)$:

$$\begin{aligned}
\int \cos^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x)\left(1 - \cos^2(x)\right)\,dx \\
\int \cos^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x) - (n-1)\int \cos^n(x)\,dx.
\end{aligned}$$

Then add $(n-1)\int \cos^n(x)\,dx$ to both sides to get

$$\begin{aligned} n\int \cos^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x) \\ \int \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\,\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x). \end{aligned}$$

Done!

1.2 Trigonometric substitution

Guidelines for trigonometric substitution:

If you see	Consider substituting
$a^2 - x^2$	$x = a\cos(\theta)$ or $x = a\sin(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$ or $x = a\cot(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$ or $x = a\csc(\theta)$

Let's do some examples.

Example 1 Evaluate $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Solution Use $x = 2\cos(\theta)$, $dx = -2\sin(\theta)\,d\theta$ to get

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx &= \int_{\pi/3}^{\pi/6} \frac{-2\sin(\theta)}{\sqrt{4-4\cos^2(\theta)}}\,d\theta \\
&= \int_{\pi/3}^{\pi/6} \frac{-2\sin(\theta)}{\sqrt{4\sin^2(\theta)}}\,d\theta \\
&= \int_{\pi/3}^{\pi/6} \frac{-2\sin(\theta)}{2\sin(\theta)}\,d\theta \\
&= -\int_{\pi/3}^{\pi/6} d\theta \\
&= -\theta\Big|_{\pi/3}^{\pi/6} \\
&= -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}.
\end{aligned}$$

Example 2 Evaluate $\int_3^{2\sqrt{3}} \frac{1}{x\sqrt{x^2-9}}\,dx$

Solution Use $x = 3\sec(\theta)$, $dx = 3\sec(\theta)\tan(\theta)\,d\theta$ to get

$$\begin{aligned}
\int_3^{2\sqrt{3}} \frac{1}{x\sqrt{x^2\,-\,9}}\,dx &= \int_0^{\frac{\pi}{6}} \frac{3\sec(\theta)\tan(\theta)}{3\sec(\theta)\sqrt{9\sec^2(\theta)\,-\,9}}\,d\theta \\
&= \int_0^{\frac{\pi}{6}} \frac{3\sec(\theta)\tan(\theta)}{3\sec(\theta)\sqrt{9\tan^2(\theta)}}\,d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{6}} d\theta \\
&= \frac{1}{3}\theta\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{18}.
\end{aligned}$$

Example 3(a) Evaluate $\int_4^{4\sqrt{3}} \frac{x}{x^2+16}\,dx$

Solution Use $x = 4\tan(\theta)$, $dx = 4\sec^2(\theta)\,d\theta$ to get

$$\begin{aligned}
\int_4^{4\sqrt{3}} \frac{x}{\sqrt{x^2+16}}\,dx &= \int_{\pi/4}^{\pi/3} \frac{16\tan(\theta)\ \sec^2(\theta)}{16\tan^2(\theta)\,+\,16}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{16\tan(\theta)\ \sec^2(\theta)}{16\sec^2(\theta)}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \tan(\theta)\,d\theta \\
&= -\ln\big|\cos(\theta)\big|\Big|_{\pi/4}^{\pi/3} \\
&= -\ln\left(\frac{1}{2}\right) + \ln\left(\frac{1}{\sqrt{2}}\right) \\
&= -\ln\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

Example 3(b) Evaluate $\int_4^{4\sqrt{3}} \frac{x}{x^2+16}\, dx$

Solution This problem is identical to the problem from the previous example. But this time, we will use the substitution $u = x^2 + 16$, $du = 2x\, dx$ to get

$$\begin{aligned}
\int_4^{4\sqrt{3}} \frac{x}{x^2+16}\, dx &= \frac{1}{2}\int_{32}^{64} \frac{1}{u}\, du \\
&= \frac{1}{2}\ln|u|\Big|_{32}^{64} \\
&= \frac{1}{2}\ln(64) - \frac{1}{2}\ln(32) \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

This is a good time to recall the **rules of logarithms**:

$$\begin{aligned}
\ln(x) + \ln(y) &= \ln(xy) \\
\ln(x) - \ln(y) &= \ln\left(\frac{x}{y}\right) \\
a\ln(x) &= \ln(x^a)
\end{aligned}$$

1 Lecture 8 - Partial Fractions, Long Division

A **rational expression** is a quotient of polynomials, i.e. has the form

$$\frac{a_n\, x^n \ + \ a_{n-1}\, x^{n-1} \ + \ \ldots \ + \ a_1 x \ + \ a_0}{b_m\, x^m \ + \ b_{m-1}\, x^{m-1} \ + \ \ldots \ + \ b_1 x \ + \ b_0}.$$

At first sight, rational expressions are difficult to integrate. However, the algebraic techniques of partial fraction expansion and long division can be used to make these complex expressions simpler.

1.1 Partial fraction expansion

Word of Warning: Our method here applies when the denominator has linear, nonrepeated factors only. This will be sufficient for our Calc II class, but some of you will see more sophisticated methods in other classes.

Partial fractionsworks when the largest power of the numerator is *smaller* than the largest power of the denominator. Here are the instructions. Start with a rational expression:

$$\frac{a_n\, x^n \ + \ a_{n-1}\, x^{n-1} \ + \ \ldots \ + \ a_1 x \ + \ a_0}{b_m\, x^m \ + \ b_{m-1}\, x^{m-1} \ + \ \ldots \ + \ b_1 x \ + \ b_0},$$

First, completely factor the bottom polynomial:

$$\frac{a_n\, x^n \ + \ a_{n-1}\, x^{n-1} \ + \ \ldots \ + \ a_1 x \ + \ a_0}{b_m\,(x - c_m)\,(x - c_{m-1})\ldots(x - c_1)\,(c - c_0)}.$$

Second, break up the fraction so that the factors of the bottom become the *denominators* of *individual* fractions, with (as of yet) unknown *constants* in the numerators:

$$\frac{A_m}{x - c_m} \ + \ \frac{A_{m-1}}{x - c_{m-1}} \ + \ \ldots \ + \ \frac{A_1}{x - c_1} \ + \ \frac{A_0}{x - c_0}.$$

Third, figure out what the constants $A_m, \ldots, A_0$ are.

Example 1 Use partial fraction expansion to simplify $\frac{x}{x^2+5x+6}$

Solution We simply follow the instructions laid out above:
First, factor the bottom:

$$\frac{x}{x^2+5x+6} = \frac{x}{(x+3)(x+2)}$$

Second, break up the fraction

$$\frac{x}{x^2+5x+6} = \frac{A_1}{x+3} + \frac{A_0}{x+2}.$$

Third, figure out what A_1 and A_0 are:

$$\begin{aligned}\frac{x}{x^2+5x+6} &= \frac{A_1}{x+3} + \frac{A_0}{x+2} \\ &= \frac{A_1(x+2) + A_0(x+3)}{(x+3)(x+2)} \\ &= \frac{(A_1+A_0)x + 2A_1 + 3A_0}{(x+3)(x+2)}.\end{aligned}$$

Thus $x = (A_1+A_0)x + 2A_1 + 3A_0$, so that $A_1+A_0 = 1$ and $2A_1+3A_0 = 0$. We can solve these two equations, to get $A_1 = 3$, $A_0 = -2$. Thus finally

$$\frac{x}{x^2+5x+6} = \frac{3}{x+3} - \frac{2}{x+2}$$

Example 2 Evaluate

$$\int \frac{2x-1}{x^2-7x+12}\,dx$$

Solution The fraction is too tough to evaluate, so we have to use partial fraction expansion to simplify it.

$$\begin{aligned}\frac{2x-1}{x^2-7x+12} &= \frac{2x-1}{(x-4)(x-3)}\\ &= \frac{A_1}{x-4} + \frac{A_0}{x-3}\\ &= \frac{(A_1+A_0)x - 3A_1 - 4A_0}{(x-4)(x-3)}.\end{aligned}$$

Thus $A_1 + A_0 = 2$ and $-3A_1 - 4A_0 = -1$. We can solve this to get $A_1 = 7$, $A_0 = -5$. Therefore

$$\frac{2x-1}{x^2-7x+12} = \frac{7}{x-4} - \frac{5}{x-3}.$$

Now we can solve our calculus problem:

$$\begin{aligned}\int \frac{2x-1}{x^2-7x+12}\,dx &= \int \frac{7}{x-4}\,dx - \int \frac{5}{x-3}\,dx\\ &= 7\ln|x-4| - 5\ln|x-3|.\end{aligned}$$

1.2 Long Division

Long division (or synthetic division) can be used when the highest power on top is equal to or larger than the highest power on bottom.

Example 3 Evaluate

$$\int \frac{x^2 - 1}{x - 2}\, dx$$

Solution The fraction is too tough to evaluate directly. Since the top power is bigger, we have to use long division. Using the long division process (which we discussed in class), we get

$$\frac{x^2 - 1}{x - 2} \;=\; x \,+\, 2 \,+\, \frac{3}{x - 2}.$$

Note: you can also use synthetic division to arrive at this conclusion. Now we can solve our calculus problem:

$$\begin{aligned}\int \frac{x^2 - 1}{x - 2}\, dx &= \int \left(x \,+\, 2 \,+\, \frac{3}{x - 2} \right) dx \\ &= \frac{1}{2}x^2 \,+\, 2x \,+\, 3\ln|x - 2| \,+\, C.\end{aligned}$$

1 Lecture 9 - Methods of approximation

Integration lets us find (signed) areas underneath graphs. Some integrals cannot be evaluated directly however, for instance

$$\int_1^2 e^{-x^2}\,dx \qquad \text{or} \qquad \int_3^{10} \sqrt{1+x^3}\,dx.$$

Nevertheless, the graphs of $y = e^{-x^2}$ or $y = \sqrt{1+x^3}$ have graphs, under which lie finite areas. We sometimes may not be able to find the area by evaluating the integral, but we can use one of several approximation methods to approximate the area.

1.1 L_n, The Right-hand rule with n intervals

We find an approximation of the integral of $f(x)$ from a to b.

Divide the interval $[a, b]$ into n many subintervals, each having length

$$\triangle x = \frac{b-a}{n}.$$

The endpoints of the subintervals are

$$x_0 = a \qquad x_1 = a + \triangle x \qquad \ldots \qquad x_n = a + n\triangle x$$

$$x_i = a + i\triangle x.$$

For example, the first subinterval is $[x_0, x_1]$, the second subinterval is $[x_1, x_2]$, and the i^{th} subinterval is $[x_{i-1}, x_i]$. We use the *right* endpoint of the i^{th} interval to construct a rectangle. The rectangle's height is $f(x_i)$ and width is $\triangle x$.

$$\begin{aligned} \text{area of the } i^{th} \text{ rectangle} &= f(x_i)\,\triangle x \\ \text{sum of the areas of all } n \text{ many rectangles} &= \sum_{i=1}^{n} f(x_i)\,\triangle x. \end{aligned}$$

1.2 R_n, The Right-hand rule with n intervals

Again, the length of the intervals $\triangle x$ and the endpoints of the intervals are given by

$$\begin{aligned} \triangle x &= \frac{b-a}{n} \\ x_i &= a + i\triangle x. \end{aligned}$$

For example, the first subinterval is $[x_0, x_1]$, the second subinterval is $[x_1, x_2]$, and the i^{th} subinterval is $[x_{i-1}, x_i]$. We use the *left* endpoint of the i^{th} interval to construct a rectangle. The rectangle's height is $f(x_{i-1})$ and width is $\triangle x$.

$$\begin{aligned} \text{area of the } i^{th} \text{ rectangle} &= f(x_{i-1})\,\triangle x \\ \text{sum of the areas of all } n \text{ rectangles} &= \sum_{i=1}^{n} f(x_{i-1})\,\triangle x. \end{aligned}$$

1.3 M_n, The Midpoint rule with n intervals

Again, the length of the intervals $\triangle x$ and the endpoints of the intervals are given by

$$\begin{aligned} \triangle x &= \frac{b-a}{n} \\ x_i &= a + i\triangle x. \end{aligned}$$

For example, the first subinterval is $[x_0, x_1]$, the second subinterval is $[x_1, x_2]$, and the i^{th} subinterval is $[x_{i-1}, x_i]$. We use the *midpoint* of the i^{th} interval to construct a rectangle. The rectangle's height is $f\left(\frac{x_{i-1}+x_i}{2}\right)$ and width is $\triangle x$.

$$\begin{aligned} \text{area of the } i^{th} \text{ rectangle} &= f\left(\frac{x_{i-1}+x_i}{2}\right)\triangle x \\ \text{sum of the areas of all } n \text{ rectangles} &= \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\triangle x. \end{aligned}$$

1.4 T_n, The Trapezoidal rule with n intervals

This time we use trapezoids, not rectangles, to approximate the area under the graph. Again, the length of the intervals $\triangle x$ and the endpoints of the intervals are given by

$$\begin{aligned} \triangle x &= \frac{b-a}{n} \\ x_i &= a + i\triangle x. \end{aligned}$$

But this time, the intervals are used to construct trapezoids, not rectangles. Recall that the area of a trapezoid is $A = \frac{1}{2}\,h\,(b_1 + b_2)$. With the width of the i^{th} trapezoid being $\triangle x$ and the two bases being $f(x_{i-1})$ and $f(x_i)$, we have

$$\begin{aligned} \text{area of the } i^{th} \text{ trapezoid} &= \frac{1}{2}\triangle x\ (f(x_{i-1}) + f(x_i)) \\ \text{sum of the areas of all } n \text{ trapezoids} & \\ = \frac{1}{2}\triangle x\ (f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n))\,. \end{aligned}$$

1 Lecture 10 - Simpson's rule and examples

1.1 S_n, Simpson's rule with n intervals

Simpson's rule uses small pieces of parabolas (that is, graphs of the kind $y = ax^2 + bx + c$) to approximate the graphs. One can easily figure out the equation of a parabola passing through three points, say $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_{i+1}, f(x_{i+1}))$. and one can easily figure out a general formula for the area under a small piece of a parabola. In the end, we get

$$\begin{aligned} \triangle x &= \frac{b-a}{n} \\ x_i &= a + i\triangle x \end{aligned}$$

and the total approximate area under the curve is

$$\begin{aligned} &\text{approximate area under the graph} \\ &\quad = \frac{1}{3}\triangle x\,\left(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right). \end{aligned}$$

1.2 Examples of uses of approximation techniques

Example 1 Approximate the integral $\int_1^3 x^2\,dx$ using L_4, R_4, M_4, T_4, and S_4.

Solution
First off, in all cases we have $n = 4$, so we can use the formulas to find

$$\begin{aligned} &\triangle x = \frac{3-1}{4} = \frac{1}{2} \\ &x_0 = 1 \quad x_1 = \frac{3}{2} \quad x_2 = 2 \quad x_3 = \frac{5}{2} \quad x_4 = 3. \end{aligned}$$

Then

$$\begin{aligned}
L_4 &= \sum_{i=1}^{4} f(x_{i-1})\triangle x \\
&= f(x_0)\triangle x \;+\; f(x_1)\triangle x \;+\; f(x_2)\triangle x \;+\; f(x_3)\triangle x \\
&= (1)^2 \cdot \frac{1}{2} \;+\; \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} \;+\; (2)^2 \cdot \frac{1}{2} \;+\; \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} \\
&= \frac{27}{4} \;=\; 6.75.
\end{aligned}$$

$$\begin{aligned}
R_4 &= \sum_{i=1}^{4} f(x_i)\triangle x \\
&= f(x_1)\triangle x \;+\; f(x_2)\triangle x \;+\; f(x_3)\triangle x \;+\; f(x_4)\triangle x \\
&= \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} \;+\; (2)^2 \cdot \frac{1}{2} \;+\; \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} \;+\; (3)^2 \cdot \frac{1}{2} \\
&= \frac{43}{4} \;=\; 10.75.
\end{aligned}$$

$$\begin{aligned}
M_4 &= \sum_{i=1}^{4} f\left(\frac{x_{i-1}+x_i}{2}\right)\triangle x \\
&= f\left(\frac{x_0+x_1}{2}\right)\triangle x \;+\; f\left(\frac{x_1+x_2}{2}\right)\triangle x \;+\; f\left(\frac{x_2+x_3}{2}\right)\triangle x \;+\; f\left(\frac{x_3+x_4}{2}\right)\triangle x \\
&= f\left(\frac{5}{4}\right)\triangle x \;+\; f\left(\frac{7}{4}\right)\triangle x \;+\; f\left(\frac{9}{4}\right)\triangle x \;+\; f\left(\frac{11}{4}\right)\triangle x \\
&= \left(\frac{5}{4}\right)^2 \cdot \frac{1}{2} \;+\; \left(\frac{7}{4}\right)^2 \cdot \frac{1}{2} \;+\; \left(\frac{9}{4}\right)^2 \cdot \frac{1}{2} \;+\; \left(\frac{11}{4}\right)^2 \cdot \frac{1}{2} \\
&= \frac{69}{8} \;=\; 8.675.
\end{aligned}$$

$$
\begin{aligned}
T_4 &= \frac{\triangle x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right) \\
&= \frac{1}{4}\left(f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3)\right) \\
&= \frac{1}{4}\left((1)^2 + 2\cdot\left(\frac{3}{2}\right)^2 + 2\cdot(2)^2 + 2\cdot\left(\frac{5}{2}\right)^2 + (3)^2\right) \\
&= \frac{35}{4} = 8.75.
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= \frac{\triangle x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) \\
&= \frac{1}{6}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + f(3)\right) \\
&= \frac{1}{6}\left((1)^2 + 4\left(\frac{3}{2}\right)^2 + 2(2)^2 + 4\left(\frac{5}{4}\right)^2 + (3)^2\right) \\
&= \frac{26}{3} = 8.\bar{6}.
\end{aligned}
$$

True value of the integral:

$$\int_1^3 x^2 dx = \frac{1}{3}x^3\Big|_1^3 = 9 - \frac{1}{3} = \frac{26}{3} = 8.\bar{6}.$$

Note that Simpson's rule is not only the best approximation, in this case it is dead on.

1 Lecture 11 - Improper Integrals

An integral $\int_{x_0}^{x_1} f(x)\,dx$ is considered 'improper' if

1) $f(x)$ is 'singular,' meaning there is an infinite discontinuity, somewhere in the closed interval $[x_0, x_1]$
2) Either $x_0 = -\infty$ or $x_1 = \infty$ or both.

1.1 First type: $f(x)$ is singular somewhere

You CANNOT integrate right up to, or across, an infinite discontinuity. You must use limits to APPROACH any discontinuities.

If $f(s)$ is singular, we have to set up the integral as

$$\int_{x_0}^{x_1} f(x)\,dx \quad = \quad \lim_{a \to s^+} \int_a^t f(x)\,dx.$$

If $f(t)$ is singular, we have to set up the integral as

$$\int_{x_0}^{x_1} f(x)\,dx \quad = \quad \lim_{a \to t^-} \int_s^a f(x)\,dx.$$

If $f(m)$ is singular for some m between x_0 and x_1, we have to set up the integral as

$$\begin{aligned} \int_{x_0}^{x_1} f(x)\,dx \quad &= \quad \int_{x_0}^{m} f(x)\,dx \;+\; \int_{m}^{x_1} f(x)\,dx \\ &= \quad \lim_{a \to m^-} \int_{x_0}^{a} f(x)\,dx \;+\; \lim_{b \to m^+} \int_{b}^{x_1} f(x)\,dx. \end{aligned}$$

1.2 Second type: and infinity in the limits of integration

If $x_1 = \infty$, set up the integral

$$\int_{x_0}^{\infty} f(x)\,dx \quad = \quad \lim_{a \to \infty} \int_{x_0}^{a} f(x)\,dx$$

If $x_0 = -\infty$, set up the integral

$$\int_{\infty}^{x_1} f(x)\,dx \quad = \quad \lim_{a \to -\infty} \int_{a}^{x_1} f(x)\,dx$$

If both $x_0 = -\infty$ and $x_1 = \infty$, set up the integral

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx \;+\; \int_{0}^{\infty} f(x)\,dx \\ &= \lim_{a \to -\infty} \int_{a}^{0} f(x)\,dx \;+\; \lim_{b \to \infty} \int_{0}^{b} f(x)\,dx. \end{aligned}$$

1.3 The comparison test

Sometimes it will not be possible to evaluate an integral directly, for example

$$\int_1^\infty \frac{1}{\sqrt{1+x^3}}\,dx.$$

Nevertheless, it is often possible to determine of the integral is finite or not, using the comparison test.

Consider an improper integral

$$\int_a^b f(x)\,dx.$$

(ie, possibly $a = -\infty$, $b = \infty$, or $f(x)$ has a singularity somewhere). The first step is to choose a comparison function $g(x)$.

To prove that the integral $\int_a^b f(x)dx$ *converges*, you must

- Prove that the integral of the comparison function converges, namely that $\int_a^b g(x)dx$ is finite
- Prove that the function is *absolutely smaller* than the comparison function, namely that $|f(x)| \leq g(x)$.

To prove that the integral $\int_a^b f(x)dx$ *diverges*, you must

- Prove that the integral of the comparison function diverges, namely that $\int_a^b g(x)dx$ is infinite
- Prove that the function is *bigger* than the comparison function, namely that $f(x) \geq g(x)$.

Choosing an appropriate comparison function is largely a matter of intuition. After you've done a few examples, you can usually tell pretty well what the comparison function should be. Basically the idea is to extract the key features of the original function.

Example 1 Is $\int_1^\infty \frac{1}{\sqrt{1+x^3}}\,dx$ finite or infinite?

Solution It is not possible to evaluate the integral directly, but we can use the comparison test.

Our function is $f(x) = \frac{1}{\sqrt{1+x^3}}$. First we try to pick a comparison function. Since we are integrating up to infinity, and since $1 + x^3 \approx x^3$ as x gets big, we should let

$$g(x) = \frac{1}{\sqrt{x^3}} \; = \; \frac{1}{x^{3/2}}$$

be our comparison function. Since $\int_1^\infty g(x)\,dx$ is FINITE (by the p-test), we guess that the original integral is also finite. Our chain of inequalities is

$$f(x) \; = \; \frac{1}{\sqrt{1+x^3}} \; < \; \frac{1}{\sqrt{x^3}} \; = \; \frac{1}{x^{3/2}} \; = \; g(x).$$

Thus $f(x) < g(x)$, so it follows that $\int_1^\infty f(x)dx < \int_1^\infty g(x)dx < \infty$.

Example 2 Is $\int_3^\infty \frac{1}{\sqrt[4]{x^4-1}}\,dx$ finite or infinite?

Solution Again, we cannot evaluate directly. Since we are integrating up to ∞, and since $x^4 - 1 \approx x^4$ when x is very big, we should let

$$g(x) \; = \; \frac{1}{\sqrt[4]{x^4}} \; = \; \frac{1}{x}$$

be our comparison function. We know, by the p-test, that $\int_3^\infty g(x)dx = \infty$. Now we try to compare $f(x)$ to $g(x)$ by using the chain of inequalities

$$f(x) \; = \; \frac{1}{\sqrt[4]{x^4-1}} \; > \; \frac{1}{\sqrt[4]{x^4}} \; = \; \frac{1}{x} \; = \; g(x).$$

Therefore $\int_3^\infty f(x)\,dx \;\; > \;\; \int_3^\infty g(x)\,dx = \infty$, so therefore the original integral diverges.

1 Lecture 12 - Areas between graphs, and volumes of rotation

1.1 Areas between graphs

The area under the graph of $f(x)$, between $x = a$ and $x = b$ is

$$\int_a^b f(x)\,dx.$$

Now consider two functions $f(x)$ and $g(x)$ which intersect at $x = x_0$ and $x = x_1$. The area under $f(x)$ between x_0 and x_1 is $\int_{x_0}^{x_1} f(x)\,dx$, and the area under $g(x)$ between x_0 and x_1 is $\int_{x_0}^{x_1} g(x)\,dx$. To get the area between the graphs, you subtract:

$$\int_{x_0}^{x_1} (f(x) - g(x))\,dx \qquad \text{or} \qquad \int_{x_0}^{x_1} (g(x) - f(x))\,dx,$$

depending on which one has the upper graph and which one has the lower graph.

Example 1 Find the area bounded between the graphs of $f(x) = 3x + 4$ and $g(x) = x^2$.

Solution First we find the points of intersection by setting the functions equal to each other:

$$\begin{aligned} f(x) &= g(x) \\ 3x + 4 &= x^2 \\ 0 &= x^2 - 3x - 4 \\ 0 &= (x-4)(x+1). \end{aligned}$$

Thus the points of intersection are $x = -1$ and $x = 4$. It can easily be seen from the graphs that $f(x) = 3x - 4$ is the upper function. Thus the bounded area is

$$
\begin{aligned}
\inf\limits_{-1}^{4} \left(3x \, + \, 4 \, - \, x^2\right) \, dx \;&=\; \frac{3}{2}x^2 \, + \, 4x \, - \, \frac{1}{3}x^3 \Big|_{-1}^{4} \\
&=\; \left(\frac{3}{2} \cdot 16 + 4 \cdot 4 - \frac{1}{3} \cdot 64\right) \, - \, \left(\frac{3}{2} \, - \, 4 \, + \, \frac{1}{3}\right) \\
&=\; 44 - \frac{3}{2} - \frac{65}{3} \;=\; \frac{125}{6}.
\end{aligned}
$$

1.2 Solids of rotation: Shells

The volume of a shell of radius R height h and thickness dx is

$$dV \;=\; 2\pi \, R \, h \, dx.$$

Example 2 Find the area of the solid obtained by rotating about the y-axis the region bounded by $f(x) = -x^2 + 5x - 6$ and the x-axis.

Solution The graph of $f(x) = -x^2 + 5x - 6 = (3 - x)(x - 2)$ is a downward opening parabola that intersects the x-axis at $x = 2$ and $x = 3$. At each value of x lies a test-rectangle of height $f(x)$ and width dx. This rectangle, when rotated about the y-axis, produces a shell of radius x, height $f(x)$, and width dx. Thus each shell has volume

$$dV \;=\; 2\pi \, f(x) \, x \, dx.$$

Summing up the infinitesimal volumes, we get

$$
\begin{aligned}
V \;&=\; \int_2^3 dV \\
&=\; \int_2^3 2\pi \, \left(-x^2 \, + \, 5x \, - \, 6\right) \, x \, dx \\
&=\; 2\pi \int_2^3 \left(-x^3 \, + \, 5x^2 \, - \, 6x\right) \, dx \\
&=\; 2\pi \left(-\frac{1}{4}x^4 \, + \, \frac{5}{3}x^3 \, - \, 3x^2\right) \Big|_2^3 \\
&=\; \frac{\pi}{12}.
\end{aligned}
$$

1 Lecture 13 - Volumes of rotation, slicing, and arclength

1.1 Volumes of Rotation - Disks and Washers

Volume of a shells:

$$dV = 2\pi\, R\, h\, dx$$

where R is the radius, h is the height, and dx is the thickness of the shell.

Volume of a disks:

$$dV = \pi\, R^2\, dx$$

where R is the disk's radius and dx is its thickness.

Volume of a washer:

$$dV = \pi\left(R^2 - r^2\right)\, dx$$

where R is the outer radius, r is the inner radius, and dx is the thickness.

Example 1 Find the volume of the solid formed by rotating the region between the graph of $y = 1 - x^2$ and $y = 0$ about the x-axis.

Solution If you were finding the area of bounded by these graphs, a test rectangle would have height $f(x) = 1 - x^2$ and width dx. Rotating a test rectangle gives a disk of radius $1 - x^2$ and thickness dx. Thus a given disk has volume $dV = \pi\left(1 - x^2\right)^2\, dx$. Adding up all the volumes gives

$$\begin{aligned} V &= \int dV \\ &= \int_{-1}^{1} \pi\left(1 - x^2\right)^2\, dx \\ &= \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4\right)\, dx \\ &= \pi\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-1}^{1} \\ &= \pi\frac{16}{15}. \end{aligned}$$

Example 2 Find the volume of the solid formed by rotating the region between the graphs of $f(x) = 4 - x^2$ and $g(x) = 6 - 3x$ around the x-axis.

Solution In the bounded region, $f(x)$ is the top graph and $g(x)$ is the bottom graph. The points of intersection are found by setting $f(x) = g(x)$ and solving:

$$\begin{aligned} 4 - x^2 &= 6 - 3x \\ 0 &= 2 - 3x + x^2 \\ 0 &= (2 - x)(1 - x) \end{aligned}$$

so the points of intersection are $x = 1$ and $x = 2$.

If you were to find area between the graphs, your test rectangle would have height $f(x) - g(x)$ and width dx. Rotating about the x-axis, you get a washer with outer radius $f(x)$, inner radius $g(x)$, and thickness dx. Thus the volume of a given washer is

$$\begin{aligned} dV &= \pi \left(f(x)^2 - g(x)^2 \right) dx \\ &= \pi \left((4 - x^2)^2 - (6 - 3x)^2 \right) dx \\ &= \pi \left(-20 + 36x - 17x^2 + x^4 \right) dx \end{aligned}$$

Thus the total volume is

$$\begin{aligned} V &= \int dV \\ &= \int_1^2 \pi \left(-20 + 36x - 17x^2 + x^4 \right) dx \\ &= \pi \left(-20x + 18x^2 - \frac{17}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_1^2 \\ &= \pi \, \frac{812}{35}. \end{aligned}$$

Example 3 Find the volume of the solid formed by rotating the region between the graphs of $f(x) = x^2$ and $g(x) = -x + 2$ around the line $y = 4$.

Solution This time $g(x)$ is the top graph and $f(x)$ is the bottom graph. The points of intersection are found by setting $f(x) = g(x)$ and solving:

$$\begin{aligned} x^2 &= -x + 2 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

so the points of intersection are $x = -2$ and $x = 1$.

If you were to find area between the graphs, your test rectangle would have height $g(x) - f(x)$ and width dx. Rotating about the line $y = 4$, you get a washer with outer radius $4 - f(x)$, inner radius $4 - g(x)$, and thickness dx. Thus the volume of a given washer is

$$\begin{aligned} dV &= \pi\left((4 - f(x))^2 - (4 - g(x))^2\right)\, dx \\ &= \pi\left((4 - x^2)^2 - (2 + x)^2\right)\, dx \\ &= \pi\left(12 + 4x - 7x^2 + x^4\right)\, dx \end{aligned}$$

Thus the total volume is

$$\begin{aligned} V &= \int dV \\ &= \int_{-2}^{1} \pi\left(12 + 4x - 7x^2 + x^4\right)\, dx \\ &= \pi\left(12x + 2x^2 - \frac{7}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-2}^{1} \\ &= \pi\,\frac{78}{5}. \end{aligned}$$

1.2 Slicing

Not every 3-dimensional shape is a solid of revolution.

Example 4 Consider the region between the graph of $y = \sqrt{1 - x^2}$ and the x-axis. A solid is formed by letting each vertical segment be the base of an equilateral triangle. Find the volume of the solid.

Solution A test rectangle would have height $f(x) = \sqrt{1 - x^2}$ and width dx. But each test rectangle forms the base of a very thin triangular prism. The formula of the area of an equilateral triangle of sidelength s is $A = \frac{\sqrt{3}}{4} s^2$. The formula for the volume of a prism is $V = (Face\ Area) \cdot (thickness)$. Thus each thin prism has side-length $\sqrt{1 - x^2}$, thickness dx, and volume

$$\begin{aligned} dV &= (Face\ Area) \cdot (thickness) \\ &= \frac{\sqrt{3}}{4} \left(\sqrt{1 - x^2} \right)^2 dx \\ &= \frac{\sqrt{3}}{4} \left(1 - 2x^2\, x^4 \right) dx \end{aligned}$$

Thus the volume of the solid is

$$\begin{aligned} V &= \int dV \\ &= \int_{-1}^{1} \frac{\sqrt{3}}{4} \left(1 - 2x^2 + x^4 \right) dx \\ &= \frac{\sqrt{3}}{4} \left(x - \frac{2}{3} x^3 + \frac{1}{5} x^4 \right) \Big|_{-1}^{1} \\ &= \frac{4\sqrt{3}}{15}. \end{aligned}$$

1.3 Arclength

The problem is to determine the *length* of a curve, not the area under the curve. As always, we look at the problem on the infinitesimal level first, where the Pythagorean Theorem indicates proves that the infinitesimal length of arc dl (called the *arclength element*) is related to the infinitesimal changes in the x- and y-coordinate values, dx and dy, by

$$dl = \sqrt{(dx)^2 + (dy)^2}.$$

If the curve in question is given by the graph of a function, $y = f(x)$, then we can manipulate this

$$\begin{aligned} dl &= \sqrt{\left(1 + \frac{(dy)^2}{(dx)^2}\right)(dx)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\sqrt{(dx)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \end{aligned}$$

Example 5 Find the length of the curve given by the graph of $y = \frac{1}{2}x^2$ for $-1 < x < 1$.

Solution We compute the arclength element

$$\begin{aligned} dl &= \sqrt{1 + \left(\frac{dy}{dx}\right)}\,dx \\ &= \sqrt{1 + x^2}\,dx, \end{aligned}$$

then integrate

$$\begin{aligned} Arclength &= \int dl \\ &= \int_{-1}^{1} \sqrt{1 + x^2}\,dx. \end{aligned}$$

A trigonometric substitution is require: use $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ to get

$$\begin{aligned}
Arclength &= \int_{-1}^{1} \sqrt{1\,+\,x^2}\,dx \\
&= \int_{-\pi/4}^{\pi/4} \sqrt{1\,+\,\tan^2\theta}\,\sec^2\theta\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \sec^3\theta\,d\theta.
\end{aligned}$$

Now we use the reduction formula for the sec function to get

$$\begin{aligned}
Arclength &= \int_{-\pi/4}^{\pi/4} \sec^3\theta\,d\theta \\
&= \frac{1}{2}\tan\theta\,\sec\theta\Big|_{-\pi/4}^{\pi/4} \,+\, \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec\theta\,d\theta \\
&= \frac{1}{2}\tan\theta\,\sec\theta\Big|_{-\pi/4}^{\pi/4} \,+\, \frac{1}{2}\ln\left(\sec\theta\,+\,\tan\theta\right)_{-\pi/4}^{\pi/4} \\
&= \sqrt{2}\,+\,\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).
\end{aligned}$$

1 Lecture 18 - Applications to Work

The principle behind the applications here is that if you can solve the infinitesimal problem, then the large-scale problem can be solved by integrating.

If a particle moves in a straight path for a distance of s under the influence of the constant force F directed either with or directly against the motion, the work done by that force is

$$W = F\,s.$$

Of course force is often nonconstant and motion is rarely in a straight line, so this formula has limited applicability. But even if the force is nonconstant, it will still be approximately constant on small enough length scales, so that the formula still holds on the infinitesimal scale:

$$dW = F\,ds.$$

Example 1 A particle moves along the x-axis from $x = 0$ to $x = 1$ under the influence of the force $F = (1 + x^2)^{-1}$. Find the work done by this force.

Solution The displacement variable is x (not s) so we use the formula $dW = F\,dx = \frac{dx}{1+x^2}$. Then

$$\begin{aligned} W &= \int dW \\ &= \int_0^1 \frac{dx}{1+x^2} \\ &= \int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta \\ &= \int_0^{\pi/4} d\theta = \frac{\pi}{4}. \end{aligned}$$

Example 2 (problem 7 from section 6.5) Suppose 2J of work is needed to stretch a spring from its natural length of 30cm to a length of 42cm. Then (a) How much work is needed to stretch it from 30cm to 40cm and (b) how far beyond its natural length will a force of 30 N keep the spring stretched?

Solution The one thing we know about springs is Hooke's law: $F = kx$ where x is the distance the spring is stretched from its natural length, and k is the (usually unknown) spring constant. The spring's natural length is 30 cm, so it is stretched 12 cm from the natural length, we can use

$$W = \int dW = \int F dx$$

$$\begin{aligned} 2\,J &= \int_0^{.12m} k\,x\,dx \\ &= \frac{k}{2}x^2\Big|_0^{.12m} \\ &= \frac{k}{2}.0144\,m^2 \\ k &= \frac{4Nm}{.0144m^2} = \frac{100}{9}N/m \end{aligned}$$

(recall $1\,J = 1\,N \cdot m$). There is one and only one piece of information that characterizes an ideal spring: its spring constant. Knowing the spring constant allows us to find out any other information we need. The solution to part (a) is

$$\begin{aligned} W &= \int dW = \int_0^{.1} F\,dx \\ &= \int_0^{.1} \frac{100}{9}\,x\,dx \\ &= \frac{50}{9}\,x^2\Big|_0^{.1} \\ &= \frac{1}{18} \end{aligned}$$

The solution to part (b) is even simpler: given that the force is 30 N, we get

$$\begin{aligned} F &= k\,x \\ 30 &= \frac{100}{9}\,x \\ x &= \frac{27}{10} \end{aligned}$$

Example 3 (problem 11 from section 6.5) A cable weighing 2 lbs/ft is used to lift 800 lbs of coal up a mineshaft 500 ft deep. find out how much work is done.

Solution We divide into two parts: the work needed to lift the coal and the work needed to lift the cable. The work lifting the coal is easy: it weighs 800 lbs and is lifted 500 ft, so

$$\begin{aligned} W &= F\,s \\ &= 800 \cdot 500 \\ &= 400000. \end{aligned}$$

Finding the work done lifting the cable is a little harder. Let x be the height from the shaft's bottom. A piece of cable, at initial position x and of length dx, must be lifted $500 - x$ feet. The force on the piece of cable due to gravity is $dF = 2 \cdot dx$, so the work done to lift it is $dW = 2dx \cdot (500 - x)$. Thus

$$\begin{aligned} W &= \int dW \\ &= \int_0^{500} 2(500 - x)\,dx \\ &= 2(500x - \frac{1}{2}x^2)\Big|_0^{500} \\ &= 250000. \end{aligned}$$

Thus the total work done is

$$\begin{aligned} W_{tot} &= W_{coal} + W_{cable} \\ &= 650000. \end{aligned}$$

1 Lecture 19 - Applications to work and finance

1.1 Pumping Water out of tanks

Knowing the geometry of a tank and the amount of fluid it holds, one can determine how much work is required to pump the fluid out. One considers thin 'slabs' of water of constant gravitational potential, and of thickness dx. To find the work required to lift that 'slab' out of the tank, you must calculate its weight (force due to gravity) and the distance it is lifted against the gravitational pull.

Example 1 A tank has semicircular cross section of radius $4m$, and length of $8m$. If the tank is filled with water up to a depth of $4m$, determine the work needed to pump the water out.

Solution Let x measure the height above the tank's bottom. Consider a 'slab' of water, at height x, of thickness dx. Using basic geometry, you can calculate the width of the slab to be $2\sqrt{8x - x^2}$. Its legth is obviously 8. Thus the slab's volume is

$$dV = 16\sqrt{8x - x^2}\,dx.$$

The density of water is $1000kg/m^3$, and weight (force due to gravity) obeys $F = mg$ with $g = 9.8m/s^2$, so we get

$$\text{Infinitesimal wieght} = 156800\sqrt{8x - x^2}\,dx.$$

The slab must be lifted $4 - x$ meters, so the work done to the slab is

$$\begin{aligned} dW &= F \cdot s \\ &= 156800\sqrt{8x - x^2}\,dx \cdot (4 - x). \end{aligned}$$

Thus the total work is

$$\begin{aligned}
W &= \int dW \\
&= 156800 \int_0^4 (4-x)\sqrt{8x\,-\,x^2}\,dx \\
&= 88400 \int_0^{16} u^{\frac{1}{2}}\,du \\
&= 88400 \frac{2}{3}\,u^{3/2}\Big|_0^{16} \\
&= \frac{10135200}{3}.
\end{aligned}$$

1.2 Applications to Finance

If you sell exactly X many items, you must charge P dollars. If you charge any more, you will sell fewer items. If you charge less, you will sell more. We call $P = P(X)$ the demand function. In addition we have the cost function $C = C(X)$ giving the total cost of producing X units, the revenue function $R = R(X)$ giving the total revenue obtained by selling X units (obviously $R(X) = X \cdot P(X)$), and the profit function $T(X) = R(X) - C(X)$.

The *marginal* cost is the derivative of the cost function $C'(X)$, and likewise for the other quantities: marginal revenue is $R'(x)$, marginal profit is $T'(X)$.

The *consumer surplus* is the total amount of value your customers receive from doing business with you. Namely, if you charge 1 dollar for something that a consumer was willing to pay 2.50 for, the consumer receives a value of 1.50.

Example 2 The demand function for your product is $P = \frac{8}{1+x^2}$. Suppose you sell 10 units. What is the consumer surplus?

Solution To sell 10 units the price you charge is $P = \frac{8}{101}$, just under 8 cents. The consumer surplus is the total amount of value received by the consumer: the integrate of the price consumers would have paid (ie, what it is worth to them), minus the price they actually paid.

$$\begin{aligned}
\text{consumer surplus} &= \int_0^{10} \left(\frac{8}{1+x^2} - \frac{8}{101} \right) dx \\
&= 8 \int_0^{10} \frac{1}{1+x^2}\, dx - \frac{800}{101} \\
&= 8 \tan^{-1}(x) \Big|_0^{10} - \frac{800}{101} \\
&\approx 3.58
\end{aligned}$$

So consumers netted a grand total of \$3.58 of value.

1 Lecture 20 - Basic concepts of differential equations

A differential equation is an equation relating a function to one or more of its derivatives.

The first main example is Newton's Law of Cooling. If a body of temperature T is immersed in surroundings of constant temperature T_{AMB}, Newton says the rate of change in the body's temperature is proportional to the difference between the body's temperature and the ambient temperature. That is

$$\frac{dT}{dt} \;=\; -k\,(T\,-\,T_{AMB})\,,$$

which is a first-order differential equation.

The second main example is Hooke's Law, which gives the force exerted by an ideal spring as $F=-kx$, where k is the spring constant and x is the displacement from the natural length. Combine this with Newton's Second Law $F = m\frac{d^2x}{dt^2}$, and we have the following equation for a mass-spring problem:

$$m\frac{d^2x}{dt^2} \;=\; -k\,x$$

which is a second-order differential equation.

Example 1 Show that $x(t)=A\cos(\sqrt{2}t)+B\sin(\sqrt{2}t)$ solves the differential equation $\frac{d^2x}{dt^2}=-2x$.

Solution We compute

$$\begin{aligned}\frac{d}{dt}\frac{d}{dt}x(t) &= \frac{d}{dt}\Big(-\sqrt{2}A\sin(\sqrt{2}t)\,+\,\sqrt{2}B\cos(\sqrt{2}t)\Big)\\ &= -2A\cos(\sqrt{2}t)\,-\,2B\sin(\sqrt{2}t)\\ &= -2x.\end{aligned}$$

Example 2 Show that $y(t) = \frac{1}{\sqrt{-2t+C}}$ solves the differential equation $\frac{dy}{dt} = y^3$.

Solution We compute

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(-2t + C)^{-\frac{1}{2}} \\ &= -\frac{1}{2}(-2t + C)^{-\frac{3}{2}} \cdot (-2) \\ &= (-2t + C)^{-\frac{3}{2}} \\ &= y^3. \end{aligned}$$

Example 3 Find the solution to $\frac{dy}{dt} = y^3$ given that $y(0) = \frac{1}{2}$.

Solution We know that $y(t) = (-2t + C)^{-\frac{1}{2}}$ solves the differential equation, but we still have the unknown constant C to deal with. To find the value of C we use the initial condition

$$\begin{aligned} y(0) &= (-2(0) + C)^{-\frac{1}{2}} \\ \frac{1}{2} &= \frac{1}{\sqrt{C}} \\ C &= 4. \end{aligned}$$

Thus the solution is

$$y(t) = \frac{1}{\sqrt{4 - 2t}}.$$

1 Lecture 21 - Separable differential equations

Any first order differential equation can be written in the form

$$\frac{dy}{dx} = F(x,y).$$

If F has the special form $F(x,y) = f(x)g(y)$, the equation

$$\frac{dy}{dx} = f(x)g(y)$$

is called a *separable first order differential equation*. To solve: get all y's and dy's on one side, and all x's and dx's on the other. Then integrate.

$$\begin{aligned}
\frac{dy}{dx} &= f(x)\cdot g(y) \\
\frac{1}{g(y)}\,dy &= f(x)\,dx \\
\int \frac{1}{g(y)}\,dy &= \int f(x)\,dx.
\end{aligned}$$

Example 1 Find the general solution to $y' = y^3 \cos(x)$.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= y^3 \cos(x) \\
y^{-3}\,dy &= \cos(x)\,dx \\
\int y^{-3}\,dy &= \int \cos(x)\,dx \\
-\frac{1}{2}y^{-2} &= \sin(x) + C \\
y^{-2} &= -2\sin(x) + C \\
y &= (-2\sin(x) + C_1)^{-\frac{1}{2}}.
\end{aligned}$$

Example 2 Find the particular solution: $\frac{du}{dt} = 2 + 2u + t + tu$, $u(1) = 2$.

Solution

$$\begin{aligned}
\frac{du}{dt} &= 2 + 2u + t + tu \\
\frac{du}{dt} &= (1 + u)(2 + t) \\
\frac{1}{(1 + u)}\, du &= (2 + t)\, dt \\
\int \frac{1}{(1 + u)}\, du &= \int (2 + t)\, dt \\
\ln\left|1 + u\right| &= 2t + \frac{1}{2}t^2 + C.
\end{aligned}$$

This gives u as an implicit function of t. To get the explicit solution we have to do more work:

$$\begin{aligned}
\ln\left|1 + u\right| &= 2t + \frac{1}{2}t^2 + C \\
e^{\ln\left|1+u\right|} &= e^{2t + \frac{1}{2}t^2 + C} \\
\left|1 + u\right| &= e^C\, e^{2t + \frac{1}{2}t^2} \\
1 + u &= \pm e^C\, e^{2t + \frac{1}{2}t^2} \\
u &= -1 \pm e^C\, e^{2t + \frac{1}{2}t^2} \\
u &= -1 + C_1\, e^{2t + \frac{1}{2}t^2}.
\end{aligned}$$

Now we use the initial condition to determine the constant:

$$\begin{aligned}
2 &= -1 + C_1\, e^{2 + \frac{1}{2}} \\
C_1 &= 3\, e^{-5/2}.
\end{aligned}$$

Thus the particular solution is

$$u(t) = -1 + 3\, e^{-\frac{5}{2} + 2t + \frac{1}{2}t^2}.$$

1 Lecture 22 - Euler's method and Center of Area

1.1 Euler's method

A first order differential equation has the form

$$\frac{dx}{dt} = F(x,t).$$

We have seen that if $F(x,t)$ separates, meaning $F(x,t) = f(x)\,g(t)$, we can solve. However, it is often not possible to solve this DE explicitly.

Euler's method is a means of approximating the solution. First pick a discrete time step, δt. We can approximate δx, the change in the variable x, using the DE:

$$\begin{aligned} \frac{\delta x}{\delta t} &\approx F(x,t) \\ \delta x &\approx \delta t \cdot F(x,t). \end{aligned}$$

Thus, given some initial condition (x_0, t_0), and choosing a discrete time step δt, we can approximate the value of x at time $t_0 + \delta t$:

$$\begin{array}{rlcl} \text{Initial} & x_0 & & \\ \text{after time } \delta t & x_1 & = & x_0 + \delta x \\ & & = & x_0 + F(x_0, t_0). \end{array}$$

After another interval of δt, the value of x has changed by $\delta x = \delta t \cdot F(x_1, t_1)$, and so on. Thus if x_i indicates the value of x after i many steps, we use

$$\begin{aligned} x_{i+1} &= x_i + \delta x \\ &= x_i + \triangle t \cdot F(x_i, t_i) \end{aligned}$$

to find the value of x after $i+1$ many steps (obviously $t_i = t_0 + i \cdot \delta t$).

1.2 Center of Area

Let A be some region of the plane. Points in the plane are described by coordinates (x, y). The center of the region A is determined by finding the average of the x- and y-values over the points of the region A.

$$\begin{aligned} x_{ave} &= \frac{1}{Area(A)} \int_A x\, dA \\ y_{ave} &= \frac{1}{Area(A)} \int_A y\, dA. \end{aligned}$$

Example 1 Consider the region bounded by $y = 1 - x^2$, $x > 0$, and $y > 0$. Find the center of area.

Solution We use the formulas above. First we find the area: the area element (ie, a test rectangle) is $dA = (1 - x^2)dx$, so we get

$$\begin{aligned} Area(A) &= \int_A dA \\ &= \int_0^1 (1 - x^2)dx \;=\; \frac{2}{3}. \end{aligned}$$

To find the average of the x-coordinate, we use the same area element $dA = (1 - x^2)dx$ to get

$$\begin{aligned} x_{ave} &= \frac{1}{Area(A)} \int_A x\, dA \\ &= \frac{3}{2} \int_0^1 x\,(1 - x^2)\, dx \\ &= \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4} \right) \;=\; \frac{3}{8} \end{aligned}$$

To find the average of the y-coordinate we use the area element $dA = \sqrt{1 - y}\, dy$

to get

$$
\begin{aligned}
y_{ave} &= \frac{1}{Area(A)} \int_A y \, dA \\
&= \frac{3}{2} \int_0^1 y \sqrt{1-y} \, dy \\
&= \frac{3}{2} \left(\frac{2}{3} - \frac{2}{5} \right) = \frac{2}{5}
\end{aligned}
$$

Thus the center of area is

$$
(x_{ave}, \, y_{ave}) = \left(\frac{3}{8}, \frac{2}{5} \right).
$$

Test I Information

Here is a list of what you have to know: Test I guidelines.

Brian's Review session will be the Math Tower, room P-131, 6-8pm on Monday

Test I room assignments:

Brian's lecture (recitations 6-10)	Old Engineering room 145, at 8:30p on Tuesday
Thomas' lecture (reciations 1-5)	Old Engineering room 143, at 8:30p on Tuesday

Test I material:

Everything in sections 5.3 through 6.2 (excepting 5.8, which we didn't do).

Test II Information

Test I room assignments:

Brian's lecture (recitations 6-10)	Old Engineering room 145, at 8:30p on Thursday November 6
Thomas' lecture (reciations 1-5)	Old Engineering room 143, at 8:30p on Thursday November 6

Test I material:

Everything in sections 6.3 through 7.3.

Extra Credit/Review Problems

Instructions

Your best 4 problems will be counted.

A full 30 points counts as one homework assignment.

No credit will be given for answers without justification.

Compute the definite integral (1 pt each).

1) $\int x\sqrt{x^2 + 1}\, dx$

Ans: $\frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + C$

2) $\int \cos^2(t)\, \sin(t)\, dt$

Ans: $-\frac{1}{3}\cos^3(t) + C$

3) $\int \frac{y^2+1}{y^3+3y}\, dy$

4) $\int \frac{b - \cos(b)\, \sin(b)}{\sqrt{b^2 + \cos^2(b)}}\, db$

Simplify (2 pts each)

5) $e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{2}}$

Ans: 0

6) $\sqrt{2}\,e^{i\frac{\pi}{4}}$

Ans: $1 + i$

7) $e^{i\frac{3\pi}{4}}$

8) $e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}$

Simplify (3 pts each)

9) $e^{1+i\pi}$

Ans: $-e$

10) $e^{2t+i\frac{\pi}{4}t}$

Ans: $e^{2t}\left(\cos(\pi t/4) + i\,\sin(t\pi/4)\right)$

11) $e^{2+i\pi/2}$

12) $e^{-x+i\,x\,\pi/8}$

Find the general solution (4 points each)

13) $y'' + 7y' + 12y = 0$

Ans: $y(t) = c_1 e^{-4t} + c_2 e^{-3t}$

14) $y'' - 5y' + 4y = 0$

15) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 0$

16) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Find the solution (5 points each)

17) $y'' + 6y' + 10y = 0$

Ans: $y(t) = C_1 e^{-3t} \cos(t) + C_2 e^{-3t} \sin(t)$

18) $x'' - 2x' + 2x = 0$

19) $\frac{d^2 s}{dt^2} - 8\frac{ds}{dt} + 20s = 0$

20) $y'' + 3y' + 7y = 0$

Work the problem (10 points each)

21) Find the length of the path

$$\begin{aligned} x &= \cos^2(t) + \sin^4(t) \\ y &= \frac{4\sqrt{2}}{3}\sin^3(t) \end{aligned}$$

for $0 \leq t \leq \pi/4$.

22) A large spring or natural length 5m is mounted to the floor. Two blocks of equal mass are placed on the spring. The spring contracts by 1.5m after the first block is placed on the spring. When the second mass is placed on the spring, the work done to the spring is $\frac{9}{4}J$. Find the mass of either one of the blocks.

Extra Credit/Review Problems II

Instructions

Your best 4 problems will be counted.

A full 30 points counts as one homework assignment.

No credit will be given for answers without justification.

Evaluate (3 points each).

1) $\int_1^{\sqrt{3}} x^{-4}\sqrt{x^2 + 1}\, dx$

Ans: $\frac{8}{3}\left(\frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{3}}\right)$

2) $\int_0^1 x^2 \left(1 + x^2\right)^{-\frac{5}{2}}\, dx$

Ans: $\frac{\sqrt{2}}{12}$

3) $\int_0^1 \left(1 + x^2\right)^{-\frac{3}{2}}\, dx$

4) $\int_1^{\sqrt{3}} x^4 \left(1 + x^2\right)^{-\frac{7}{2}}\, dx$

Evaluate (3 points each)

5) $\int_{e}^{e^e} \frac{1}{x\ \ln(x)}\,dx$

Ans: 1

6) $\int_{e^e}^{e^{e^e}} \frac{1}{x\ \ln(x)\ \ln(\ln(x))}\,dx$

Ans: 1

7) $\int_{e^e}^{e^{e^2}} \frac{1}{x\ \ln(x)\,(\ln(\ln(x)))^2}\,dx$

8) $\int_{e^e}^{e^{e^3}} \frac{(\ln(\ln(x)))^2}{x\ \ln(x)}\,dx$

Let $\Gamma(x)$ be the function of x defined by

$$\Gamma(x) \;=\; \int_0^1 s^x\, e^{-s}\, ds.$$

9) (3 pts) Evaluate $\Gamma(1)$

Ans: 1

10) (3 pts) Evaluate $\Gamma(2)$

Ans: 2

11) (5 pts) Evaluate $\Gamma(3)$

12) (5 pts) Evaluate $\Gamma(4)$

13) (10 pts) Find the Taylor series centered at $x = -1$ for $f(x) = (2 - x)^{-2}$, and determine the radius of convergence.

14) (10 pts) Find the Taylor series centered at $x = 1$ for $f(x) = x^2 \ln(x)$, and determine the radius of convergence.

Notes on Second Order Linear Differential Equations

Stony Brook University Mathematics Department

1. The general second order homogeneous linear differential equation with constant coefficients looks like

$$Ay'' + By' + Cy = 0,$$

where y is an unknown function of the variable x, and A, B, and C are constants. If $A = 0$ this becomes a first order linear equation, which we already know how to solve. So we will consider the case $A \neq 0$. We can divide through by A and obtain the equivalent equation

$$y'' + by' + cy = 0$$

where $b = B/A$ and $c = C/A$.

"Linear with constant coefficients" means that each term in the equation is a constant times y or a derivative of y. "Homogeneous" excludes equations like $y'' + by' + cy = f(x)$ which can be solved, in certain important cases, by an extension of the methods we will study here.

2. In order to solve this equation, we guess that there is a solution of the form

$$y = e^{\lambda x},$$

where λ is an unknown constant. Why? Because it works!

We substitute $y = e^{\lambda x}$ in our equation. This gives

$$\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0.$$

Since $e^{\lambda x}$ is never zero, we can divide through and get the equation

$$\lambda^2 + b\lambda + c = 0.$$

Whenever λ is a solution of this equation, $y = e^{\lambda x}$ will automatically be a solution of our original differential equation, and if λ is not a solution, then $y = e^{\lambda x}$ cannot solve the differential equation. So the substitution $y = e^{\lambda x}$ transforms the differential equation into an algebraic equation!

Example 1. Consider the differential equation

$$y'' - y = 0.$$

Plugging in $y = e^{\lambda x}$ give us the associated equation

$$\lambda^2 - 1 = 0,$$

which factors as

$$(\lambda + 1)(\lambda - 1) = 0;$$

this equation has $\lambda = 1$ and $\lambda = -1$ as solutions. Both $y = e^x$ and $y = e^{-x}$ are solutions to the differential equation $y'' - y = 0$. (You should check this for yourself!)

Example 2. For the differential equation

$$y'' + y' - 2y = 0,$$

we look for the roots of the associated algebraic equation

$$\lambda^2 + \lambda - 2 = 0.$$

Since this factors as $(\lambda - 1)(\lambda + 2) = 0$, we get both $y = e^x$ and $y = e^{-2x}$ as solutions to the differential equation. Again, you should check that these are solutions.

3. For the general equation of the form

$$y'' + by' + cy = 0,$$

we need to find the roots of $\lambda^2 + b\lambda + c = 0$, which we can do using the quadratic formula to get

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If the *discriminant* $b^2 - 4c$ is positive, then there are two solutions, one for the plus sign and one for the minus.

This is what we saw in the two examples above.

Now here is a useful fact about linear differential equations: if y_1 and y_2 are solutions of the homogeneous differential equation $y'' + by' + cy = 0$, then so is the linear combination $py_1 + qy_2$ for any numbers p and q. This fact is easy to check (just plug $py_1 + qy_2$ into the equation and regroup terms; note that the coefficients b and c do not need to be constant for this to work. This means that for the differential equation in Example 1 ($y'' - y = 0$), any function of the form

$$pe^x + qe^{-x} \qquad \text{where } p \text{ and } q \text{ are any constants}$$

is a solution. Indeed, while we can't justify it here, *all* solutions are of this form. Similarly, in Example 2, the general solution of

$$y'' + y' - 2y = 0$$

is

$$y = pe^x + qe^{-2x}, \qquad \text{where } p \text{ and } q \text{ are constants.}$$

4. If the discriminant $b^2 - 4c$ is negative, then the equation $\lambda^2 + b\lambda + c = 0$ has no solutions, unless we enlarge the number field to include $i = \sqrt{-1}$, i.e. unless we work with complex numbers. If $b^2 - 4c < 0$, then since we can write any positive number as a square k^2, we let $k^2 = -(b^2 - 4c)$. Then ik will be a square root of $b^2 - 4c$, since $(ik)^2 = i^2k^2 = (-1)k^2 = -k^2 = b^2 - 4c$. The solutions of the associated algebraic equation are then

$$\lambda_1 = \frac{-b + ik}{2}, \;\; \lambda_2 = \frac{-b - ik}{2}.$$

Example 3. If we start with the differential equation $y'' + y = 0$ (so $b = 0$ and $c = 1$) the discriminant is $b^2 - 4c = -4$, so $2i$ is a square root of the discriminant and the solutions of the associated algebraic equation are $\lambda_1 = i$ and $\lambda_2 = -i$.

Example 4. If the differential equation is $y'' + 2y' + 2y = 0$ (so $b = 2$ and $c = 2$ and $b^2 - 4c = 4 - 8 = -4$). In this case the solutions of the associated algebraic equation are $\lambda = (-2 \pm 2i)/2$, i.e. $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$.

5. Going from the solutions of the associated algebraic equation to the solutions of the differential equation involves interpreting $e^{\lambda x}$ as a function of x when λ is a complex number. Suppose λ has real part a and imaginary part ib, so that $\lambda = a + ib$ with a and b real numbers. Then

$$e^{\lambda x} = e^{(a+ib)x} = e^{ax}e^{ibx}$$

assuming for the moment that complex numbers can be exponentiated so as to satisfy the law of exponents. The factor e^{ax} does not cause a problem, but what is e^{ibx}? Everything will work out if we take

$$e^{ibx} = \cos(bx) + i\sin(bx),$$

and we will see later that this formula is a necessary consequence of the elementary properties of the exponential, sine and cosine functions.

6. Let us try this formula with our examples.

Example 3. For $y'' + y = 0$ we found $\lambda_1 = i$ and $\lambda_2 = -i$, so the solutions are $y_1 = e^{ix}$ and $y_2 = e^{-ix}$. The formula gives us $y_1 = \cos x + i \sin x$ and $y_2 = \cos x - i \sin x$.

Our earlier observation that if y_1 and y_2 are solutions of the linear differential equation, then so is the combination $py_1 + qy_2$ for any numbers p and q holds even if p and q are complex constants.

Using this fact with the solutions from our example, we notice that $\frac{1}{2}(y_1 + y_2) = \cos x$ and $\frac{1}{2i}(y_1 - y_2) = \sin x$ are both solutions. *When we are given a problem with real coefficients it is customary, and always possible, to exhibit real solutions.* Using the fact about linear combinations again, we can say that $y = p\cos x + q \sin x$ is a solution for any p and q. This is the general solution. (It is also correct to call $y = pe^{ix} + qe^{-ix}$ the general solution; which one you use depends on the context.)

Example 4. $y'' + 2y' + 2y = 0$. We found $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Using the formula we have

$$y_1 = e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}(\cos x + i \sin x),$$
$$y_2 = e^{\lambda_2 x} = e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}(\cos x - i \sin x).$$

Exactly as before we can take $\frac{1}{2}(y_1 + y_2)$ and $\frac{1}{2i}(y_1 - y_2)$ to get the real solutions $e^{-x}\cos x$ and $e^{-x}\sin x$. (Check that these functions both satisfy the differential equation!) The general solution will be $y = pe^{-x}\cos x + qe^{-x}\sin x$.

7. *Repeated roots.* Suppose the discriminant is zero: $b^2 - 4c = 0$. Then the "characteristic equation" $\lambda^2 + b\lambda + c = 0$ has one root. In this case both $e^{\lambda x}$ **and** $xe^{\lambda x}$ are solutions of the differential equation.

Example 5. Consider the equation $y'' + 4y' + 4y = 0$. Here $b = c = 4$. The discriminant is $b^2 - 4c = 4^2 - 4 \times 4 = 0$. The only root is $\lambda = -2$. Check that **both** e^{-2x} and xe^{-2x} are solutions. The general solution is then $y = pe^{-2x} + qxe^{-2x}$.

8. *Initial Conditions.* For a first-order differential equation the undetermined constant can be adjusted to make the solution satisfy the initial condition $y(0) = y_0$; in the same way the p and the q in the general solution of a second order differential equation can be adjusted to satisfy initial conditions. Now there are two: we can specify both the value and the first derivative of the solution for some "initial" value of x.

Example 5. Suppose that for the differential equation of Example 2, $y'' + y' - 2y = 0$, we want a solution with $y(0) = 1$ and $y'(0) = -1$. The general solution is $y = pe^x + qe^{-2x}$, since the two roots of the characteristic equation are 1 and -2. The method is to write down what the initial conditions mean in terms of the general solution, and then to solve for p and q. In this case we have

$$1 = y(0) = pe0 + qe^{-2\times 0} = p + q$$
$$-1 = y'(0) = pe0 - 2qe^{-2\times 0} = p - 2q.$$

This leads to the set of linear equations $p + q = 1, p - 2q = -1$ with solution $q = 2/3, p = 1/3$. You should check that the solution

$$y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}$$

satisfies the initial conditions.

Example 6. For the differential equation of Example 4, $y'' + 2y' + 2y = 0$, we found the general solution $y = pe^{-x}\cos x + qe^{-x}\sin x$. To find a solution satisfying the initial conditions $y(0) = -2$ and $y'(0) = 1$ we proceed as in the last example:

$$-2 = y(0) = pe^{-0}\cos 0 + qe^{-0}\sin 0 = p$$

$$1 = y'(0) = -pe^{-0}\cos 0 - pe^{-0}\sin 0 - qe^{-0}\sin 0 + qe^{-0}\cos 0 = -p + q.$$

So $p = -2$ and $q = -1$. Again check that the solution

$$y = -2e^{-x}\cos x - e^{-x}\sin x$$

satisfies the initial conditions.

Problems cribbed from Salas-Hille-Etgen, page 1133

In exercises 1-10, find the general solution. Give the real form.

1. $y'' - 13y' + 42y = 0.$
2. $y'' + 7y' + 3y = 0.$
3. $y'' - 3y' + 8y = 0.$
4. $y'' - 12y = 0.$
5. $y'' + 12y = 0.$
6. $y'' - 3y' + \frac{9}{4}y = 0.$
7. $2y'' + 3y' = 0.$
8. $y'' - y' - 30y = 0.$
9. $y'' - 4y' + 4y = 0.$
10. $5y'' - 2y' + y = 0.$

In exercises 11-16, solve the given initial-value problem.

11. $y'' - 5y' + 6y = 0,\ \ y(0) = 1,\ \ y'(0) = 1$
12. $y'' + 2y' + y = 0,\ \ y(2) = 1,\ \ y'(2) = 2$
13. $y'' + \frac{1}{4}y = 0,\ \ y(\pi) = 1,\ \ y'(\pi) = -1$
14. $y'' - 2y' + 2y = 0,\ \ y(0) = -1,\ \ y'(0) = -1$
15. $y'' + 4y' + 4y = 0,\ \ y(-1) = 2,\ \ y'(-1) = 1$
16. $y'' - 2y' + 5y = 0,\ \ y(\pi/2) = 0,\ \ y'(\pi/2) = 2$

Brian Weber

MWF Lecturer, Course coordinator

Class: Lec 2 - Harriman Hll 137 - MWF 9:35am-10:30am

Office: Math Tower 3-121

Office Phone: 632-8264

Office Hours: Wednesdays 4-5, Thursdays 2-3, and in the MLC: Mondays 1-2

Email: brweber (at math dot sunysb dot edu)

Jan Gutt

TuTh Recitation

Class: R06 - SB Union 237 - TuTh 2:20pm-3:15pm

Office: S-240C (in the lowest level of the math tower)

Office Phone:

Office Hours: Wed 10:30-11:30
Wed 11:30-1:30 in the MLC

Email: jgutt (at math dot sunysb dot edu)

Evan Wright

MW Recitation

Class: R07 - Chemistry 128 - MW 11:45pm-12:40pm
R09 - Physics P117 - MW 3:50pm-4:45pm

Office: S-240C (in the lowest level of the math tower)

Office Phone:

Office Hours: Mon 12:50-1:50 in the MLC
Tues 11:30-12:30 in S240C
Wed 12:40-1:50 in the MLC

Email: evanpw (at math dot sunysb dot edu)

Webpage: math.stonybrook.edu/~evanpw/

Ye Sle Cha

TuTh Recitation

Class: R08 - Library N4006 - TuTh 8:20am-9:40am

Office: 4-118 Math Tower

Office Phone:

Office Hours: Mon 5:00-6:00
Mon 6:00-8:00 in the MLC

Email: ycha (at math dot sunysb dot edu)

Brandon Williams

TuTh Recitation

Class: R10 - SB Union 231 - TuTh 11:20am-12:40pm

Office: 2-105 Math Tower

Office Phone:

Office Hours: Wed 11:00a-1:00p
Wed 10:00a-11:00a in the MLC

Email: mbw (at math dot sunysb dot edu)

Webpage: math.sunysb.edu/~mbw

Thomas Poole

MW Lecturer

Class: Lec 1 - Old Chem 116 - MW 5:20pm-6:45pm

Office: Math Tower 2-120

Office Phone:

Office Hours: Mon 4:00-5:00 and Wed 4:00-5:00
Mon 12:50-1:50 in the MLC

Email: tpoole (at math dot sunysb dot edu)

Jiansong Chen

MW Recitation

Class: R01 - Physics P112 - MW 6:50pm-7:45pm
R03 - Physics P115 - MW 11:45am-12:40pm

Office: Math Tower 2-106

Office Phone:

Office Hours: Mon 1:30-2:30
Mon 4:50-6:50 in the MLC

Email: jschen (at math dot sunysb dot edu)

Joseph Walsh

TuTh Recitation

Class: R02 - Physics P122 - TuTh 5:20pm-6:40pm

Office: Math Tower 2-105

Office Phone:

Office Hours: Mon 7:00-8:00
Wed 7:00-8:00
Mon 6:00-7:00 in the MLC

Email: jwalsh (at math dot sunysb dot edu)

Prachi Bemalkhedkar

MF Recitation

Class: R04 - Lgt Engr Lab 152 - MF 12:50pm-1:45pm

Office: Light Engineering Building

Office Phone:

Office Hours: Mon 3:00-5:00 in the MLC
Wed 3:00-5:00 in the MLC

Email: pbemalkh (at ic dot sunysb dot edu)

Frank Palladino

MW Recitation

Class: R05 - SB Union 231 - MW 3:50pm-4:45pm

Office: Math Tower 2-112

Office Phone:

Office Hours: Mon 2:00-3:00
Mon 1:00-2:00 in the MLC
Wed 1:00-2:00 in the MLC

Email: fpalladino (at math dot sunysb dot edu)