Math 131: Calculus I - Spring 2007

The final exam scores and final course grades have been posted on Blackboard.

Lectures

Lec 1	MWF	9:35 a.m. - 10:30 a.m.	Humanities 1003	Redden, Corbett
Lec 2	TuTh	5:20 p.m. - 6:40 p.m.	Humanities 1003	Stimpson, Andrew

Recitations

Lec 1	R01	MW	11:45 a.m. - 12:40 p.m.	Chemistry 128	Jaggi, Amit
	R02	TuTh	3:50 p.m. - 4:45 p.m.	Harriman Hall 115	Lyberg, Ivar
	R03	TuTh	12:50 p.m. - 2:10 p.m.	Physics P122	Lyberg, Ivar
Lec 2	R06	TuTh	6:50 p.m. - 8:10 p.m.	Physics P127	Chen, Xiaojun

Contact Info and Office Hours

Corbett Redden	redden AT math.sunysb.edu	
Andrew Stimpson	stimpson AT math.sunysb.edu	
Amit Jaggi	amitsingh.jaggi AT gmail.com	T Th 2:20 - 3:20 p.m. in MLC
Ivar Lyberg	ilyberg AT math.sunysb.edu	
Xiaojun Chen	chen AT math.sunysb.edu	

Prerequisites

To take MAT 131, you must have either passed MAT 123 or MAT 130 with a C or better, or you must have received a score of 5 or better on the Mathematics Placement Exam. For more information about the differences between MAT 131 and other calculus courses, please see the First Year Mathematics Courses document.

Textbook

The course textbook is James Stewart's *Single Variable Calculus: Concepts and Contexts (Third Edition; Stony Brook University Edition)* (Brooks/Cole 2006, ISBN 0-495-08145-0). The book is available at the campus bookstore.

Course Grade

The course grade will be determined by the exam grades along with a homework/recitation grade using the following percentages:

Exam 1	25%
Exam 2	25%
Final Exam	35%
Homework/Recitation	15%

A grade of C or better is required to take MAT 132.

Homework

Homework assignments will be assigned weekly and collected in the recitation sections. The homework problems, along with the course schedule, are found on the Homework webpage; each assignment will be due the week after we cover the material. For instance, we cover sections 1.1-1.6 the first week, and the assignment is due the second week (usually in the second meeting of the week). Though homework is not a large portion of your final grade, it is *extremely* important to take seriously. It is impossible to learn the material by merely paying attention in lecture, and failing to do homework will greatly affect your exam performance. You are welcome and encouraged to work with others on homework; however, the actual writing up of solutions must be your own work. Finally, only a few problems in each homework assignment will actually be graded. This will allow the recitation instructor to give useful comments as opposed to merely checking the final answer. No late homework will be accepted, but the lowest homework grade will be dropped.

Recitations

In addition to the lectures, there are also two recitation sections per week. You must enroll in the recitation section you attend. In addition to collecting and returning homeworks, recitation leaders will go over some of the homework problems and also administer periodic quizzes. These quizzes contribute to the Homework/Recitation grade, and they are designed to prepare you for exams.

Calculators

No calculators will be permitted during the exams. However, you may need to use a calculator for a few of the homework problems. Otherwise, we strongly recommend that you do your homework without using a calculator.

Don't Miss Exams

The exam dates and times are not flexible, and there will be no make-up exams except in the case of a *serious, documented* emergency. It is your responsibility to properly arrange your schedule. In particular, work conflicts are not a valid excuse for missing an exam. In the case of a serious illness or death in the family, please notify your professor as soon as physically possible and then bring documentation once you are able to return.

Office Hours and MLC

Lecturers and recitation instructors will hold three office hours per week. These are listed on the Sections webpage. Another source of help is the Math Learning Center (MLC), located in Room S-240A of the Mathematics Building. It is staffed by experienced mathematics tutors, including professors, graduate students, and advanced undergraduate students. Students may drop in without an appointment. Also, your recitation instructor will hold at least one office hour per week at the MLC. Check out the MLC website for more information.

Important Dates

Friday, February 2	Last Day for Add/Drop
Tuesday, February 20	Midterm 1
Friday, March 2	Last Day to Drop Down to Lower Math Course
Monday, March 19	Midterm 2
Friday, May 11	Final Exam

Disabilities

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services , ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential. Students requiring emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information, go to the following web site: http://www.stonybrook.edu/ehs/fire/disabilities.shtml.

The numbers in black are the required homework problems which must be turned in. It is strongly suggested that you work the grey problems as well, though they do not need to be turned in. For information about homework policies, see the Policies webpage. Please note that the course outline is tentative.

Week Of	Material Covered and HW Assigned	Week HW Due
1/22 - 1/26	1.1: 7, 42, 46, 52, 2, 5, 6, 8, 19, 24 1.2: 4, 3, 10 1.3: 4ad, 18, 36, 3, 6, 7, 11, 13, 14, 39, 47, 63 1.5: 9, 18, 10, 15, 19 1.6: 24, 34, 48, 5, 6, 12, 17, 20, 37, 52, 53 Appendix C: 18, 37, 1, 4, 15, 30	1/29-2/2
1/29 - 2/2	2.1: 4, 6, 3, 5 2.2: 4, 5, 6, 12, 3, 8, 9, 10, 17, 23a 2.3: 2acd, 4, 8, 10, 15, 28, 31, 3, 7, 11, 16, 17, 29, 33, 35, 43	2/5-2/9
2/5 - 2/9	2.4: 4, 10, 14, 18, 25, 31, 33, 39, 49, 3, 6, 7, 15, 17, 27, 30, 36, 37 2.5: 4, 8, 17, 20, 23, 28, 31, 46, 3, 10, 15, 21, 24, 29, 47	2/12 - 2/16
2/12 - 2/16	2.6: 2, 8, 9, 14, 18, 7, 13, 17, 20, 24 2.7: 3, 6, 8, 13, 16, 20, 22, 27, 30, 17, 21, 35, 36	2/19 - 2/23
2/19 - 2/23	**Midterm 1 (2/20)**; Review 2.8: 4, 9, 30, 32, 39, 45, 7, 8, 11, 31, 37 2.9: 1, 5, 8, 10, 11, 15, 16, 2, 6, 12, 17	2/26-3/2
2/26 - 3/2	3.1: 2, 4, 5, 6, 8, 11, 13, 16, 17, 20, 21, 24, 26, 37, 38, 41, 43, 46, 48, 3 through 24, 25, 44, 45 3.2: 1, 2, 3, 6, 8, 9, 12, 13, 16, 19, 22, 29, 32, 33, 36, 38, 41 3 through 20, 21, 31, 35, 37, 42	3/5-3/9

	3.4: 1, 2, 4, 5, 8, 10, 11, 13, 14, 15	
3/5 - 3/9	3.4: 17, 19a, 24, 27, 30, 31, 33, 17, 28, 29, 34ab 3.5: 1, 3, 4, 7, 10, 12, 13, 17, 23, 25, 27, 30, 34, 35, 59, 69, 71, 1 through 32 3.6: 1, 3, 6, 7, 11, 15, 18, 29, 30, 39, 43, 3 through 20	3/12-3/16
3/12 - 3/16	3.7: 2, 4, 6, 11, 14, 17, 19, 23, 27, 29, 31, 2-18, 27-36 3.8: 1, 5, 6, 15, 16 Review	3/19-3/23
3/19 - 3/23	**Midterm 2 (3/19)**; Review 4.1: 1, 5, 8, 9, 11, 13, 14, 25 4.2: 1, 3, 6, 8, 9, 13, 25, 28, 29, 31, 36, 37, 38, 43, 45	3/26-3/30
3/26- 3/30	4.3: 7, 13, 19, 23, 27, 31, 32, 9, 11, 29, 30, 4.5: 1, 2, 5, 9, 13, 21, 23, 25, 45, 55, 7, 11, 15, 17, 19, 27, 29, 47	4/9 - 4/13
4/2 - 4/6	Spring Recess: No Classes	
4/9 - 4/13	4.6: 5, 7, 9, 13, 21, 28, 30, 38, 3, 4, 11, 12, 18, 23, 32, 41, 44 4.7: 15, 17 4.8: 4cd, 5, 9, 11, 27, 30, 1, 6, 10, 12, 23, 29	4/16 - 4/20
4/16 - 4/20	4.9: 1, 4, 6, 7, 9, 10, 12, 15, 19, 20, 23, 25, 28, 40, 42, 45 5.1: 1, 4, 17, 19, 20 5.2: 1, 17, 18, 21, 23, 24, 27, 31, 33, 34, 37, 41, 42, 43	4/23 - 4/27
4/23 - 4/27	**Extra Credit Quiz** 5.3: 1, 5, 9, 15, 17, 19, 21, 25, 27, 29, 41, 47, 55, 57 5.4: 3, 5, 7, 9, 13, 15, 19	4/30 - 5/4 5.4 may be later.

4/30 - 5/4	5.5: 1, 5, 9, 13, 17, 21, 27, 29, 39, 43, 47, 49, 53, 60	Ask TA
	Review	

Exam Schedule

Midterm 1	Tuesday, February 20	8:30 p.m. - 10:00 p.m.
Midterm 2	Monday, March 19	8:30 p.m. - 10:00 p.m.
Final Exam	Friday, May 11	11:00 a.m. - 1:30 p.m.

Remember that these exams times are not flexible, and there will be no make-up exams (except for severe documented emergencies). More information, including exam locations and review sheets, will be posted throughout the semester.

Final Exam - ESS 001

Friday, May 11. 11:00a.m. - 1:30p.m.

The final exam is cumulative; it covers all the material from the class. There will be a slight emphasis on the material covered since the second midterm.

Old Exams:
Fall 2003 Final Exam (Solutions)
Fall 2006 Final Exam (Solutions)
Spring 2003 Final Review

Grades:
The average score on the final exam was an 81 (out of 140). Below is a chart of the score distributions. Your final exam scores, along with your final course grades, are available at Blackboard.

Extra Credit Quiz

There will be an Extra Credit Quiz covering Chapter 4. For Lec 1(Redden), this will take place in class Monday, April 23. For Lec 2 (Stimpson), this will take place in recitation on Tuesday the 24th. The score from this quiz, which is worth 20 possible points, will be added to the point total (out of 200) from the two midterms. Also, the final exam covers a lot of material, and this quiz should motivate you to review the Chapter 4 material now as opposed to immediately before the final. If you treat this quiz seriously, you will be better prepared for the final and score higher.

The quiz (at least for Lec 1) contains one of each of the following types of problems: related rates, graphing a function, optimization.
Solutions to Extra Credit Quiz (Redden)

Midterm 2

R02, R03: Harriman 137
R01, R06: Javitz 102 (see sections for list of recitations)

The exam covers 2.8-2.9, 3.1-3.8. There are no calculators or notes allowed. You do not need to memorize the formulas for derivatives of inverse trig functions (e.g. arcsin, arctan). Below are a couple of older exams. Out of the three, this year's exam will be most similar to Fall 1999 Midterm 2. The textbook also provides a large number of practice problems.

Old Exams:
Fall 2003 Midterm 2 Practice (and solutions)
Fall 1999 Midterm 2
Fall 2006 Midterm 2 (and solutions)

Solutions:
Spring 2007 Midterm 2 Solutions.

Grades:
The average score on the second midterm was a 73. Below is a chart of the grade distributions.

Here is an *approximate* guide to grades:
>86 ~ A
73-86 ~ B
55-72 ~ C
40-54 ~ D
< 40 ~ F

Midterm 1

Exam location: Javitz 100

Exam format and information: Bring pencils and Stony Brook ID! The exam focuses on 2.1 - 2.7, though students are also responsible for the material covered from chapter 1. No calculators will be needed or allowed. The exam will consist of 10 multiple choice problems worth 5 points each and 5 free response questions worth 10 points each. Partial credit will be given on the free response questions. The midterm will be similar to homework problems, and the best way to prepare is to work through homework problems and old exams.

Old Exams:
Spring 2003 Midterm 1 Review
Fall 2003 Midterm 1 Sample Exam (and solutions)
Spring 2006 Midterm 1 (and solutions)

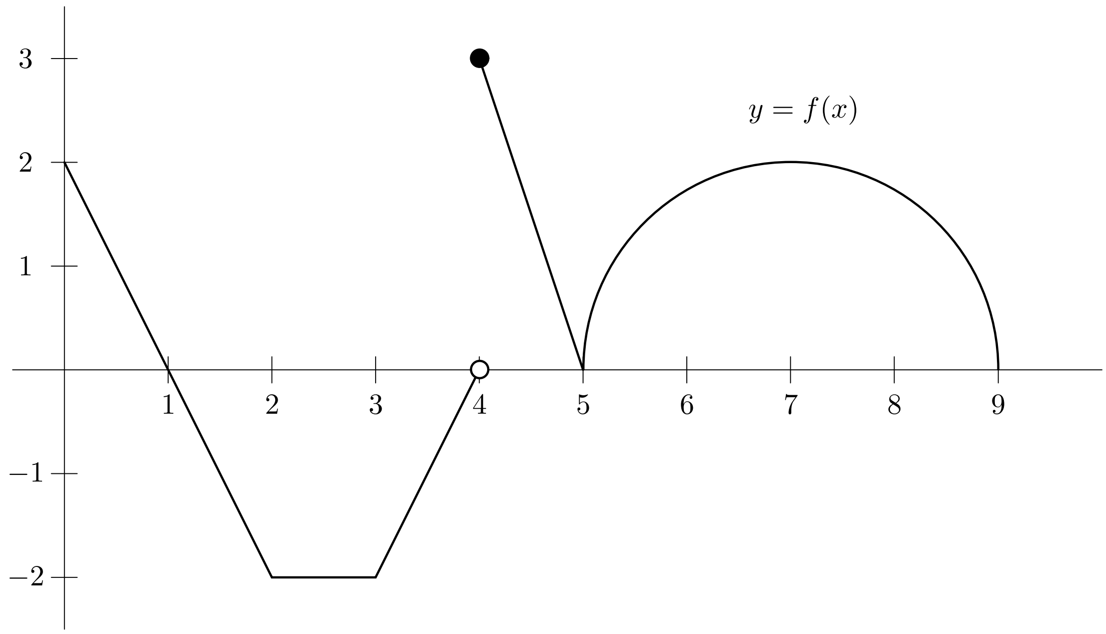
Solutions:
Spring 2007 Midterm 1 Solutions.

Grades:
The average score on the midterm was a 50. There is a curve, but it is calculated as a final curve for the class. This means that all of your raw scores (midterms, final, and homework, weighted appropriately) will be added together to produce one score. The course grades will be based on a curve for this total score. This means it is difficult to say precisely what a numerical midterm score means as a letter grade, and it is more relevant to see how you compared with your classmates. The graph below shows the grade distribution for Midterm 1.

Also, here is an *approximate* guide to what the numerical scores mean.
>70 ~ A
55-70 ~ B
40-55 ~ C
25-40 ~ D for Danger! You need to improve performance!
< 25 ~ F. Drastic improvements must be made.
Please know that Friday, March 2 is the last day to drop down to a lower math class.

Problem 1. [**12 points**] Let $h(x) = \dfrac{1}{x^2+1}$. On what intervals is the graph of h concave up?

Problem 2. **[3 points each]** Use the sketch of $y = f(x)$ below to find the following:



(a) $f'(1) =$

(b) $\displaystyle\int_0^3 f(x)dx =$

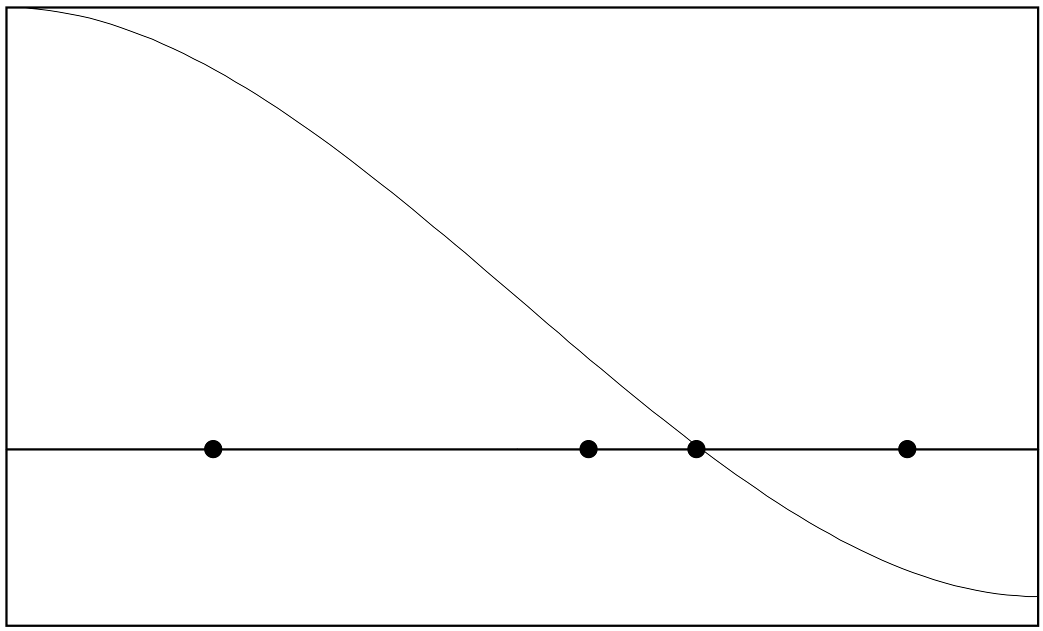
(c) $f(4) =$

(d) $\displaystyle\lim_{x\to 4^-} f(x) =$

(e) $\displaystyle\int_5^9 f(x)dx =$

(f) $f'(5) =$

Problem 3. Below is the graph of a function g. Consider Newton's method for solving the equation $g(x) = 0$.



(a) **[6 points]** The four points in the figure above represent an initial guess and three values produced from successive iterations of Newton's method. But which are which? Correctly label the points in the picture: mark the initial guess as x_0 and the other points as x_1, x_2, and x_3.

(b) **[6 points]** Here, actually,

$$x_0 = \frac{2}{5} \quad g\left(\frac{2}{5}\right) = \frac{43}{50} \text{ and } g'\left(\frac{2}{5}\right) = -\frac{16}{25}.$$

Use this information to calculate the value of x_1.

Problem 4. [3 points each] True or False.

(a) Suppose that f is differentiable and $f(2) = f(6)$. Then there must be at least one point $c \in (2, 6)$ with $f'(c) = 0$.

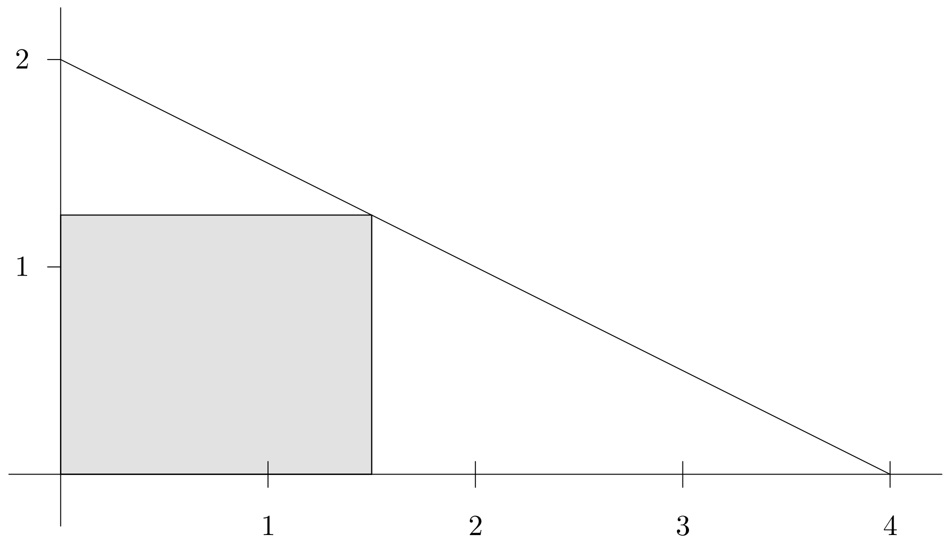
(b) Suppose that g is continuous, $g(1) = 5$ and $g(5) = 10$. Then the equation $g(c) = 7$ must have a solution $c \in (1, 5)$.

(c) If $f(x) > x$ for all x, then $\int_0^{10} f(x)dx > 5$.

(d) If $f'(x) = g'(x)$ then $f(x) = g(x)$.

(e) $\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$ for any f and g.

Problem 5. **[12 points]** Pictured below is the triangle formed by the intersection of the line $y = -\frac{1}{2}x + 2$, the x-axis, and the y-axis, together with an inscribed rectangle.



Find the area of the largest rectangle that can be inscribed in this triangle.

Problem 6. **[4 points each]** Compute:

(a) $\lim\limits_{x \to 0} \dfrac{x \sin(x)}{\cos(x) - 1}$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\, dx$

(c) $\dfrac{d}{dx} \sin\left(e^x\right)$

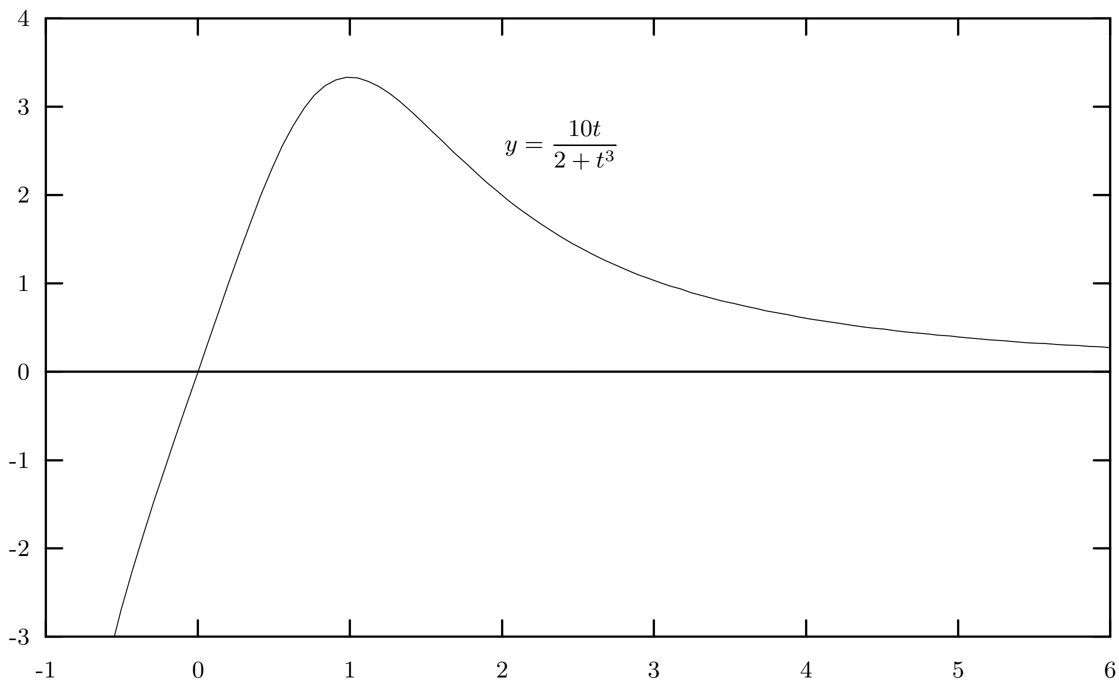
(d) $\displaystyle\int x^2 + 3x + 5\, dx$

(e) $\lim\limits_{x \to \infty} \dfrac{3x - e^x}{8x^2 + \ln(x)}$

Problem 7. Let A be the function defined by

$$A(x) = \int_1^x \frac{10t}{2+t^3}dt \text{ for } t \geq -\sqrt[3]{2}.$$

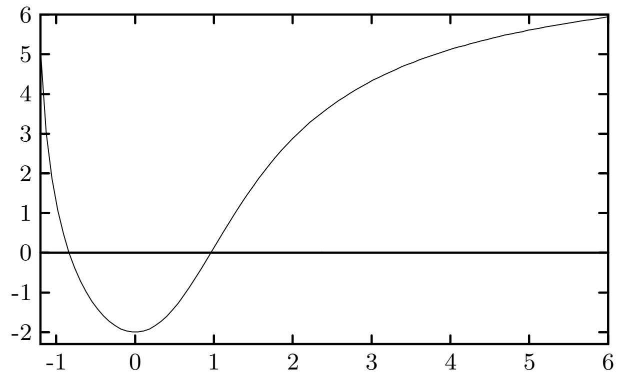
This sketch might be helpful:

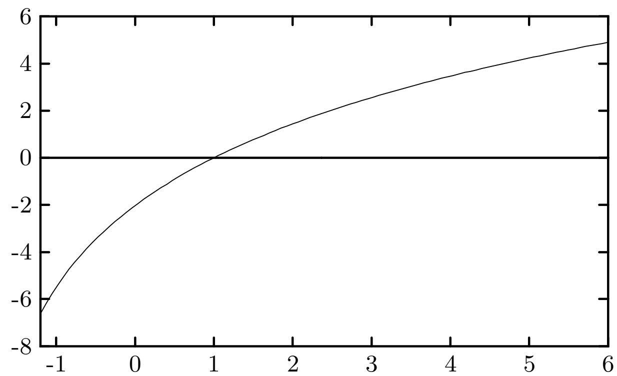


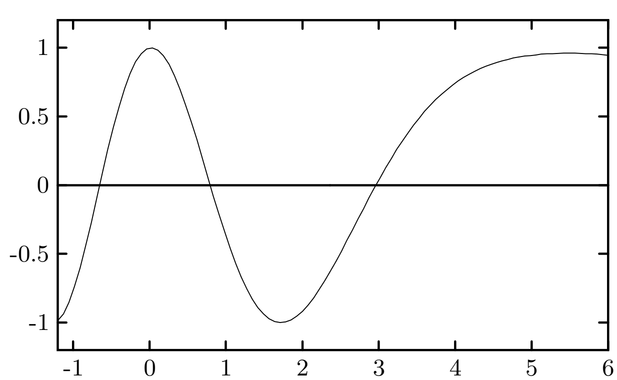
(a) **[6 points]** Approximate $A(3)$ using two circumscribed rectangles.

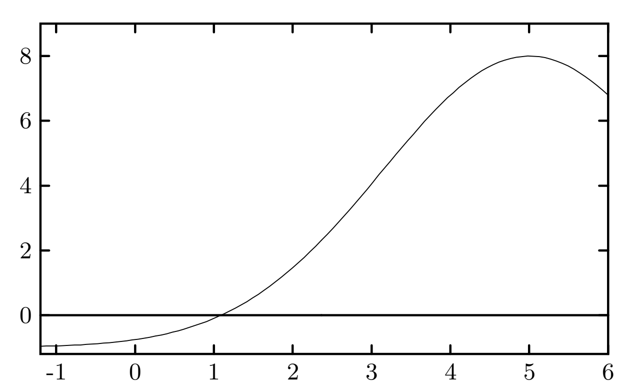
Problem 7. (Continued.)

(b) **[6 points]** Which of the following is the graph of $y = A(x)$?









EXAM

Final Exam

Math 131

Tuesday, December 16, 2003

- Name ....
- Student ID ....
- Lecture Section ....
- Recitation Section ....

1	/12
2	/18
3	/12
4	/15
5	/12
6	/20
7	/12
total	/101

EXAM

Final Exam

Math 131

Tuesday, December 16, 2003

ANSWERS

Problem 1. **[12 points]** Let $h(x) = \dfrac{1}{x^2+1}$. On what intervals is the graph of h concave up?

Answer:
We compute

$$h'(x) = \frac{-2x}{(x^2+1)^2}$$

and

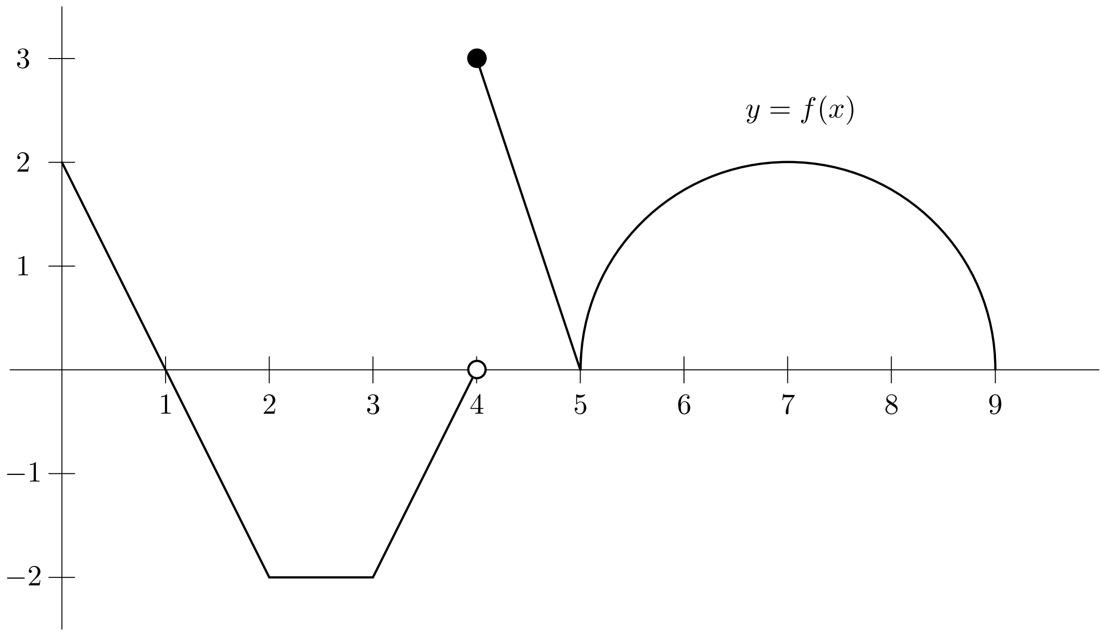
$$h''(x) = \frac{(x^2+1)^2(-2) - (-2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{-2(x^2+1)+(8x^2)}{(x^2+1)^3} = \frac{6x^2-2}{(x^2+1)^3}.$$

The graph of h is concave up when

$$h''(x) = \frac{6x^2-2}{(x^2+1)^3} > 0 \Leftrightarrow 6x^2 - 2 > 0 \Leftrightarrow x^2 > \frac{1}{3}.$$

Thus, h is concave on the intervals $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

Problem 2. **[3 points each]** Use the sketch of $y = f(x)$ below to find the following:



Answer:

(a) $f'(1) = -2$

(d) $\lim_{x \to 4^-} f(x) = 0$

(b) $\int_0^3 f(x)dx = -2$

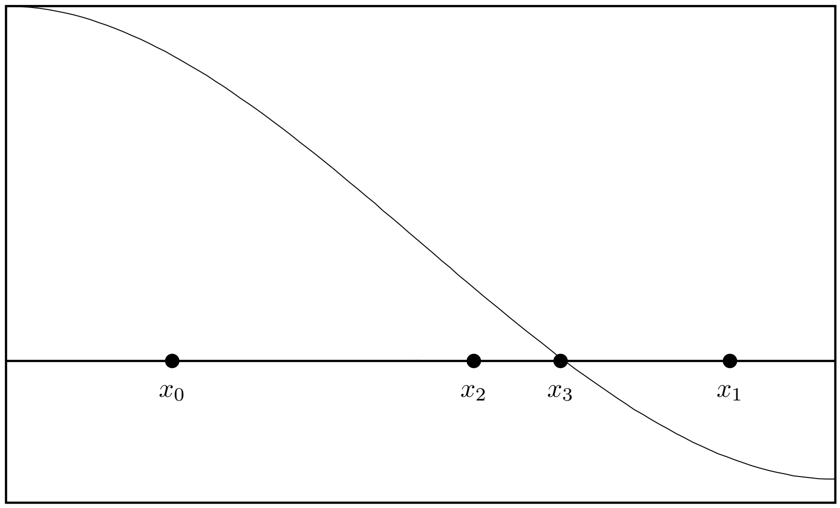
(e) $\int_5^9 f(x)dx = 2\pi$

(c) $f(4) = 3$

(f) $f'(5) =$ does not exist.

Problem 3. Below is the graph of a function g. Consider Newton's method for solving the equation $g(x) = 0$.

Answer:



(a) **[6 points]** The four points in the figure above represent an initial guess and three values produced from successive iterations of Newton's method. But which are which? Correctly label the points in the picture: mark the initial guess as x_0 and the other points as x_1, x_2, and x_3.

(b) **[6 points]** Here, actually,

$$x_0 = \frac{2}{5} \quad g\left(\frac{2}{5}\right) = \frac{43}{50} \text{ and } g'\left(\frac{2}{5}\right) = -\frac{16}{25}.$$

Use this information to calculate the value of x_1.

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = \frac{2}{5} - \frac{\frac{43}{50}}{-\frac{16}{25}} = \frac{2}{5} + \frac{43}{32} = \frac{279}{160} = 1.74375.$$

Problem 4. **[3 points each]** True or False.

Answer:

(a) Suppose that f is differentiable and $f(2) = f(6)$. Then there must be at least one point $c \in (2, 6)$ with $f'(c) = 0$. **True** by the mean value theorem.

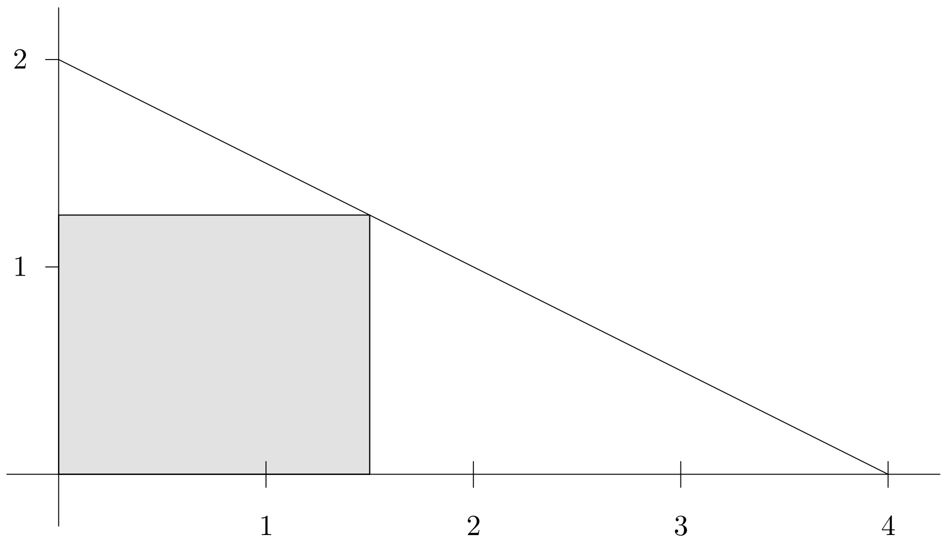
(b) Suppose that g is continuous, $g(1) = 5$ and $g(5) = 10$. Then the equation $g(c) = 7$ must have a solution $c \in (1, 5)$. **True** by the intermediate value theorem.

(c) If $f(x) > x$ for all x, then $\displaystyle\int_0^{10} f(x)dx > 5$. **True**: since $5 = \int_0^1 0xdx$.

(d) If $f'(x) = g'(x)$ then $f(x) = g(x)$. **False.**

(e) $\displaystyle\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$ for any f and g. **False.**

Problem 5. **[12 points]** Pictured below is the triangle formed by the intersection of the line $y = -\frac{1}{2}x + 2$, the x-axis, and the y-axis, together with an inscribed rectangle.



Find the area of the largest rectangle that can be inscribed in this triangle.

Answer:
We wish to maximize $A = xy$ subject to the constraint that $y = -\frac{1}{2}x+2$, and $x \in [0, 4]$. Substituting $-\frac{1}{2}x + 2$ in for y gives

$$A(x) = -\frac{1}{2}x^2 + 2x.$$

In order to maximize A, we find

$$A'(x) = -x + 2.$$

So the critical points for A are $x = 0, 2, 4$. We note that A is continuous on $[0, 4]$ and thus will have an absolute maximum among $A(0) = 0, A(2) = 2, A(4) = 0$. So, the largest rectangle that can be inscribed in this triangle has dimensions $x = 2$, $y = 1$, and has an area of 2.

Problem 6. **[4 points each]** Compute:

Answer:

(a) $\lim_{x\to 0} \dfrac{x\sin(x)}{\cos(x)-1} \stackrel{L}{=} \lim_{x\to 0} \dfrac{x\cos(x)+\sin(x)}{-\sin(x)} \stackrel{L}{=} \lim_{x\to 0} \dfrac{\cos(x)-x\sin(x)+\cos(x)}{-cos(x)} = -2.$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\,dx = \cos(x)\Big]_0^{\frac{\pi}{3}} = 1 - \frac{1}{2} = \frac{1}{2}.$

(c) $\dfrac{d}{dx}\sin(e^x) = e^x\cos(e^x).$

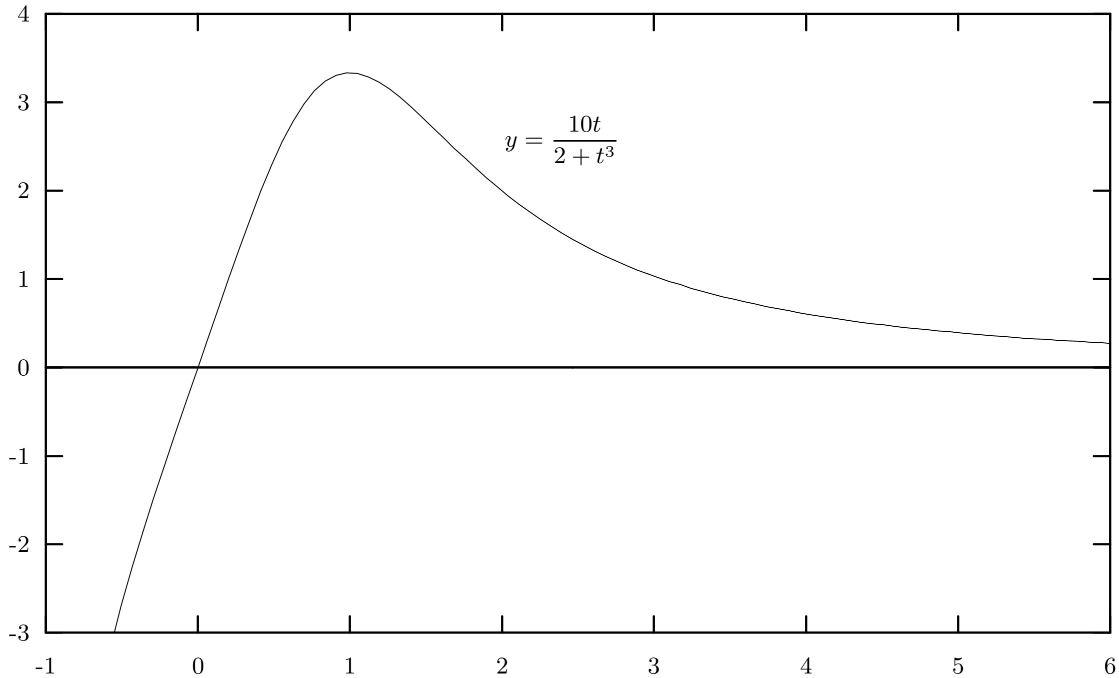
(d) $\displaystyle\int x^2+3x+5dx = \frac{x^3}{3} + \frac{3x^2}{2} + 5x + c.$

(e) $\lim_{x\to\infty} \dfrac{3x-e^x}{8x^2+\ln(x)} = -\infty.$

Problem 7. Let A be the function defined by

$$A(x) = \int_1^x \frac{10t}{2+t^3} dt \text{ for } t \geq -\sqrt[3]{2}.$$

This sketch might be helpful:



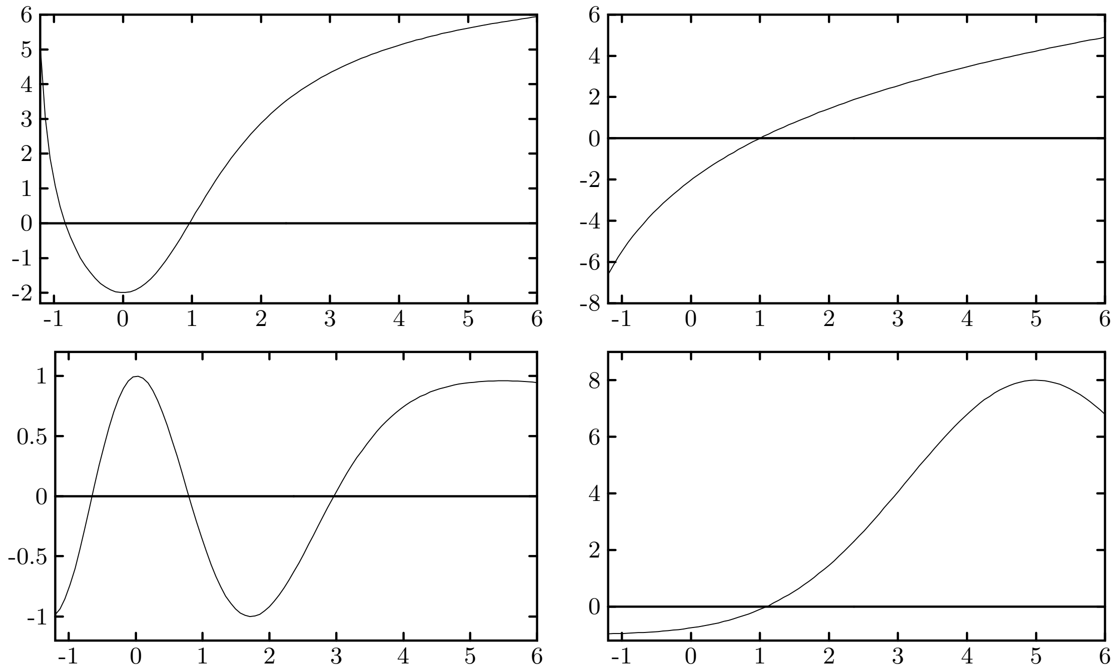
(a) **[6 points]** Approximate $A(3)$ using two circumscribed rectangles.

Answer:

$$A(3) \approx (height_1)(width_1) + (height_2)(width_2) = \left(\frac{10}{3}\right)(1) + \left(\frac{20}{10}\right)(1) = \frac{16}{3} = 5.33333\ldots.$$

Problem 7. (Continued.)

(b) **[6 points]** Which of the following is the graph of $y = A(x)$?



Answer:

The graph of A is the first one. Notice that $A'(x) = \frac{10x}{2+x^3}$. Hence, A is increasing for $x > 0$ and decreasing for $x < 0$. This makes our choice conclusive.

Name: ____________________________ Recitation: ____________ ID#: _________________

MAT131: Final Exam

Wednesday, December 20 2006
11:00am-1:30pm

Humanities 1003: R01, R03 and R05
Humanities 1006: R02 and R04
Harriman 137: LEC2
Old Chemistry 116: LEC3

Problems:	1	2	3	4	5	6	7	Total
Points:								

There are seven problems. Do all work on these pages. **No** calculators, cell phones or notes may be used. The point value (out 200) of each problem is marked in the margin. Except for problem 1, you are expected to write the appropriate computations and justifications to get full credit. For each question, please put a rectangle around your final answer.

(**10 pts**) **1.** For each of the question below, circle (**T**) if the statement is true and (**F**) if it is false. Each correct answer gives 1 point and each incorrect answer or unanswered question gives 0 point.

T **F** (a) If a function f has an antiderivative F, then F is necessarily continuous.

T **F** (b) If f is an even function on the interval $[-\pi, \pi]$, then necessarily

$$\int_{-\pi}^{\pi} f(x)dx = 0.$$

T **F** (c) $\displaystyle\sum_{i=1}^{20} i = 210.$

T **F** (d) The equation $x^2 + 2x + 1 = 0$ has a solution in the interval $[0, 1]$.

T **F** (e) If a function f is differentiable on an interval $[a, b]$, then f has a global minimum on $[a, b]$.

T **F** (f) If f and g are functions differentiable at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}, \quad \text{provided } g'(a) \neq 0.$$

T **F** (g) If $f(x) > 1$ for all $x \in [0, 1]$, then

$$\int_{0}^{1} f(x)dx \geq 0.$$

T **F** (h) If f is concave up on an open interval (a, b), then f necessarily has a global minimum on (a, b).

T **F** (i) If $f(x) = (x^3 + 1)^3$, then there exists a number $c \in (0, 1)$ such that

$$f'(c) = 7.$$

T **F** (j) If $\lim_{x \to a} f(x) = 3$ and $\lim_{x \to a} g(x) = 2$, then

$$\lim_{x \to a} (f(x) + g(x)) = 6.$$

2. Compute the following derivatives.

(5pts) **(a)** $\frac{d}{dx}\frac{\cos(e^x+x^2)}{2x}$

(5pts) **(b)** $\frac{dy}{dx}$ if $2(x^2+y^2)^2 = 25(x^2-y^2)$

(5pts) **(c)** $\frac{d}{dx}(\ln x)^{3x}$

(5pts) **(d)** $\frac{d}{dx}\int_x^{x+2} t^2 e^{t^2}\, dt$

3. Consider the function $f(x) = xe^{-\frac{x^2}{2}}$.

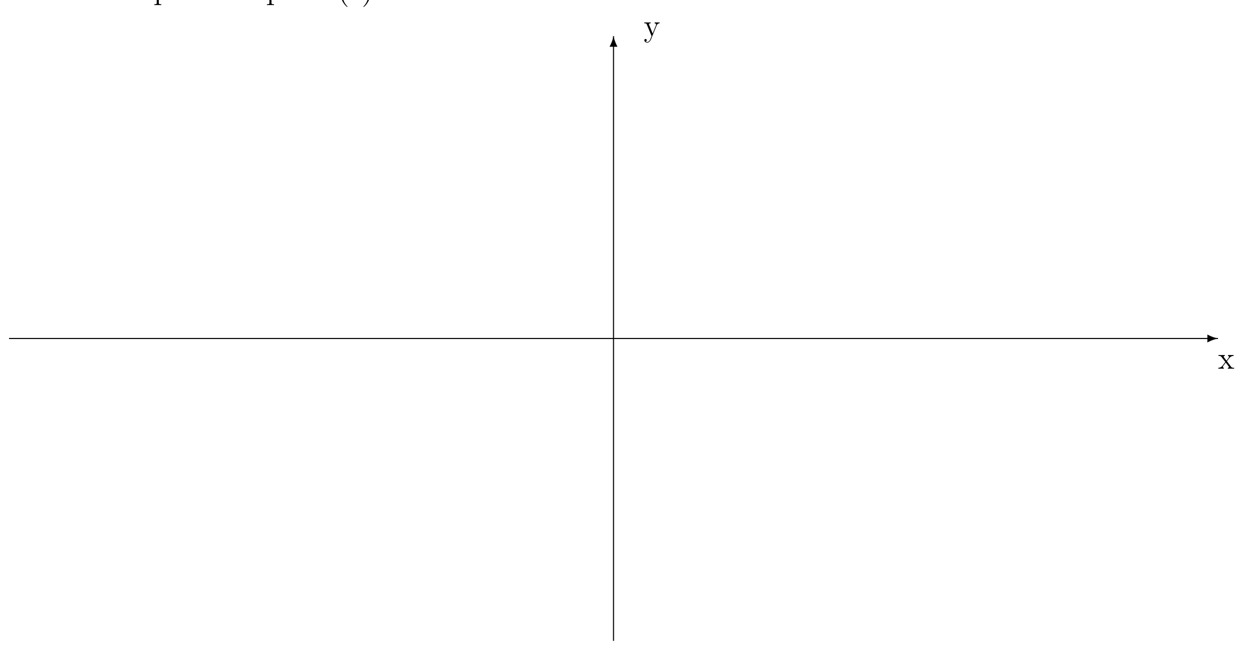
(5pts) **(a)** Is it an even function, an odd function or is it neither even nor odd?

(10pts) **(b)** Find the horizontal and vertical asymptote(s) of the function f.

(10pts) **(c)** Find the critical number(s), if any, and the intervals of increase and decrease of the function f.

(10pts) **(d)** Find the inflection point(s), if any, and the intervals of concavity of the function f.

(15pts) **(e)** From steps (a), (b), (c) and (d), draw the graph of the function f. Label all important point(s).



(40pts) **4.** A cheesemaker wants to make a cheese of volume $V = 6L$ (6 liters correspond to $6000cm^3$) shaped like a circular cylinder. This cheese will be cut into six equal pieces, each piece being obtained by cutting a sector of the cylinder of angle $\frac{\pi}{3}$. The cheesemaker would like to wrap each piece in plastic foil. What are the radius and the height of the cylinder that will minimize the amount of plastic foil needed?

5.

(10pts) **(a)** Use the linear approximation of the function $f(x) = \sqrt[3]{x}$ at $x = 1$ to get an approximation of the number $\frac{1}{\sqrt[3]{2}} = \sqrt[3]{\frac{1}{2}}$.

(10pts) **(b)** Use Newton's method with the equation $\frac{1}{x^3} - 2 = 0$ and starting with $x_1 = 1$ to find the third approximation x_3 of the number $\frac{1}{\sqrt[3]{2}}$.

(5pts) **(c)** Is the approximation in (b) bigger or smaller than the actual value of $\frac{1}{\sqrt[3]{2}}$?

6. Compute the following limits.

(5pts) **(a)** $\lim_{x \to 0^-} \frac{\cos^2 x - 1}{\sin(2x)}$

(5pts) **(b)** $\lim_{x \to +\infty} \frac{e^{-x}}{x}$

(5pts) **(c)** $\lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{3}{n} \left(\frac{3i}{n} + 1 \right)^3 \right)$

(5pts) **(d)** $\lim_{x\to 0^+} (\tan x)^x$

(5pts) **(e)** $\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

7. Compute the following integrals.

(5pts) **(a)** $\int \left(\cos(x) + \sec^2(x)\right) dx$

(5pts) (b) $\int \frac{(\ln x)^2}{x} dx$

(5pts) (c) $\int \frac{\arcsin x}{\sqrt{1-x^2}} dx$

(5pts) **(d)** $\int_1^3 (x-2)(x+3)dx$

(5pts) **(e)** $\int_0^{\frac{\pi}{4}} (\sec^2 x)(\tan x)dx$

(5pts) (f) $\int_{-3}^3 \sqrt{1-\frac{x^2}{9}}dx$

Recitations of MAT131

Recitation	Time	Location	TA
R01	TuTh 12:50pm- 1:45pm	Earth and Space 183	Disconzi, Marcelo
R02	MF 2:20pm- 3:15pm	Earth and Space 79	Cheng, Jonathan
R03	TuTh 8:20am- 9:15am	Earth and Space 181	Disconzi, Marcelo
R04	WF 11:45am-12:40pm	Earth and Space 181	Lehrer, Raquel
R05	MF 2:20pm- 3:15pm	Harriman Hll 115	Jaggi, Amit
R06	MW 10:40am-11:35am	Earth and Space 183	Weng, Luoying
R07	MW 6:50pm- 8:10pm	Earth and Space 183	Li, Yinghua
R08	MW 3:50pm- 4:45pm	Chemistry 128	Dutta, Satyaki
R09	TuTh 5:20pm- 6:15pm	Lgt Engr Lab 154	Hassan, Mohamed
R10	WF 9:35am-10:30am	Earth and Space 183	Weng, Luoying
R11	TuTh 3:50pm- 4:45pm	Lgt Engr Lab 154	Hassan, Mohamed
R12	TuTh 8:20am- 9:15am	Library N4000	Fong, Chee
R13	MF 12:50pm- 1:45pm	Library N4006	Li, Yinghua
R14	TuTh 3:50pm- 5:10pm	Physics P125	Fong, Chee

MAT131 Review for the final

1. Evaluate the integral

   $$a.\ \int_1^8 (t^{2/3} - 2t^{4/3})dt$$

   $$b.\ \int_0^{\pi} (-\sin x + \cos x)dx$$

   $$c.\ \int_{-1}^{0} \frac{7}{\sqrt{1-x^2}}dx$$

2. A rectangular playground is to be fenced off and divided into two by another fence parallel to one side of the playground. Six hundred feet of fencing is used. Find the dimension of the playground that will enclose the greatest total area.
3. For the function f(x)=x/(1+x)
   a. Find the vertical and horizontal asymptotes
   b. Find the intervals of increase or decrease.
   c. Find the local maximum and minimum values
   d. Find the intervals of concavity and the inflexion points
   e. Use the information from parts a. to d., to sketch the graph of f.
4. Find the linear approximation of the function $f(x)=(2+x)^{3/2}$ at x=2 and use it to approximate the numbers $(4.1)^{3/2}$, $(3.9)^{3/2}$ and f(2.2).
5. Let f be a function such that f(1)=-3 and $f'(x)=(2+x^3)^{-1}$. Use linear approximation to estimate the value of f(1.3).

6. Differentiate

$a.\ \ y = \dfrac{e^x \cos x}{3x - \sin x}$

$b.\ \ y = \cos(\ln x)$

$c.\ \ y = \ln(\cos x)$

$d.\ y = 3^x$

$e.\ \ y = \tan^{-1}(2x - 4)$

$f.\ \ y = (\sin(x^2))^{3x}$

$g.\ \ y = \sqrt[3]{\dfrac{x^2 + 2x}{x^5 - 27}}$

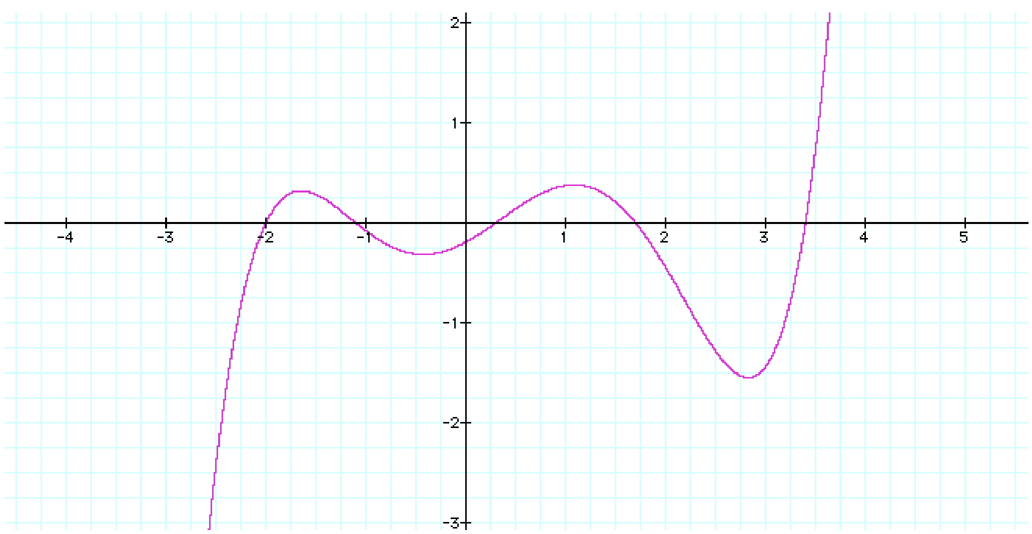
$h.\ \ y = |\, x \,|$

$i.\ \ y = \ln |\, x \,|$

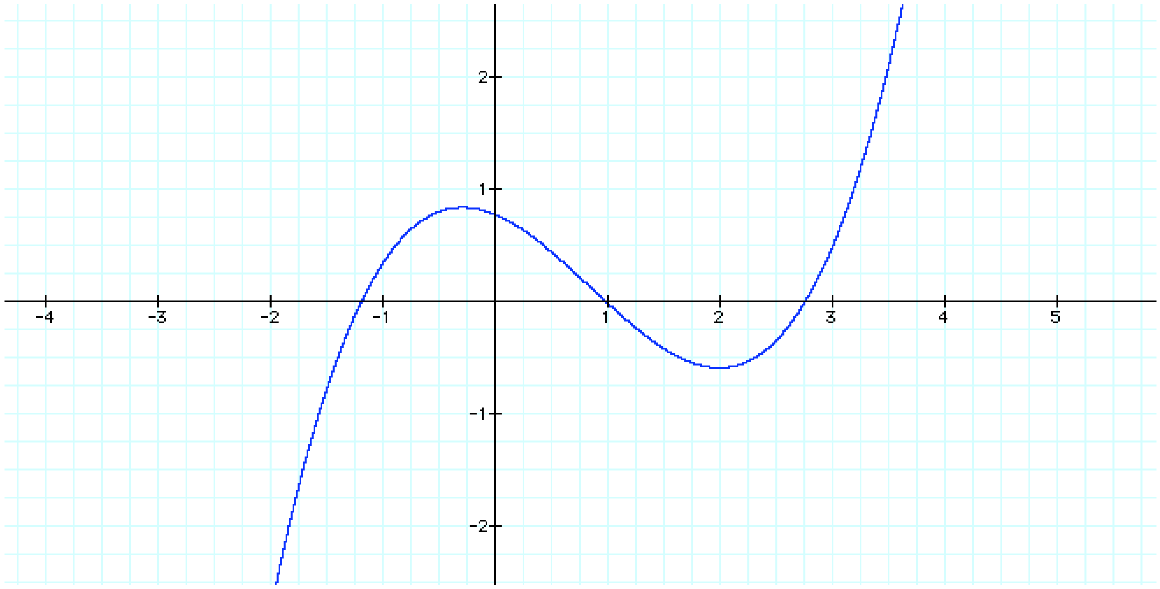
$j.\ \ y = \sin(\ln |\, x \,|)$

7. Consider the curve √x+√y=3
   a. Find y' and y'' by implicit differentiation.
   b. Find an equation of the tangent line at the point $(2,(3-\sqrt{2})^2)$
   c. Find all the points on the curve where the tangent has slope –1/2.

8. The graph of f is given. Sketch a graph of the derivative of f.



9. The graph of the derivative of f, is given. Sketch the graph of f if f(3)=0.5.



10. Sketch the graph of a function G with the following properties

a. The domain of G is the interval [-4,5]

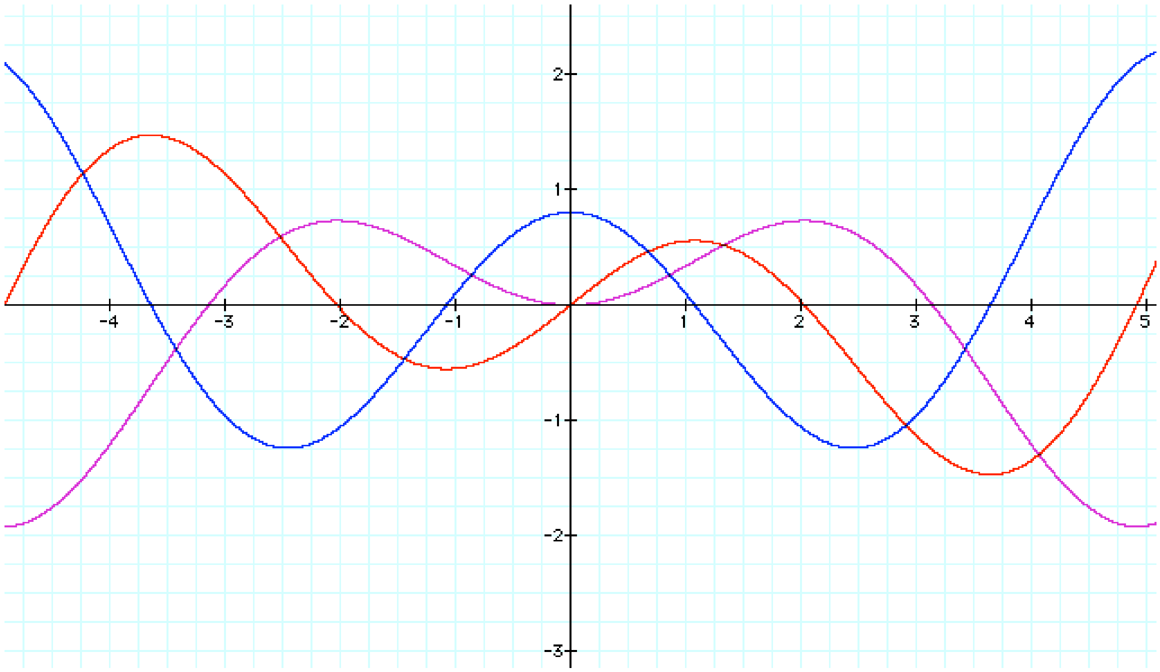
b. $G(0)=1$ $G'(0)=1$

c. $G(1.2)=1.7$, (1 point) $G'(1.2)=0$

d. $G'(x)>0$ on (-2.5,1.2)

e. $G''(x)<0$ on (-0.8,4]

11. Determine which is f, f'' and f''



12. Evaluate the limit, if it exists. If you use the fact that the function is continuous to evaluate the limit (that is, to "plug in" the value), state it. If the limit it does not exist, explain why.

a. $\lim_{x \to 3} e^{x^3 - 3x^2}$

b. $\lim_{x \to -4} \dfrac{x^2 - 16}{x^2 - 7x + 12}$

c. $\lim_{x \to 10} \dfrac{3 - \sqrt{x-1}}{x - 10}$

d. $\lim_{x \to \infty} \dfrac{3x^2 - x + 2}{2x^2 + 5x^2 + 4}$

e $\lim_{x \to -\infty} \dfrac{x^5 + 4x}{-x^6 + 15x^2 + 41}$

f $\lim_{x \to \infty} \dfrac{x^7 + 4x + 12}{-x^2 - 15x^3 + 31x}$

13. Consider the function $f(x) = \begin{cases} -x + 2 & if \ x < -3 \\ x^2 - c & if \ -3 \le x < 2 \\ x + 1 & if \ x \ge 2 \end{cases}$

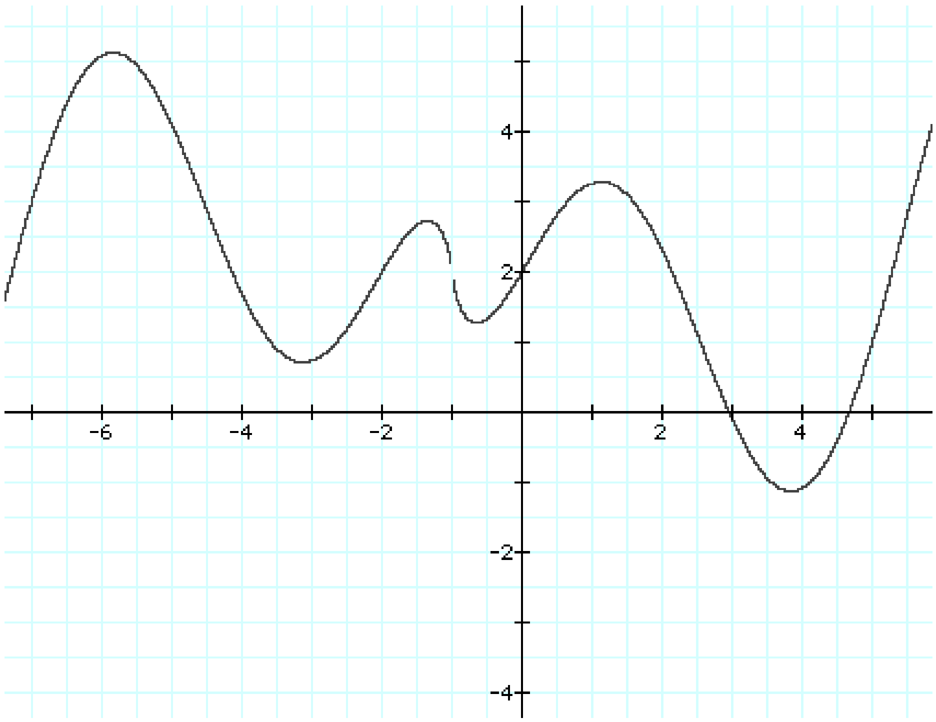
a. Sketch the graph of f for c=-2. Find the numbers at which f is discontinuous when c=-2.

b. Find the values of c for which f is continuous at all real numbers.

14. Use the intermediate value theorem to show that $f(x) = \frac{1}{2}x^3 - x^2 - x + 1$ the equation f(x)=0 has at least three solutions. For each of these solutions, find an interval of length at most 1 that contains it.

15. A particle moves with acceleration function $a(t) = 6 + 4t - 3t^2$. Its initial velocity is v(0)=2m/s ad its initial displacement is s(0)=5m. Find its position after 2 seconds.

16. The graph of f(x) is given.



a. Shade the area represented by $\int_{-4}^{-2} f(x)dx$.

b. List the numbers $\int_{-4}^{-2} f(x)dx,\ \int_{-6}^{-4} f(x)dx,\ and\ \int_{4}^{5} f(x)dx,$ in increasing order. Explain your reasoning.

c. Sketch a graph of a tangent line to the curve y=f(x) at x=4.

d. List the numbers f'(-6), f'(-4), and f'(-2) in increasing order. Explain your reasoning

It is strongly advisable that you solve more exercises than the ones on these pages before the final, with emphasis in the ones that you have more difficulty.

You can find more exercises in the book. Also, in the following webpages, you will find all kind of solved problems. http://www.math.ucdavis.edu/~kouba/ProblemsList.html

In particular, in http://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/maxmindirectory/MaxMin.html you'll find solved problems about optimization. In

http://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/graphingdirectory/Graphing.html you'll find solved problems about graphing functions using first and second derivatives.

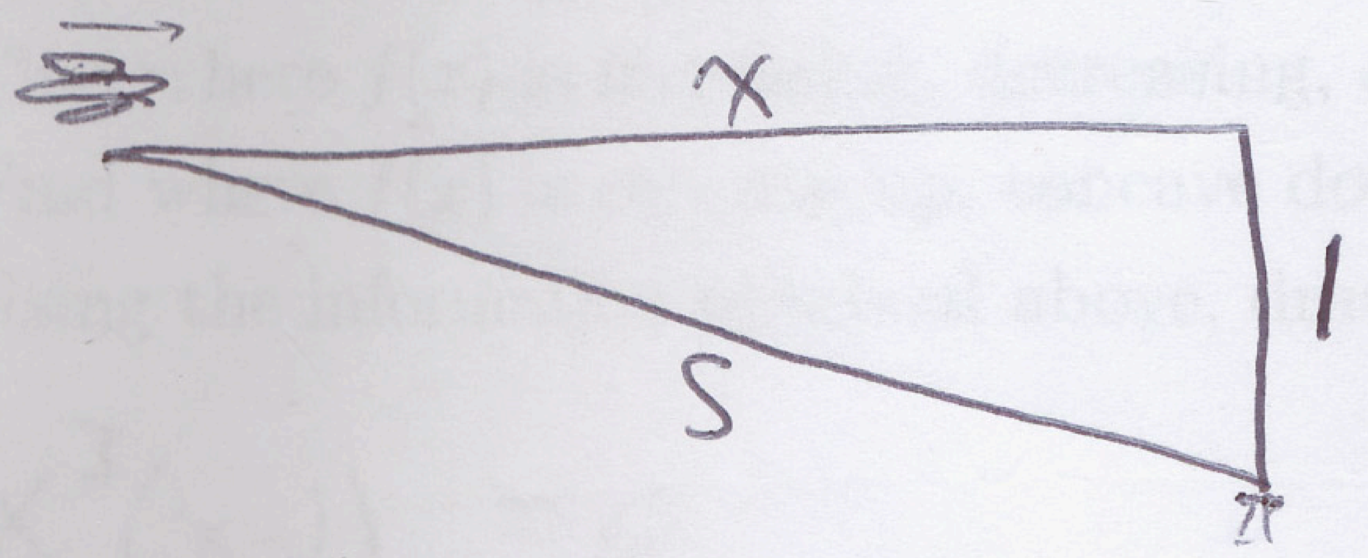
Another link with exercises is: http://archives.math.utk.edu/visual.calculus/

Name ANSWER KEY Section ____________

Math 131 Extra Credit Quiz

Clearly show all work! You may leave answers in radical or fraction form.

Problem 1 An airplane is flying horizontally at an altitude of 1 mile. A radar station (located on the ground) measures the distance between the airplane and the station. When the distance is 2 miles, the distance is decreasing at a rate of 500 miles per hour. What is the airplane's (horizontal) speed?



What is $\frac{dx}{dt}$?

Know: When $s = 2$, $\frac{ds}{dt} = -500$

$$x^2 + 1^2 = s^2$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

When $s = 2$

$$x^2 + 1^2 = 2^2$$

$$x^2 = 3$$

$$x = \sqrt{3}$$

$$\Rightarrow \frac{dx}{dt} = \frac{(2)}{(\sqrt{3})}(-500) = \frac{-1000}{\sqrt{3}}$$

$\boxed{\text{The plane's horizontal speed is } \frac{1000}{\sqrt{3}} \text{ miles/hr}}$

$= \frac{1000\sqrt{3}}{3}$ mph.

Problem 2 You are given the function $f(x)$ and its derivatives.

$$f(x) = x^3(x-2)$$
$$f'(x) = 2x^2(2x-3)$$
$$f''(x) = 12x(x-1)$$

a • Find the x-intercepts of $f(x)$.

b • Find where $f(x)$ is increasing, decreasing, and where it has local extrema.

c • Find where $f(x)$ is concave up, concave down, and where it has inflection points.

d • Using the information obtained above, draw the graph of $f(x)$.

a) $f(x) = x^3(x-2) = 0$

$x^3 = 0$ or $x - 2 = 0$

$\underline{x = 0, 2}$

$\boxed{\underline{x\text{-int}} \quad x = 0, 2}$

b) $f' = 0 = 2x^2(2x-3)$

$2x^2 = 0$; $2x - 3 = 0$

$x = 0, 3/2$ ← Critical Points

$-$ $\quad$ $-$ $\quad$ Min $\quad$ $+$ $\quad$ f'

$-\infty$ $\quad$ 0 $\quad$ $3/2$ $\quad$ $+\infty$

$f'(-1) = + \cdot - = -$ $\quad$ $f'(1) = +(-1) = -$ $\quad$ $f'(2) = ++ = +$

f is

$(-\infty, 0), (0, 3/2)$ Decreas. ~~Increas.~~

$(3/2, +\infty)$ Incr.

Rel Min at $x = 3/2$.

Rel Extrema occurs when f' changes sign.

c) $f'' = 0 = 12x(x-1)$

$x = 0, 1$ ← Crit pts for concavity

$+$ IP $\quad$ $-$ IP $\quad$ $+$ $\quad$ f''

$-\infty$ $\quad$ 0 $\quad$ 1 $\quad$ $+\infty$

$f''(-2) = -- = +$ $\quad$ $f'(\tfrac{1}{2}) = +- = -$ $\quad$ $f''(2) = ++ = +$

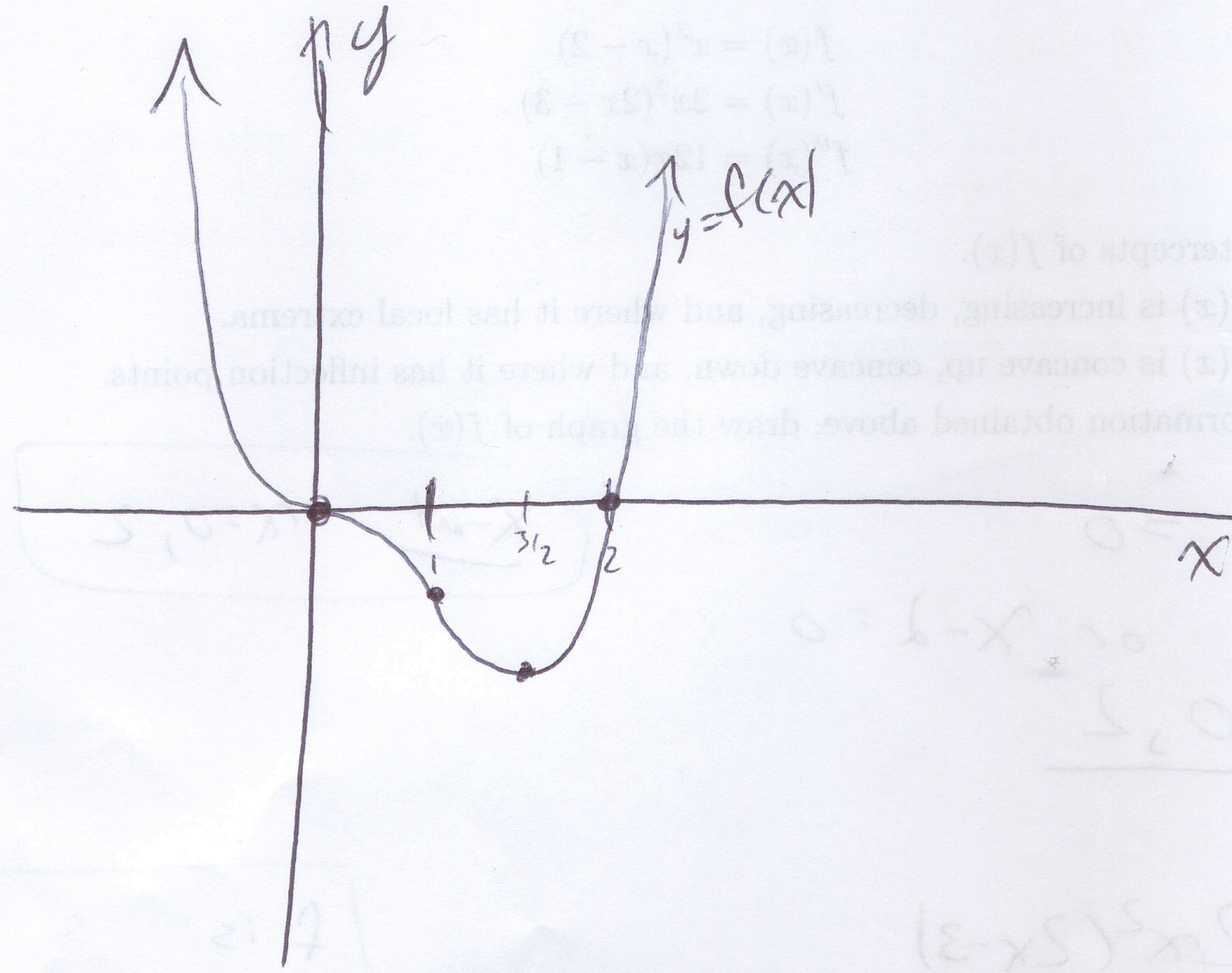
Conc Up $(-\infty, 0), (1, +\infty)$

Conc Down $(0, 1)$

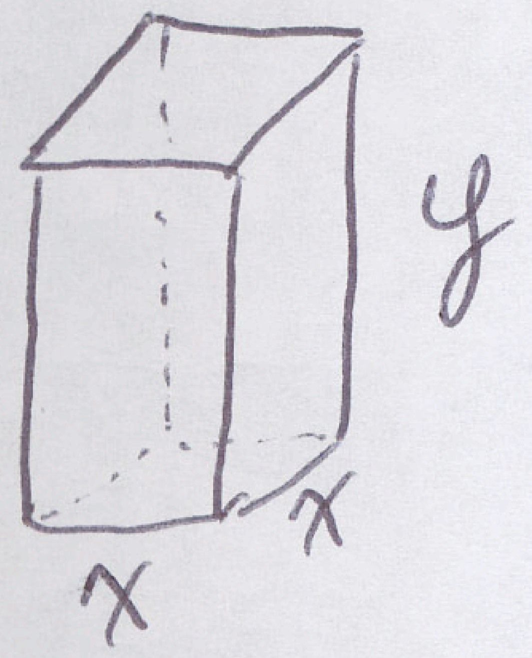
Infl Points at $x = 0, 1$.

Graph on Next Page

e)



Problem 3 A company wishes to ship crates containing 8 cubic meters of goods (i.e. the volume is 8 m^3). They decide the crates should be rectangular solids with a square base. To minimize cost, they want to build the boxes using the smallest amount of material (i.e. the smallest surface area). What should be the dimensions of the crates?



$V = x^2 y = 8$

$A = 2x^2 + 4xy$

Want to minimize A

$x^2 y = 8 \Rightarrow \underline{y = 8/x^2}$

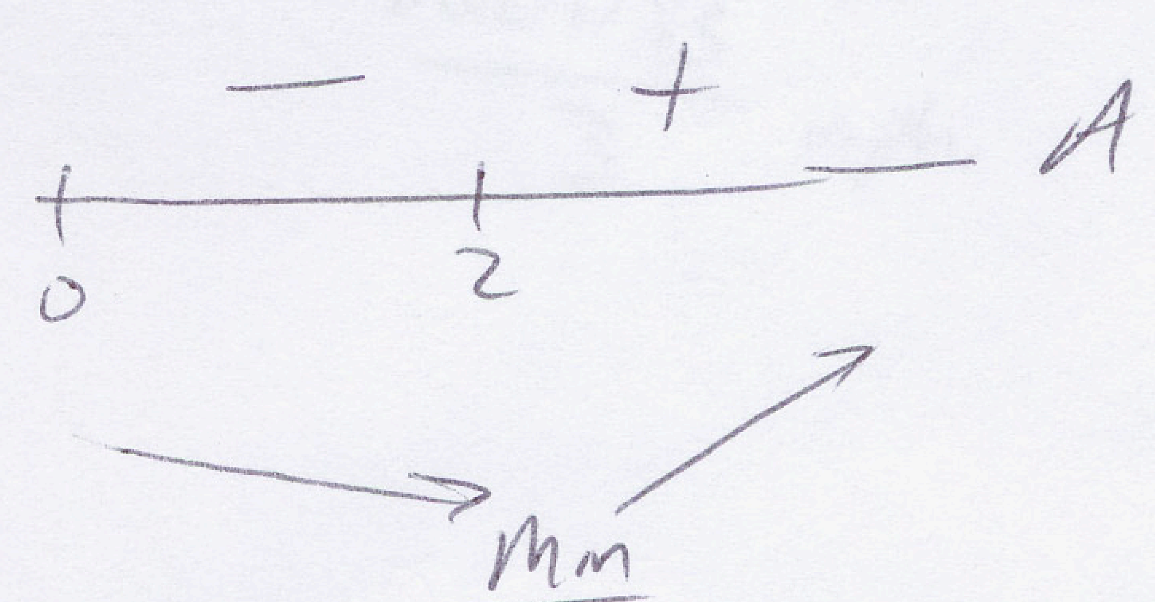
$$A = 2x^2 + 4xy = 2x^2 + 4x(8/x^2)$$
$$= 2x^2 + 32x^{-1}$$

$$0 = \frac{dA}{dx} = 4x - 32x^{-2}$$

$4x = 32/x^2$

$x^3 = 8$

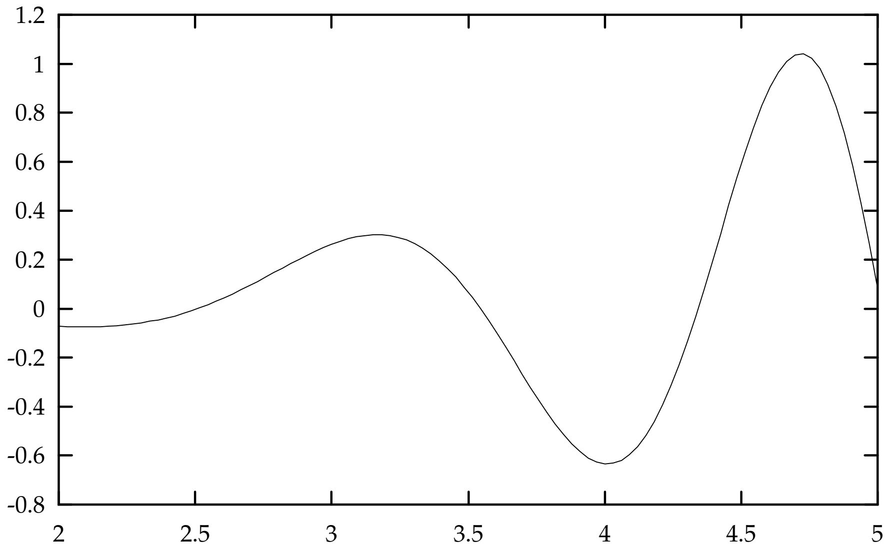
$\underline{x = 2}$



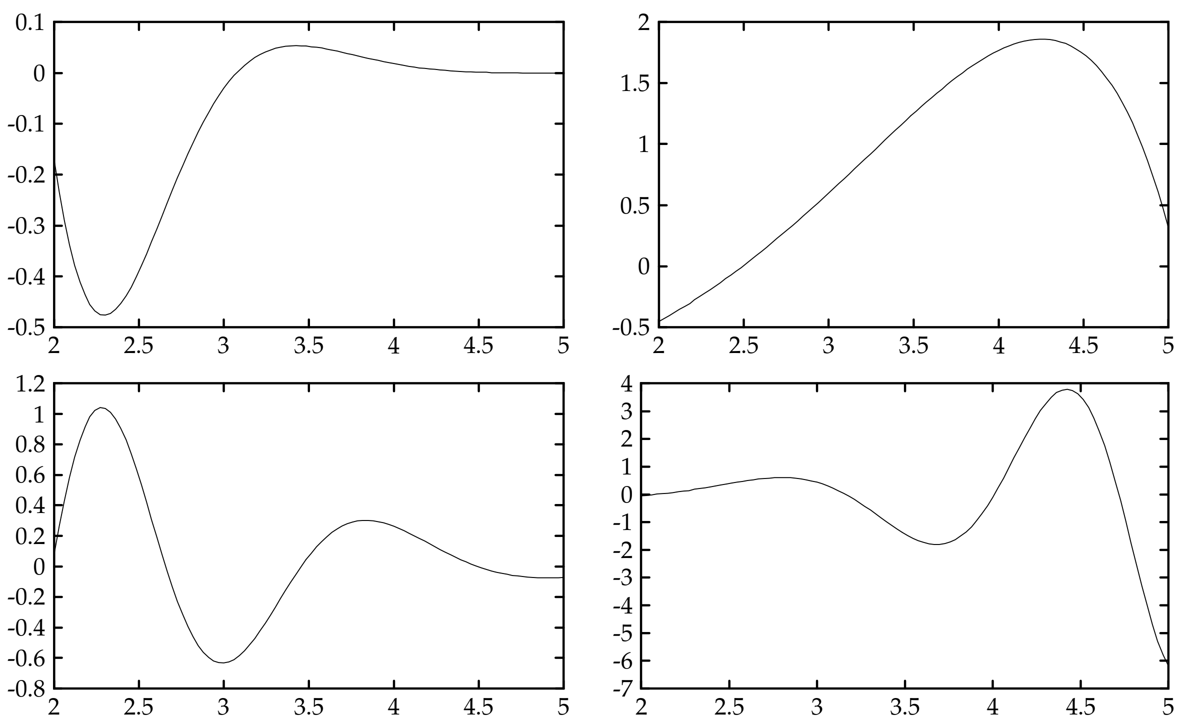
If $x=2$, then $y = 8/2^2 = 2$

$\Rightarrow$ Dimensions of $2' \times 2' \times 2'$ minimize surface area.

Problem 1. Below is a sketch of the graph of a function f.



Which is the graph of f'?



Problem 2. Let f and g be two functions satisfying $f(2) = 5$, $f'(2) = 2$, $g(2) = -3$, and $g'(2) = 4$. Find

(a) $\displaystyle \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$

(b) $\displaystyle \lim_{x \to 2} f(x)$

(c) $\displaystyle \left(\frac{f}{g}\right)'(2)$

(d) $\displaystyle \lim_{h \to 0} \frac{f(2+h)g(2+h) + 15}{h}$

Problem 3. Compute

(a) $g'(x)$ if $g(x) = \dfrac{x^5 \cos(x)}{1 + x + x^8}$.

(b) $\dfrac{d^2 y}{dx^2}$ if $y = e^{\sin(x)}$.

(c) $\lim_{x \to 0} \dfrac{\ln(1 + x)}{x}$.

(d) $\lim_{x \to 0} \dfrac{\cos(x) - 1}{\sin(x)}$.

Problem 4. Suppose that air is being pumped into a spherical ballon at a rate of 20cm^3 per second.

(a) How fast is the radius growing when the volume is $1000\pi\text{cm}^3$?

(b) How fast is the surface area growing at this time?

Problem 5. Consider a function A with the following properties:

- $A(1) = \frac{\pi}{4}$,
- $\lim_{x\to\infty} A(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} A(x) = -\frac{\pi}{2}$
- $A'(x) = \frac{1}{1+x^2}$ for all $x \in (-\infty, \infty)$.

(a) There isn't a nice formula for $A(x)$ so, it is not possible to determine $A(1.2)$ exactly. Use an approximation by differentials to estimate $A(1.2)$.

(b) On what interval is the graph of A concave down?

(c) Multiple choice: circle the correct one. $\frac{d}{dx}\left(A\left(2\frac{\sin(x)}{\cos(x)}\right)\right) =$

$$\frac{2}{1+3\sin^2(x)} \qquad \frac{2\sec(x)\tan(x)}{1+x^2} \qquad 4x^2 \qquad \frac{1}{1+4\tan^2(x)}$$

EXAM

Sample Midterm 2

Math 131

November 3, 2003

EXAM

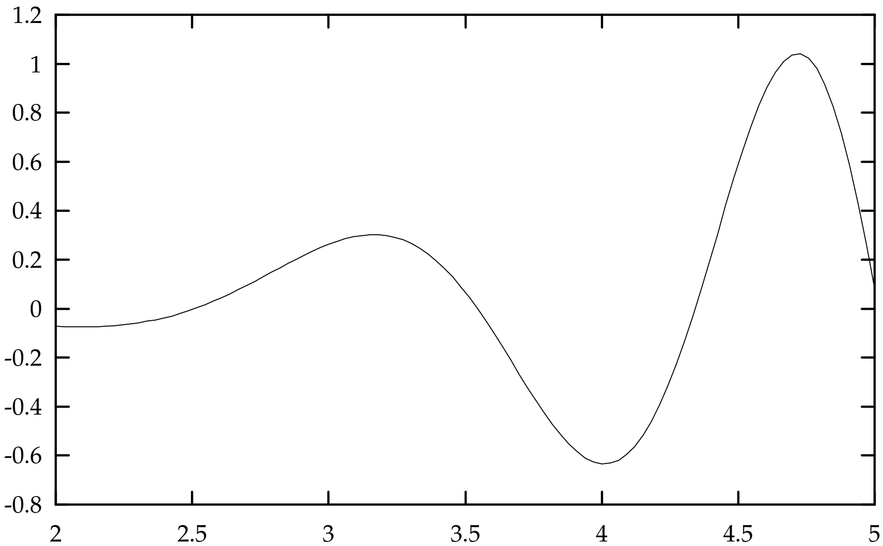
Sample Midterm 2

Math 131

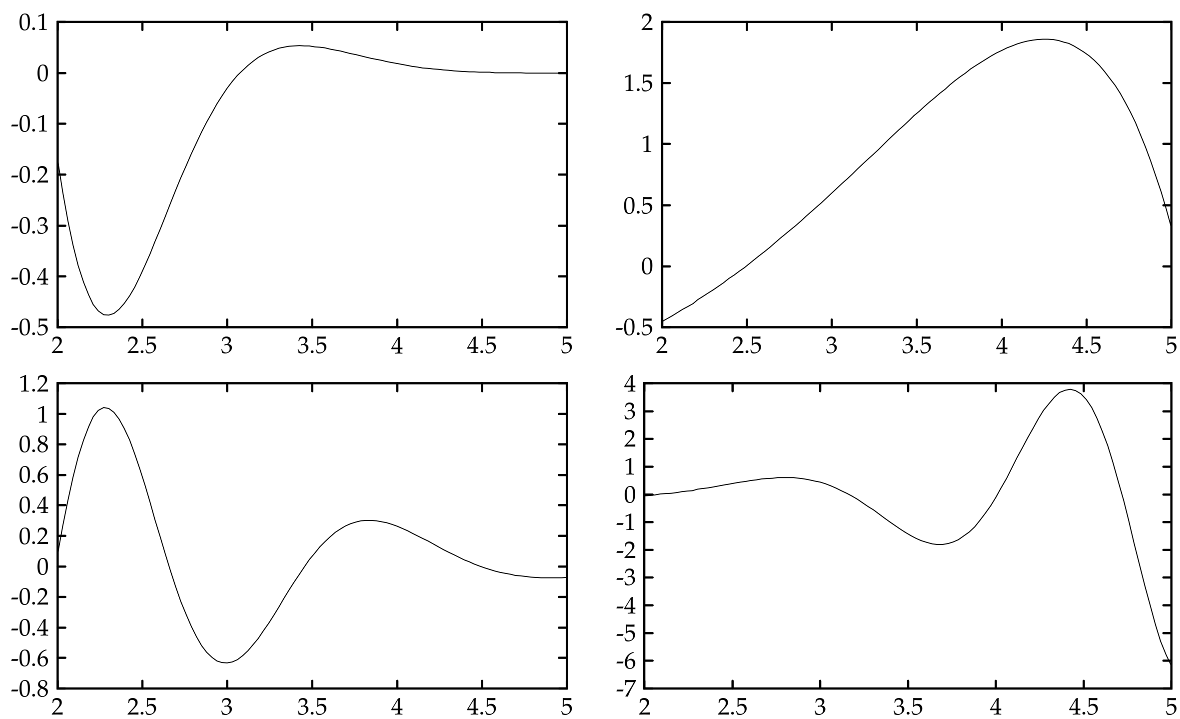
November 4, 2003

ANSWERS

Problem 1. Below is a sketch of the graph of a function f.



Which is the graph of f'?



Answer:
The graph of f' is the one on the bottom right. There are many ways to tell. For one, f is decreasing from, say 3.2 to 3.9. So, $f'(x) < 0$ for $x \in (3.2, 3.9)$; and there's only one graph that has that property.

Problem 2. Let f and g be two functions satisfying $f(2) = 5$, $f'(2) = 2$, $g(2) = -3$, and $g'(2) = 4$. Find

Answer:

(a) $\displaystyle\lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = g'(2) = 4.$

(b) $\displaystyle\lim_{x\to 2} f(x) = 5$. This requires a sentence of explanation. Because f is differentiable at 2, f is continuous at 2. Thus, $\displaystyle\lim_{x\to 2} f(x) = f(2) = 5$.

(c) $\displaystyle\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \frac{(-3)(2)-(5)(4)}{(-3)^2} = -\frac{26}{9}.$

(d) $\displaystyle\lim_{h\to 0} \frac{f(2+h)g(2+h)+15}{h} = (fg)'(2) = f'(2)g(2) + f(2)g'(2) = -6+20 = 14.$

Problem 3. Compute

Answer:

(a) $g'(x)$ if $g(x) = \dfrac{x^5\cos(x)}{1+x+x^8}$.

$$g'(x) = \frac{(1+x+x^8)\left(5x^4\cos(x) - x^5\sin(x)\right) - \left(x^5\cos(x)\right)(1+8x^7)}{(1+x+x^8)^2}.$$

(b) $\dfrac{d^2y}{dx^2}$ if $y = e^{\sin(x)}$.

$$\frac{dy}{dx} = \cos(x)e^{\sin(x)} \text{ and } \frac{d^2y}{dx^2} = -\sin(x)e^{\sin(x)} + \cos^2(x)e^{\sin(x)}.$$

(c) $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x}$.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \ln'(1) = \frac{1}{1} = 1.$$

(d) $\displaystyle\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)}$.

$$\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)} = \lim_{x\to 0}\frac{\cos(x)-1}{x}\,\frac{x}{\sin(x)} = (0)\left(\frac{1}{1}\right) = 0.$$

Problem 4. Suppose that air is being pumped into a spherical ballon at a rate of 20cm^3 per second.

(a) How fast is the radius growing when the volume is $1000\pi\text{cm}^3$?

Answer:

Note that for all times, we have $V = \frac{4}{3}\pi r^3$. Differentiating with respect to time t, gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Note that when the volume is equal to $1000\pi\text{cm}^3 = \frac{4}{3}\pi r^3$, we have $r^3 = 750 \Rightarrow r = \sqrt[3]{750}\text{cm}$. Substituting $20cm^3$ per second in for $\frac{dV}{dt}$ and $r = \sqrt[3]{750}$ gives

$$20\text{cm}^3/\text{sec} = 4\pi \left(\sqrt[3]{750}\right)^2 \text{cm}^2 \frac{dr}{dt}.$$

Thus,

$$\frac{dr}{dt} = \frac{5}{\pi \left(\sqrt[3]{750}\right)^2} \text{cm/sec}.$$

(b) How fast is the surface area growing at this time?

Answer:

We have $S = 4\pi r^2$ and differentiating with respect to time t gives

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Substituting the numbers

$$\frac{dr}{dt} = \frac{5}{\pi \left(\sqrt[3]{750}\right)^2} \text{cm/sec and } r = \sqrt[3]{750}\text{cm}$$

give

$$\frac{dS}{dt} = \frac{40}{\sqrt[3]{750}} \text{cm}^2/\text{sec}.$$

Problem 5. Consider a function A with the following properties:

- $A(1) = \frac{\pi}{4}$,
- $\lim_{x\to\infty} A(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} A(x) = -\frac{\pi}{2}$
- $A'(x) = \frac{1}{1+x^2}$ for all $x \in (-\infty, \infty)$.

(a) There isn't a nice formula for $A(x)$ so, it is not possible to determine $A(1.2)$ exactly. Use an approximation by differentials to estimate $A(1.2)$.

Answer:

We have

$$A(1.2) \simeq A(1) + A'(1)(0.2) = \frac{\pi}{4} + \left(\frac{1}{1+1^2}\right)(0.2) = \frac{\pi}{4} + 0.1 = .885398\ldots.$$

(b) On what interval is the graph of A concave down?

Answer:

The graph of A will be concave down when $A''(x) = \frac{-2x}{(1+x^2)^2} < 0$. That is, when $x > 0$.

(c) Multiple choice: circle the correct one. $\frac{d}{dx}\left(A\left(2\frac{\sin(x)}{\cos(x)}\right)\right) =$

$$\frac{2}{1+3\sin^2(x)} \qquad \frac{2\sec(x)\tan(x)}{1+x^2} \qquad 4x^2 \qquad \frac{1}{1+4\tan^2(x)}$$

Answer:

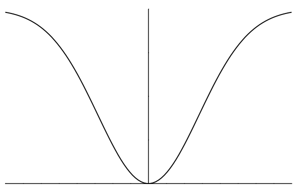
$$\begin{aligned}\frac{d}{dx}\left(A\left(2\frac{\sin(x)}{\cos(x)}\right)\right) &= \left(\frac{1}{1+\left(2\frac{\sin(x)}{\cos(x)}\right)^2}\right)\left(2\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\right)\\ &= \frac{2}{\cos^2(x)+4\sin^2(x)}\\ &= \frac{2}{1+3\sin^2(x)}.\end{aligned}$$

MAT 131 – CALCULUS II – FALL, 1999
MIDTERM EXAMINATION II

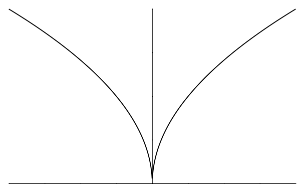
November 10, 1999

Name:	**ID Number:**	**Section:**

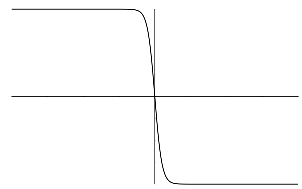
Problem 1. (15 points) Here are the graphs of five functions.



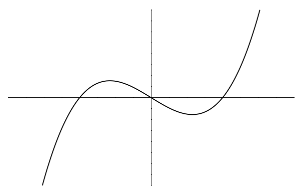
(a) ________



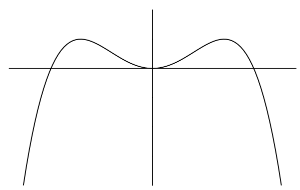
(b) ________



(c) ________

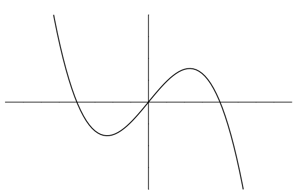


(d) ________

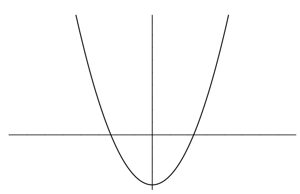


(e) ________

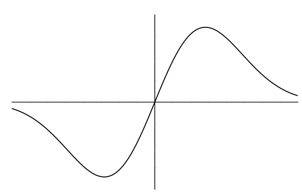
For each of the graphs above, indicate which of the following graphs is its derivative.



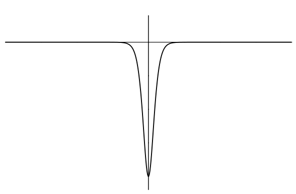
(1)



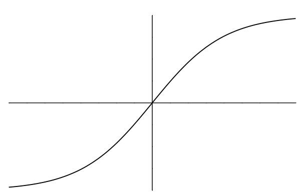
(2)



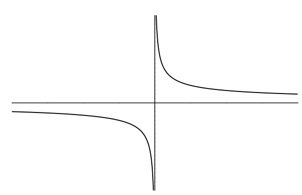
(3)



(4)



(5)



(6)

Problem 2. (30 points) Calculate the derivatives of the following functions.

(a) $f(x) = 7^x$:

(b) $g(x) = \sin(3x)$:

(c) $h(x) = x^{17} - 3x^4 + 6x + 5$:

(d) $k(x) = \tan(2x)$:

(e) $f(x) = \cos(x)\sin(x)$:

(f) $g(x) = \dfrac{3x^2}{x^3 - 17}$:

(g) $h(x) = x^2 e^{x-1}$:

(h) $k(x) = \sec(2^x)$:

(i) $f(x) = \dfrac{\sin(6x)}{\sin(7x)}$:

(j) $k(x) = (x^3 + x - 1)^{101}$:

(k) $F(t) = Ae^{kt} + Be^{-kt}$:

(l) $G(y) = \sqrt{1 + \sin(y/2)}$:

(m) $H(x) = 3x^e$:

(n) $K(z) = e^{\cos z}$:

(o) $R(c) = Kc(r - c)$:

Problem 3. (5 points) Which one of the following limits gives $f'(2)$, where $f(x) = \sin(\pi e^x)$?

(a) $\displaystyle\lim_{x\to 2} \frac{\sin(\pi e^x) - \sin(2\pi)}{h}$

(b) $\displaystyle\lim_{h\to 0} \frac{\sin(\pi e^2 e^h) - \sin(\pi e^2)}{h}$

(c) $\displaystyle\lim_{h\to 0} \frac{\sin(\pi e^x) - \sin(\pi e^a)}{h}$

(d) $\displaystyle\lim_{h\to 0} \frac{\sin(\pi(e^x + e^h)) - \sin(\pi e^x)}{h}$

Problem 4. (10 points) Find the equation of the line tangent to the graph of the function $f(x) = 1/(1+x^2)$ at $x = 2$.

Problem 5. (10 points) Let $g(z) = -2z^3 - 3z^2 + 2$.

(a) On what interval(s) is g increasing?

(b) On what interval(s) is g concave up?

Problem 6. (10 points) Suppose that the deer population $P(t)$ (*i.e.*, number of deer) in a certain game preserve varies periodically throughout the year according to the formula

$$P(t) = 25\sin(\pi(t-5)/6) + 75,$$

where t denotes the number of months since January 1.

(a) What is the deer population on September 1?

(b) On June 1, is the population growing or shrinking?

(c) On June 1, how fast is the population changing? (Be sure to specify the units for your answer. A decimal approximation to your answer is not necessary.)

Problem	1	2	3	4	5	6	**Total**
Score							

Name: ______________________ Recitation: __________ ID#: ______________

MAT131: Midterm II
Solutions

Thursday, November 9 2006
8:30pm-10pm

Javits 100: LEC I and LEC III
Old Engineering 143: LEC II

Problems:	1	2	3	4	5	Total
Points:						

There are five problems. Do all work on these pages. **No** calculators, cell phones or notes may be used. The point value (out 100) of each problem is marked in the margin. Except for problems 1 and 4, you are expected to write the appropriate computations and justifications to get full credit. For each question, please put a rectangle around your final answer.

(10 pts) 1. For each of the question below, circle (**T**) if the statement is true and (**F**) if it is false. Each correct answer gives 1 point and each incorrect answer or unanswered question gives 0 point.

T **F** (a) A function may have a global maximum at three different numbers.

T **F** (b) If a function f is such that $f'(2) = 0$, then it necessarily has a local minimum or maximum at $x = 2$.

T **F** (c) $\lim\limits_{h\to 0} \frac{2^h-1}{h} = \log_2 e$.

T **F** (d) If f and g are two functions differentiable at $x = 15$, then their quotient $\frac{f}{g}$ is also differentiable at $x = 15$.

T **F** (e) $\frac{d^{31}}{dx^{31}} \cos x = \sin x$.

T **F** (f) $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$.

T **F** (g) $\frac{d}{dx} \ln(-e^x) = 1$.

T **F** (h) The linear approximation of $\sqrt{x+3}$ at $x = 6$ is

$$L(x) = 3 + \frac{x}{6}.$$

T **F** (i) If f is a continuous function on the interval $[0, 1]$, then f has a global minimum on this interval.

T **F** (j) If f and g are differentiable functions at $x = 3$, then their composite $f \circ g$ is differentiable at $x = 3$.

2. Compute the derivatives of the following functions.

(5pts) (a) $\frac{e^x \cos x}{x^3+1}$

(5pts) (b) $\sin(x^5 + 1)$

(5pts) (c) $x^4 - 23\cos(3x) + \ln(x^2)$

(5pts) (d) $(2x)^{x\cos(x)}$

(5pts) (e) $\arctan(\ln x)$.

(5pts) (f) $\frac{e^{sinx}(x^2+2)^5}{\sqrt{x}(\tan x)^2}$

3. Consider the parametric curve $\mathcal{C}$ given by the equation

$$x(t) = e^{2t} - e^{-2t}, \quad y(t) = e^{2t} + e^{-2t}.$$

Alternatively, it is given implicitly by the equation: $x^2 - y^2 = -4, \quad y > 0$.

(10pts) (a) Compute the slope $\frac{dy}{dx}$ of the tangent to the curve $\mathcal{C}$ at the point $(x(t_0), y(t_0))$ for $t_0 \in \mathbb{R}$.

(10pts) (b) The curve $\mathcal{C}$ intersects the ellipse $\mathcal{E}$ given by the equation $\frac{x^2}{9} + \frac{y^2}{25} = \frac{1}{2}$ at the points $(-\frac{3}{2}, \frac{5}{2})$ and $(\frac{3}{2}, \frac{5}{2})$. By computing the slope of the tangent of $\mathcal{E}$ at these points, check whether or not the curves $\mathcal{C}$ and $\mathcal{E}$ are orthogonal to each other.

(20pts) **4.** Sketch the graph of a function f with domain $(-2, 2)$ satisfying the following conditions:

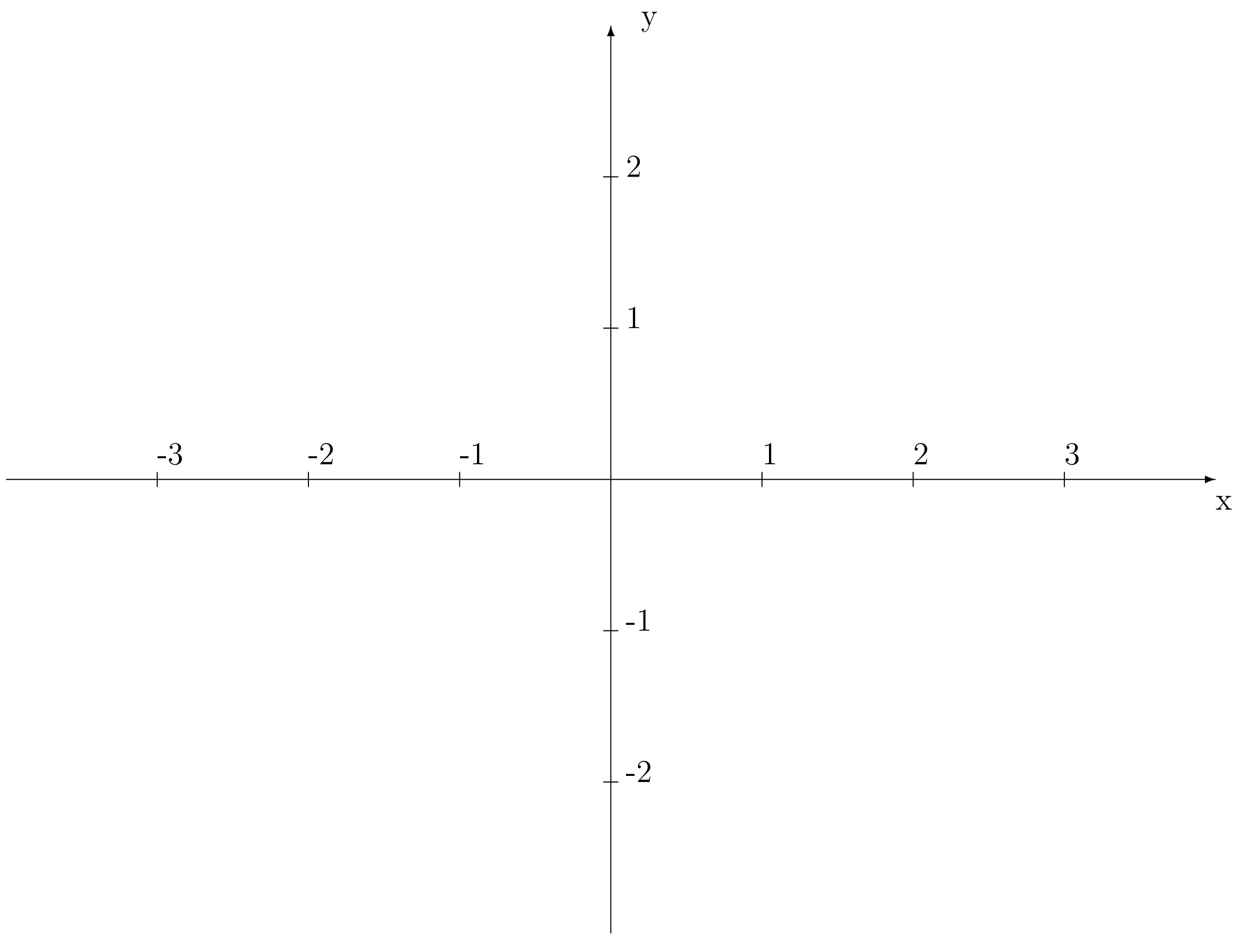
i. The **linear approximation** of f at $x = 1$ is $L(x) = 2$;

ii. $f'(x) > 0$ for $0 \leq x < 1$;

iii. $f''(x) < 0$ for $0 < x < 2$;

iv. $\lim_{x \to 2^-} f(x) = -\infty$;

v. f is an odd function.

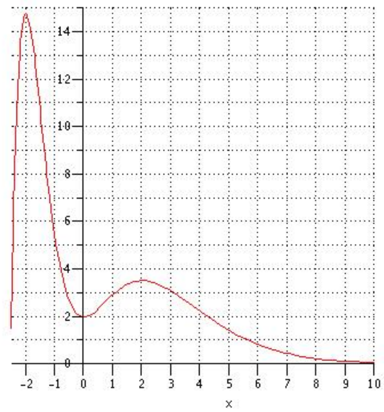


(20pts) **5.** Captain Nemo is in his motionless submarine (the Nautilus) 300 meters under the sea level waiting for his good old friend Captain Ahab to discuss about their misanthropic vision of the world over a cup of tea. Captain Ahab is in his whaling ship (the Pequod) moving at a constant speed toward the point of the water surface which is exactly above the Nautilus. Using his sonar, Captain Nemo is able to measure that the distance between the Nautilus and the Pequod is decreasing at a rate of $10m/s$ when this distance is 500 meters. What is the speed of the Pequod?

Recitations of MAT131

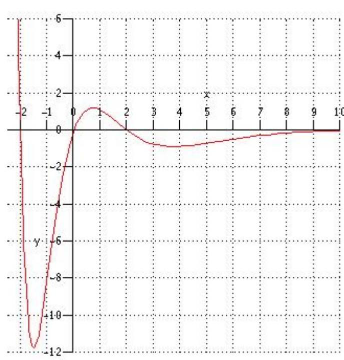
Recitation	Time	Location	TA
R01	TuTh 12:50pm- 1:45pm	Earth and Space 183	Disconzi, Marcelo
R02	MF 2:20pm- 3:15pm	Earth and Space 79	Cheng, Jonathan
R03	TuTh 8:20am- 9:15am	Earth and Space 181	Disconzi, Marcelo
R04	WF 11:45am-12:40pm	Earth and Space 181	Lehrer, Raquel
R05	MF 2:20pm- 3:15pm	Harriman Hll 115	Jaggi, Amit
R06	MW 10:40am-11:35am	Earth and Space 183	Weng, Luoying
R07	MW 6:50pm- 8:10pm	Earth and Space 183	Li, Yinghua
R08	MW 3:50pm- 4:45pm	Chemistry 128	Dutta, Satyaki
R09	TuTh 5:20pm- 6:15pm	Lgt Engr Lab 154	Hassan, Mohamed
R10	WF 9:35am-10:30am	Earth and Space 183	Weng, Luoying
R11	TuTh 3:50pm- 4:45pm	Lgt Engr Lab 154	Hassan, Mohamed
R12	TuTh 8:20am- 9:15am	Library N4000	Fong, Chee
R13	MF 12:50pm- 1:45pm	Library N4006	Li, Yinghua
R14	TuTh 3:50pm- 5:10pm	Physics P125	Fong, Chee

Spring 2007 Midterm 2 Solutions
Part I: Multiple choice questions (5 points each)

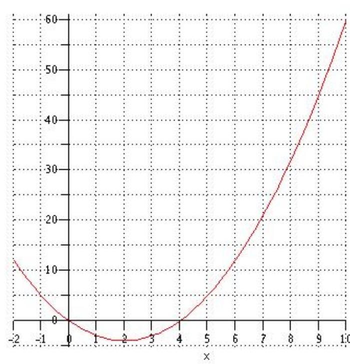


1. Which of the following is the graph of $f'(x)$?
   (a) When $f(x)$ has horizontal tangent lines, $f'(x) = 0$. Notice that this occurs at $x = -2, 0, 2$ and $f'(x)$ should be small but negative for larger x-values.

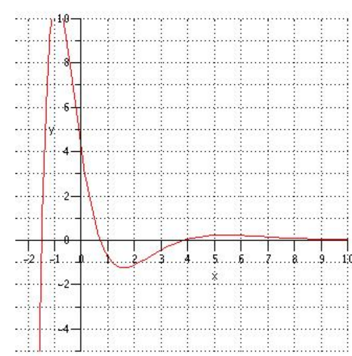
2. Which of the following is the graph of $f''(x)$?
   (c) When $f(x)$ changes concavity, $f''(x) = 0$. This occurs at around $x \approx -1.5, .75, 4$.



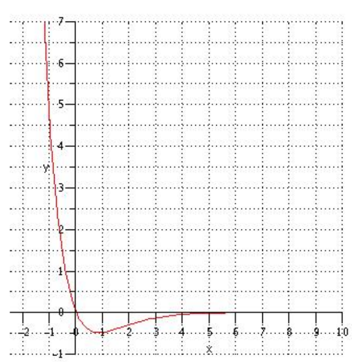
(a)



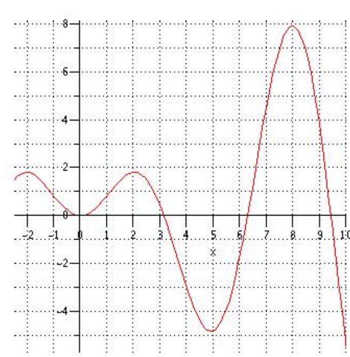
(b)



(c)



(d)



(e)

3. $\lim_{h\to 0} \dfrac{\cos h - 1}{h} =$

(b) 0.
$= f'(0)$ where $f(x) = \cos x$.
$f'(0) = -\sin(0) = 0$.

4. $\dfrac{d}{dx}(e^{2\pi}) =$
(e) 0
$e^{2\pi}$ is a constant.

5. Let $f(x) = (e^x)(x^5 + 4)$. What is $f'(x)$?

(d) $(e^x)(x^5 + 5x^4 + 4)$

$$\begin{aligned} f'(x) &= \left[(e^x)(x^5+4)\right]' = (e^x)'(x^5+4) + (e^x)(x^5+4)' = \\ &= (e^x)(x^5+4) + (e^x)(5x^4) = (e^x)(x^5+5x^4+4). \end{aligned}$$

6. If $y(t) = \sin(\sqrt{3t})$, then $\dfrac{dy}{dt} =$

(b) $\dfrac{3\cos(\sqrt{3t})}{2\sqrt{3t}}$

$$\begin{aligned} \frac{dy}{dt} &= \cos(\sqrt{3t}) \cdot \frac{d}{dt}(\sqrt{3t}) = \cos(\sqrt{3t})\frac{1}{2}(3t)^{-\frac{1}{2}} \cdot \frac{d}{dt}(3t) \\ &= \cos(\sqrt{3t})\frac{1}{2}(3t)^{-\frac{1}{2}}3 = \frac{3\cos(\sqrt{3t})}{2\sqrt{3t}}. \end{aligned}$$

7. Let $h(x) = \ln\left[\dfrac{e^{2x}(x-1)^2}{(x-3)^4}\right]$. Find $h'(x)$. (You may assume $x \neq 1, 3$.)

(c) $2 + \dfrac{2}{x-1} - \dfrac{4}{x-3}$

$$\begin{aligned} h(x) &= 2x + 2\ln(x-1) - 4\ln(x-3) \\ h'(x) &= 2 + 2\frac{(x-1)'}{x-1} - 4\frac{(x-3)'}{x-3} = 2 + \frac{2}{x-1} - \frac{4}{x-3} \end{aligned}$$

8. $W(x) = \cos^4\left(\dfrac{\sin x}{x^2+1}\right)$. What is $\dfrac{dW}{dx}$?

There was a mistake. The correct answer does not appear.

$$\begin{aligned}
\frac{dW}{dx} &= \frac{d}{dx}\left[\cos^4\left(\frac{\sin x}{x^2+1}\right)\right] = 4\cos^3\left(\frac{\sin x}{x^2+1}\right)\cdot\frac{d}{dx}\left[\cos(\frac{\sin x}{x^2+1})\right] \\
&= 4\cos^3\left(\frac{\sin x}{x^2+1}\right)\left(-\sin(\frac{\sin x}{x^2+1})\right)\cdot\frac{d}{dx}\left[\frac{\sin x}{x^2+1}\right] \\
&= 4\cos^3\left(\frac{\sin x}{x^2+1}\right)\left(-\sin(\frac{\sin x}{x^2+1})\right)\left[\frac{(x^2+1)\frac{d}{dx}(\sin x)-(\sin x)\frac{d}{dx}(x^2+1)}{(x^2+1)^4}\right] \\
&= -4\cos^3\left(\frac{\sin x}{x^2+1}\right)\left(\sin(\frac{\sin x}{x^2+1})\right)\left[\frac{(x^2+1)\cos x-(\sin x)(2x)}{(x^2+1)^2}\right]
\end{aligned}$$

9. If $y = x^{2x}$, then $\dfrac{dy}{dx} =$

(b) $x^{2x}(2\ln x + 2)$

$$\begin{aligned}
y &= x^{2x} \\
\ln y &= \ln(x^{2x}) = 2x\ln x \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(2x\ln x) \\
\frac{1}{y}\frac{dy}{dx} &= 2\ln x + 2x\frac{1}{x} \\
\frac{dy}{dx} &= y(2\ln x + 2) \\
\frac{dy}{dx} &= x^{2x}(2\ln x + 2)
\end{aligned}$$

10. If $h(\theta) = 2^{\cos\theta}$, then $\dfrac{dh}{d\theta} =$

(a) $-2^{\cos\theta}(\ln 2)(\sin\theta)$

$$\begin{aligned}
\frac{dh}{d\theta} &= \frac{d}{d\theta}(2^{\cos\theta}) \\
&= 2^{\cos\theta}\ln 2\frac{d}{d\theta}(\cos\theta) \\
&= 2^{\cos\theta}\ln 2(-\sin\theta)
\end{aligned}$$

Part II: Partial credit questions (10 points each)

Show all of your work!! Write clearly!

11. Let $g(x) = -\dfrac{1}{3}x^3 + x^2 + 5$.

(a) For what values of x is $g(x)$ increasing?

$g(x)$ is increasing when $g'(x) > 0$.
$g'(x) = -x^2 + 2x = -x(x-2)$.
$g'(x)$ is a quadratic function and (by several possible means) we can see that

$$-x(x-2) > 0 \text{ when } 0 < x < 2.$$

(b) For what values of x is $g(x)$ concave up?

$g(x)$ is concave up when $g''(x) > 0$.
$g''(x) = -2x + 2$.
$-2x + 2 > 0$ when $x < 1$.

12. Let $f(x) = \sqrt{x}$. Write the equation for the tangent line to $f(x)$ at $x = 25$.

The equation of the tangent line will be

$$y - f(25) = f'(25)(x - 25).$$

$f(25) = \sqrt{25} = 5$.
$f'(x) = \dfrac{1}{2}(x)^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$
$f'(25) = \dfrac{1}{2\sqrt{25}} = \dfrac{1}{10} = .1$
Therefore, the equation of the tangent line at $x = 25$ is

$$y - 5 = .1(x - 25)$$

or

$$y = .1(x - 25) + 5.$$

(b) Use *linear approximation* to estimate $\sqrt{27}$.

We will use linear approximation for the function $f(x) = \sqrt{x}$ at $x = 25$ since we have an exact decimal value for $\sqrt{25} = 5$. The linearization is exactly the tangent line from part (a). For x close to 25,

$$f(x) \approx L(x) = .1(x - 25) + 5$$
$$\sqrt{27} = f(27) \approx L(27) = .1(27 - 25) + 5 = 5.2$$

13. The equation

$$\frac{x^2}{2} + y^2 = 3$$

gives an ellipse. Find the *slope* of the tangent line to the ellipse at the point $(2, 1)$.

$$\begin{aligned} \frac{d}{dx}\left[\frac{x^2}{2} + y^2\right] &= \frac{d}{dx}(3) \\ x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{2y} \end{aligned}$$

At the point $(2, 1)$,

$$\frac{dy}{dx} = -\frac{2}{2 \cdot 1} = -1.$$

14. Suppose a particle is moving along a circle of radius 1 cm. The angle (in radians) at time t seconds is given by

$$\theta(t) = (t-1)\pi.$$

(a) Calculate $\dfrac{d\theta}{dt}$ (the angular velocity of the particle) at $t = 1$ second.

$$\frac{d\theta}{dt} = \frac{d}{dt}\left[(t-1)\pi\right] = \pi.$$

Therefore, $\dfrac{d\theta}{dt} = \pi$ (radian/sec) when $t = 1$.

(b) The (x, y) coordinates of the particle at time t are given by usual trigonometric functions

$$x(\theta) = \cos\theta, \qquad y(\theta) = \sin\theta.$$

Find $\dfrac{dy}{dt}$ (the vertical velocity of the particle) at $t = 1$ second. Include units.

$$\frac{dy}{dt} = \frac{d}{dt}\left[\sin\theta\right] = \cos\theta\frac{d\theta}{dt}$$

When $t = 1$, $\theta = (1-1)\pi = 0$. Therefore, when $t = 1$ second,

$$\frac{dy}{dt} = (\cos 0)\pi = \pi \text{ cm/sec.}$$

Alternatively, we can write

$$y(\theta(t)) = \sin(t\pi - \pi).$$

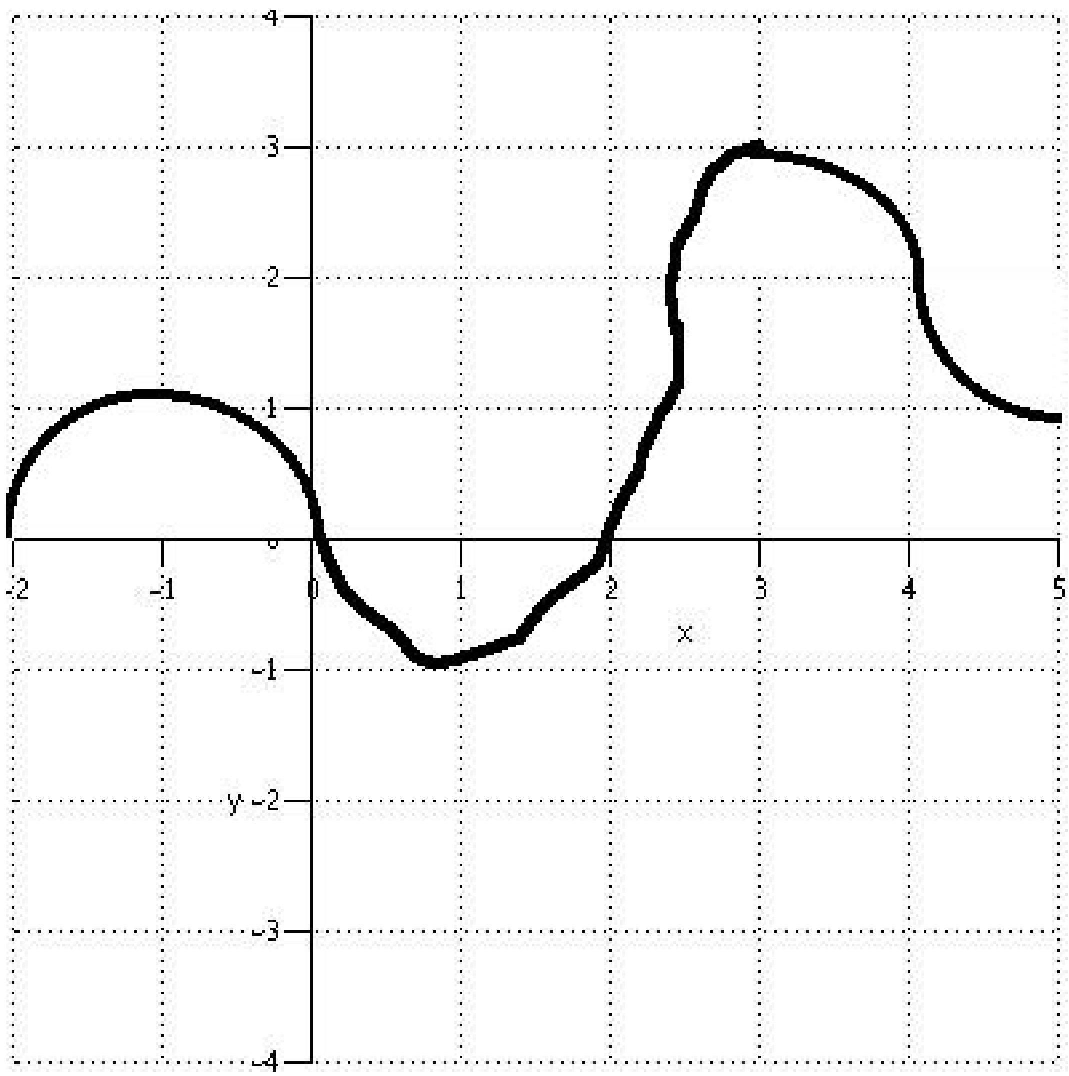
Then, taking the derivative, we see that

$$\frac{dy}{dt} = \cos(t\pi - \pi)\pi$$

and when $t = 1$, $\dfrac{dy}{dt} = (\cos 0)\pi = \pi$ cm/sec.

15. Draw the graph of a continuous function $h(x)$ that satisfies the following properties:

- $h(3) = 3$
- $h(-2) = h(0) = h(2) = 0$
- $h'(-1) = h'(1) = h'(3) = 0$
- $h''(0) = h''(2.5) = h''(4) = 0$



There are multiple correct graphs one could draw.

MAT131-REVIEW Midterm 1

Instructions: The exam will consist in six questions (like the ones bellow). You will have 90 minutes to answer all six questions. You will not be allowed to use any books or notes, but you may use your calculator.

(1) You are given the following information about the function f(x)

(i) The domain of f(x) is the interval [-3,4]

(ii) The function f(x) is continuous in the intervals (-3,1) and (1,4) and it is not continuous at x=1.

(iii) f(2)=37

Which, if any, of the following limits exists? Which limits, if any, can you find using this information. Justify your answer.

(a) $\lim_{x\to 2} f(x)$

(b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 4^-} f(x)$

(d) $\lim_{x\to 1+} f(x)$

(2) Let c be a real number and let f(x) be the function

$$f(x) = \begin{cases} x^{20}+4x & \text{if } x < -1. \\ x+c & \text{if } x \geq -1. \end{cases}$$

(a) Find $\lim_{x\to 0} f(x)$

(b) Find $\lim_{x\to -1^+} f(x)$

(c) Find $\lim_{x\to -1^-} f(x)$

(d) Find $f(-1)$

(e) Does there exist any value of c which makes the function f(x) continuous?

(3) Let f(x) be given by

$$f(x) = \begin{cases} \sqrt{x+5} & \text{if } x \geq -5. \\ x^2-25 & \text{if } -10 \leq x \leq -5. \\ \frac{1}{x+10} & \text{if } x < -10 \end{cases}$$

(a) Find right and left hand limits of this function at x=0, x=-5 and x=-10.

(b) Does $\lim_{x\to 0} f(x)$ exist?

(c) Does $\lim_{x\to -5} f(x)$ exist?

(d) Does $\lim_{x \to -10} f(x)$ exist?

(4) Evaluate the following limits if possible. If the limit does not exist, explain why it doesn't.

(a) $\lim_{x \to \frac{\pi}{2}} e^{\cos x}$
(b) $\lim_{x \to 1} \frac{x^3 - x^2}{x^2 - 1}$
(c) $\lim_{x \to -2} \frac{2x^3 - x^2 + 3x}{x^2 - 4}$
(d) $\lim_{t \to 0} \frac{(t+4)^{\frac{1}{2}} - 2}{t}$
(e) Evaluate the following limits if possible. If the limit does not exist, explain why it doesn't.
(f) $\lim_{x \to 0} \sin(\frac{1}{x})$
(g) $\lim_{x \to 3}(\frac{1}{x-3} - \frac{6}{x^2-9})$
(h) $\lim_{x \to 7} \frac{x-7}{|x-7|}$
(i) $\lim_{t \to 37} \frac{t}{t-37}$
(j) $\lim_{x \to -1} \frac{x^4 - x^3 + x^2}{x^3 - x}$

(5) State the intervals where each of the following functions are continuous

(a) $f(x) = |x+3|$
(b) $f(x) = \frac{|x-4|}{x-4}$
(c) $\ln(x^2 - 3x)$
(d) $\cos(x)$
(e) $\frac{x^2+x+2}{x^2-9}$

(6) Sketch the graph of a function with the following properties:

(i) $f(x)$ is one-to-one.
(ii) The domain of f is the inverval (-3,7)
(iii) The range of f is the interval (-4,6)
(iv) f(x) is continuous everywhere except at x=3 and x=4,
(v) f(x) is continuous from the left at x=4 and it is continuous from the right at x=3.
(vi) $f^{-1}(4) = 5$

(7) Prove that the following equations have at least one real solution. Find an interval of length at most 2 that contains a solution.

(a) $\log(x) + x = 3$
(b) $x^4 - x + 3 = 0$

(c) $\sin(\tan x) = 0.5$

(8) If $\lim_{x\to 2}(f(x)+g(x)) = 4$ and $\lim_{x\to 2}(f(x)-g(x)) = -3$, find

(a) $\lim_{x\to 2} f(x)$

(b) $\lim_{x\to 2} g(x)$

(c) $\lim_{x\to 2} 3f(x) - g(x)$

(d) $\lim_{x\to 2} f(x)g(x)$

(9) Find all values of c such that f is continuous on all the real numbers.

$$f(x) = \begin{cases} x^2 - 5 & \text{if } x \geq c. \\ x - 5 & \text{if } x < c. \end{cases}$$

(10) Sketch the graph of a function f that satisfies the following conditions

(i) $\lim_{x\to 2^-} f(x) = -1$, $\lim_{x\to 2^+} f(x) = 2$

(ii) $f(1)$ is not 2.

(iii) $\lim_{x\to 3} f(x) = 4$

(iv) f is increasing an defined everywhere.

(11) An aeroplane is flying from New York to Paris. At t hours, it has covered d(t) miles.

(a) Give a formula for the average velocity during the second hour of flight.

(b) State what is the instantaneous velocity at t=2 in terms of the graph of $y = d(t)$

(12) (Hard) Let $[[x]]$ denote the largest integer that is less than or equal to x (For example, $[[2.7]] = 2$ and $[[3]] = 3$).

(a) Find $\lim_{x\to 4^-} f(x)$

(b) Find $\lim_{x\to 4^+} f(x)$

(c) Sketch a graph of f.

(d) For which values of x, $[[x]]$ is not continuous?

(13) For which values of a does $\lim_{x\to 1} \frac{x^2-ax-1}{x-1}$ exists?

(14) For $f(x) = \frac{\sqrt{x}-\sqrt{1-x}}{2x-1}$

(a) Find its domain

(b) Calculate $\lim_{x\to \frac{1}{2}} f(x)$.

(15) For

$$f(x) = \begin{cases} x & \text{if } x > 0. \\ \cos x & \text{if } x \leq 0. \end{cases}$$

(a) Sketch the graph of $f(x)$

(b) State the intervals where $f(x)$ is continious.

(16) The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why. Justify all your steps.

(a) $\lim_{x\to 1}(2f(x) - 3g(x))$

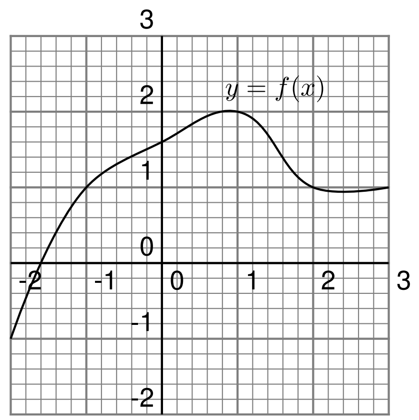
(b) $\lim_{x\to -1} f(g(x))$

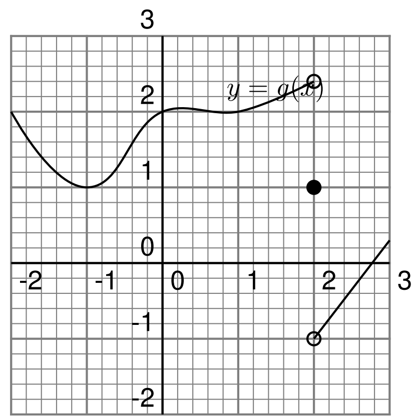
(c) $\lim_{x\to 0} \frac{f(x)}{g(x)}$

(d) $\lim_{x\to 0} \frac{f(x)}{x+3}$

(e) $\lim_{x\to 2} g(x)$

(f) $\lim_{x\to 0}[f(x)]^2$





Problem 1. Let $r(x) = \dfrac{2x}{x^2+1}$.

(a) Use the definition of the derivative to compute $r'(a)$.

(b) There are two points at which the tangent line to the graph of r is horizontal. Give the coordinates of these points.

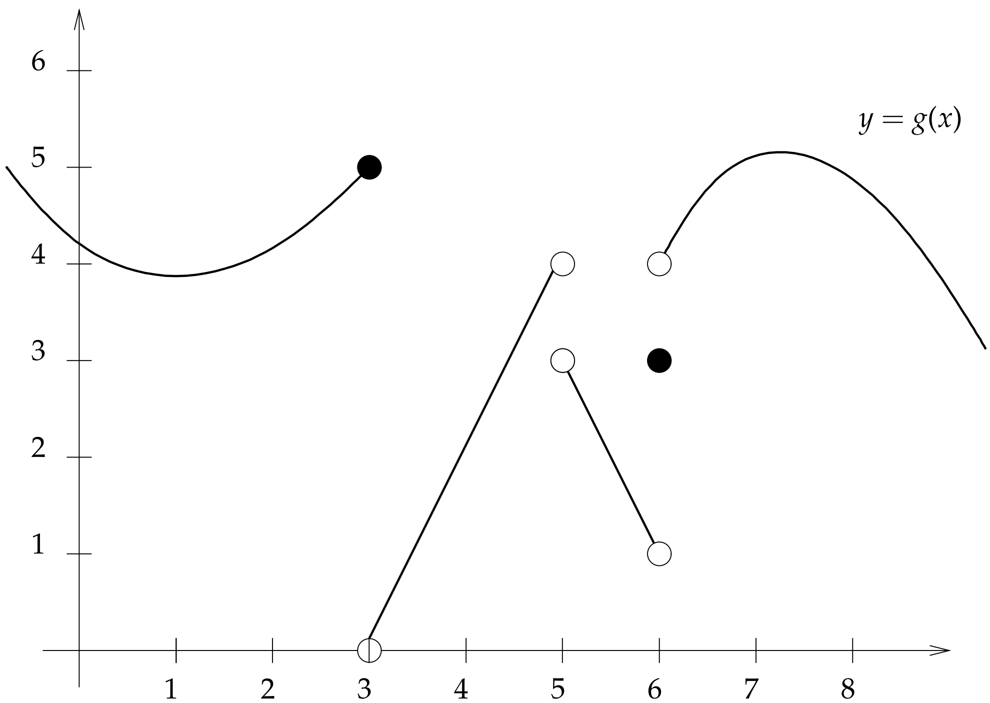
Problem 2. Compute:

(a) $\lim_{x \to \infty} \frac{1}{x^2} \sin(x)$

(b) $\lim_{t \to 4} \frac{\sqrt{t+5} - 3}{\sqrt{2t+1} - 3}$

(c) $\lim_{x \to 2^-} \frac{x^2 - 3x}{x^2 - 4}$

Problem 3.



Use the picture to find:

(a) $\lim_{x\to 6^+} g(x)$

(b) $g(6)$

(c) $\lim_{x\to 6} g(x)$

(d) $g(5)$

(e) $\lim_{x\to 3^-} g(x)$

(f) $\lim_{x\to 3^+} \dfrac{1}{g(x)}$

(g) $\lim_{x\to 3^+} \dfrac{x-3}{g(x)}$

(h) $\lim_{h\to 0} \dfrac{g(4+h)-2}{h}$

(i) $\lim_{x\to 3^-} g(2x)$

Problem 4. A recent study of toxin levels in a stream has yielded the following data:

t	1	2	2.1	4	6.5	7.2	8	9	10	10.9	11
$a(t)$	3.12	5.4	5.7	8.15	10.10	9.63	9.21	9.31	8.98	7.82	7.62

Here, $a(t)$ is the amount of toxin measured at time t. Use the table to estimate $a'(2)$ and $a'(11)$.

Problem 4. True or False

(a) If f is any function with $f(3) = 5$ and $f(6) = 3$, then there must exist a number x between 3 and 6 with $f(x) = 4$.

(b) The function f defined by

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \leq 1, \\ 5x - 1 & \text{if } x > 1 \end{cases}$$

is continuous at $x = 1$.

(c) $\lim_{r \to 0} \frac{1}{r^2} = \infty$.

(d) If $|x - 3| < 0.1$ then $9.9 < 3x + 1 < 10.1$.

(e) If $\lim_{x \to -\infty} g(x) = 14$ then the line $y = 14$ is a horizontal asymptote of the graph of g.

(f) The curve $y = \frac{x^3 - 1}{x^2 - 1}$ has a vertical asymptote at $x = 1$.

(g) The $\lim_{t \to 0} \frac{e^t - 1}{t - 1}$ exists and is finite.

EXAM

Sample Midterm 1

Math 131

October 3, 2003

EXAM

Sample Midterm 1

Math 131

October 9, 2003

ANSWERS

Problem 1. Let $r(x) = \dfrac{2x}{x^2+1}$.

(a) Use the definition of the derivative to compute $r'(a)$.

Answer:

$$\begin{aligned}
r'(a) &= \lim_{h\to 0} \frac{r(a+h) - r(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(a+h)}{(a+h)^2+1} - \frac{2a}{a^2+1}}{h} \\
&= \lim_{h\to 0} \frac{2(a+h)(a^2+1) - 2a\left((a+h)^2+1\right)}{h(a^2+1)\left((a+h)^2+1\right)} \\
&= \lim_{h\to 0} \frac{(2a^3+2ha^2+2a+2h) - \left(2a^3+4ha^2+2h^2a+2a\right)}{h(a^2+1)\left((a+h)^2+1\right)} \\
&= \lim_{h\to 0} \frac{-2ha^2+2h-2h^2a}{h(a^2+1)\left((a+h)^2+1\right)} \\
&= \lim_{h\to 0} \frac{-2a^2+2-2ha}{(a^2+1)\left((a+h)^2+1\right)} \\
&= \frac{-2a^2+2}{(a^2+1)^2}.
\end{aligned}$$

(b) There are two points at which the tangent line to the graph of r is horizontal. Give the coordinates of these points.

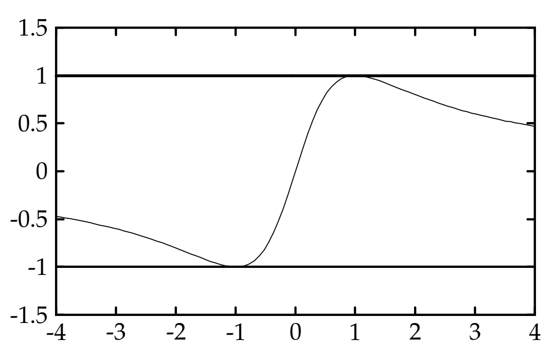
Answer:

The tangent line to $y = r(x)$ will be horizontal at $(a, r(a))$ when the slope $r'(a) = \dfrac{-2a^2+2}{(a^2+1)^2}$ is zero. Setting $r'(a) = 0$ gives

$$r'(a) = 0 \Rightarrow \frac{-2a^2+2}{(a^2+1)^2} = 0 \Rightarrow -2a^2+2 = 0 \Rightarrow a = 1 \text{ or } a = -1.$$

So, the tangent line to the graph of r is horizontal at the points $(-1,-1)$ and $(1,1)$.

Here's a little sketch of $y = r(x)$ with its two horizontal tangent lines:



Problem 2. Compute:

(a) $\lim_{x\to\infty} \frac{1}{x^2}\sin(x)$

Answer:

Because $-1 \le \sin(x) \le 1$ for all x, we have $-\frac{1}{x^2} \le \frac{1}{x^2}\sin(x) \le \frac{1}{x^2}$. Since $\lim_{x\to\infty} -\frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^2} = 0$, the squeeze theorem says that $\lim_{x\to\infty} \frac{1}{x^2}\sin(x) = 0$ too.

(b) $\lim_{t\to 4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3}$

Answer:

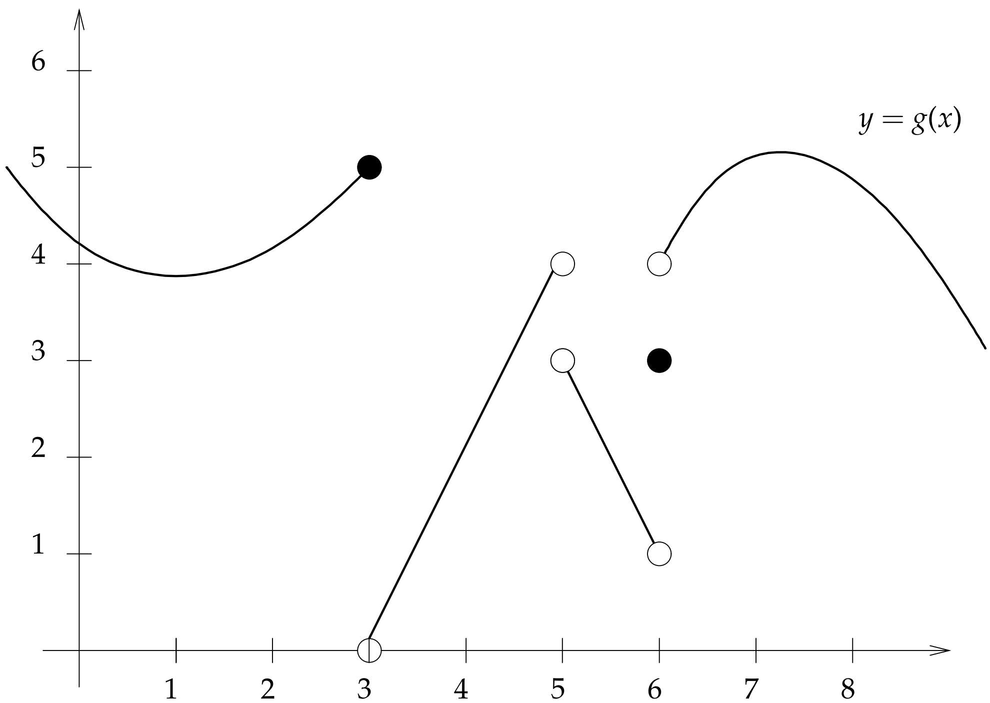
$$\begin{aligned}
\lim_{t\to 4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3} &= \lim_{t\to 4} \frac{\sqrt{t+5}-3}{\sqrt{2t+1}-3}\left(\frac{\sqrt{2t+1}+3}{\sqrt{2t+1}+3}\right)\left(\frac{\sqrt{t+5}+3}{\sqrt{t+5}+3}\right) \\
&= \lim_{t\to 4} \frac{(t+5-9)\left(\sqrt{2t+1}+3\right)}{(2t+1-9)\left(\sqrt{t+5}+3\right)} \\
&= \lim_{t\to 4} \frac{(t-4)}{2(t-4)}\left(\frac{\sqrt{2t+1}+3}{\sqrt{t+5}+3}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

(c) $\lim_{x\to 2^-} \frac{x^2-3x}{x^2-4}$

Answer:

Notice that $\lim_{x\to 2^-} x^2-3x = -2$ and $\lim_{x\to 2^-} x^2-4 = 0$. When x is near, but less than, 2, $x^2-4<0$. So, we find that both the numerator and denominator of $\frac{x^2-3x}{x^2-4}$ are negative for x near but less than 2, with the denominator tending to zero, and the numerator bounded away from zero. We conclude that $\lim_{x\to 2^-} \frac{x^2-3x}{x^2-4} = +\infty$.

Problem 3.



Use the picture to find:

Answer:

(a) $\lim_{x\to 6^+} g(x) = 4$.

(b) $g(6) = 3$.

(c) $\lim_{x\to 6} g(x)$ does not exist.

(d) $g(5)$ does not exist.

(e) $\lim_{x\to 3^-} g(x) = 5$.

(f) $\lim_{x\to 3^+} \dfrac{1}{g(x)} = +\infty$.

(g) $\lim_{x\to 3^+} \dfrac{x-3}{g(x)} = \dfrac{1}{2}$ (note that for x near, but greater than, 3 $g(x)$ looks like $2(x-3)$.)

(h) $\lim_{h\to 0} \dfrac{g(4+h)-2}{h} = 2$. (here, this is the slope of the tangent line to the graph of g at $(2,2)$.)

(i) $\lim_{x\to 3^-} g(2x) = 1$. ($= \lim_{x\to 6^-} g(x)$.)

Problem 4. A recent study of toxin levels in a stream has yielded the following data:

t	1	2	2.1	4	6.5	7.2	8	9	10	10.9	11
$a(t)$	3.12	5.4	5.7	8.15	10.10	9.63	9.21	9.31	8.98	7.82	7.62

Here, $a(t)$ is the amount of toxin measured at time t. Use the table to estimate $a'(2)$ and $a'(11)$.

Answer:
Since

$$a'(2) = \lim_{x \to 2} \frac{a(x) - a(2)}{x - 2},$$

it follows that

$$a'(2) \approx \frac{a(x) - a(2)}{x - 2} \text{ if } x \approx 2.$$

So,

$$a'(2) \approx \frac{a(2.1) - a(2)}{2.1 - 2} = \frac{.3}{.1} = 3.$$

Similarly, we approximate

$$a'(11) \approx \frac{a(10.9) - a(11)}{10.9 - 11} = \frac{.2}{-.1} = -2.$$

Problem 5. True or False

(a) If f is any function with $f(3) = 5$ and $f(6) = 3$, then there must exist a number x between 3 and 6 with $f(x) = 4$.

Answer:

False. For example, look at the function whose graph is sketched in problem 3. There, $g(3) = 5$ and $g(6) = 3$ but there is no $x \in (3, 6)$ with $g(x) = 4$.

(b) The function f defined by

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \leq 1, \\ 5x - 1 & \text{if } x > 1 \end{cases}$$

is continuous at $x = 1$.

Answer:

True. We check, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 2x + 1 = 4$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5x - 1 = 4$ and $f(1) = 1$ and find that $\lim_{x\to 1} f(x) = 1 = f(1)$.

(c) $\displaystyle\lim_{r\to 0} \frac{1}{r^2} = \infty$.

Answer:

True. $\displaystyle\lim_{r\to 0^-} \frac{1}{r^2} = \infty$ and $\displaystyle\lim_{r\to 0+} \frac{1}{r^2} = \infty$. So, we have the two sided limit $\displaystyle\lim_{r\to 0} \frac{1}{r^2} = \infty$.

(d) If $|x - 3| < 0.1$ then $9.9 < 3x + 1 < 10.1$.

Answer:

False. Let $x = 2.91$ then $|x - 3| = .09 < 0.1$ but $3x + 1 = 9.73$, which does not lie between 9.9 and 10.1.

(e) If $\displaystyle\lim_{x\to -\infty} g(x) = 14$ then the line $y = 14$ is a horizontal asymptote of the graph of g.

Answer:

True.

(f) The curve $y = \dfrac{x^3 - 1}{x^2 - 1}$ has a vertical asymptote at $x = 1$.

Answer:

False. $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$ which is finite.

(g) The $\displaystyle\lim_{t\to 0} \frac{e^t - 1}{t - 1}$ exists and is finite.

Answer:

True. This limit is the slope of the tangent line to $y = e^t$ at $(0, 1)$, which by inspection exists and is finite.

First Midterm
MAT 131

February 21, 2006
8:30 – 10:00 pm

Name:____________________________________ ID number: ____________________

Recitation number (e.g., R01):__________
(for evening lecture, use “ELC 4”)

Lecture 1	MWF 9:35am–10:30am	Stephen Preston
R01	MW 11:45am–12:40pm	Yinghua Li
R02	TuTh 3:50pm–4:45pm	Bryan Kim
R03	TuTh 12:50pm–2:10pm	Ariel Hitron
Evening Lec 4	TuTh 5:20pm–7:10pm	Wenchuan Hu

1	
2	
3	
4	
5	
6	
7	
8	
Sum	

Instructions.

- Show all work to get full credit; a correct answer with incorrect justification will not get credit.
- All electronic devices, including cell phones, beepers, and other noisemakers, must be turned off.
- Calculators of any sort are unnecessary and not allowed.
- When finished, give your exam to your own TA or lecturer. If we do not recognize you, we will ask to see your student ID.
- Remember: *calm, confident, and calculating correctly.*

Do not open the exam until instructed!

1. The U.S. national debt was about \$4 trillion in 1995 and about \$8 trillion in 2005.

   (a) If we model the debt with a linear function of time, what prediction will we have for the debt in 2015?

   (b) If we model the debt with an exponential function of time, what prediction will we have in 2015?

2. Suppose $f(x) = 3x^2 - x$. Simplify the equation

$$(f \circ f)(x) = x$$

   and find all solutions x.

   (Note: if you get an equation you can't solve, you probably did something wrong.)

3. Suppose the function $f(x)$ has a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 2$. What are the two asymptotes of the function $y = 2f(2x) + 1$? Explain your method.

4. Consider the function $f(x) = \dfrac{1}{1 - e^{-x}}$.

   (a) Find the inverse function in the form $y = f^{-1}(x)$.

   (b) Write down the domain and range of f. Explain your reasoning.

5. Compute the limits:

   (a) $\displaystyle\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 + 4x + 5}$

   (b) $\displaystyle\lim_{x \to -1} \frac{x^2 + 9x + 8}{x^2 - 1}$

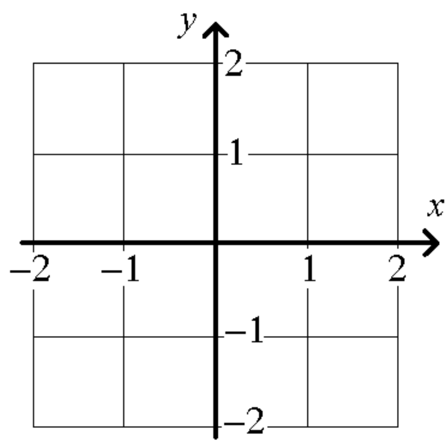
   (c) $\displaystyle\lim_{x \to -1} \frac{x^2 - 5x + 4}{x^2 + 1}$

6. (a) How do you know the function $f(x) = x^4 - 17x + 3$ is continuous for all x? (One sentence only.)

   (b) Find an interval of length 1 that contains a solution of $f(x) = 0$. What technique are you using?

7. On the given axes, sketch the graph of a function defined at all points of $[-2, 2]$ such that

- f is continuous at all points of $[-2, 2]$ except for $x = 1$ and $x = -1$.
- $\lim_{x \to -1} f(x) = -1$
- $\lim_{x \to 1^+} f(x) = 0$
- $\lim_{x \to 1^-} f(x) = 1$
- $f(1) = 2$



8. Compute the limits.

(a) $\lim_{x \to -\infty} \sqrt{x^2 + x - 1} - x$

(b) $\lim_{x \to \infty} \sqrt{x^2 + x - 1} - x$

MAT 131 Midterm 1 solutions

1. The U.S. national debt was about \$4 trillion in 1995 and about \$8 trillion in 2005.

   (a) If we model the debt with a linear function of time, what prediction will we have for the debt in 2015?

   Answer After 10 years, the debt has increased by \$4 trillion, so after another 10 years, it will increase by another \$4 trillion. Thus in 2015, the debt would be \$12 trillion.

   (b) If we model the debt with an exponential function of time, what prediction will we have in 2015?

   Answer After 10 years, the debt has doubled. Therefore after another 10 years, the debt will double again, to \$16 trillion.

2. Suppose $f(x) = 3x^2 - x$. Simplify the equation

$$(f \circ f)(x) = x$$

   and find all solutions x.

   (Note: if you get an equation you can't solve, you probably did something wrong.)

   Answer We compute

$$\begin{aligned}(f \circ f)(x) = f\big(f(x)\big) = 3f(x)^2 - f(x) &= 3(3x^2 - x)^2 - (3x^2 - x) \\ &= 27x^4 - 18x^3 + 3x^2 - 3x^2 + x = 27x^4 - 18x^3 + x.\end{aligned}$$

   So the solutions of $(f \circ f)(x) = x$ are the solutions of

$$27x^4 - 18x^3 = 0,$$

   which are $x = 0$ and $x = 2/3$.

3. Suppose the function $f(x)$ has a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 2$. What are the two asymptotes of the function $y = 2f(2x) + 1$? Explain your method.

Answer The graph is being compressed horizontally by a factor of 2; therefore the vertical asymptote at $x = 3$ gets moved to $x = 3/2$. The graph is also being stretched vertically by a factor of 2 and then shifted up by 1 unit, so that the horizontal asymptote $y = 2$ gets moved to $y = 4$ and then shifted to $y = 5$.

4. Consider the function $f(x) = \dfrac{1}{1 - e^{-x}}$.

   (a) Find the inverse function in the form $y = f^{-1}(x)$.

   Answer First we solve for x in terms of y:

$$\begin{aligned} y &= \frac{1}{1 - e^{-x}} \\ 1 - e^{-x} &= \frac{1}{y} \\ e^{-x} &= 1 - \frac{1}{y} \\ e^{x} &= \frac{y}{y-1} \\ x &= \ln\left(\frac{y}{y-1}\right) \end{aligned}$$

   Now we interchange x and y to get

$$f^{-1}(x) = \ln\left(\frac{x}{x-1}\right).$$

   (b) Write down the domain and range of f. Explain your reasoning.

   Answer The domain of f is the set of all x such that $f(x)$ does not involve dividing by 0, which is

$$D = \{x \mid x \neq 0\}.$$

   The range of f is the domain of f^{-1}, which we find by the requirement that $y/(y-1)$ must be positive. This implies that either $y > 0$ and $y - 1 > 0$ (so that $y > 1$) or that $y < 0$ and $y - 1 < 0$ (so that $y < 0$). Therefore the range is

$$R = \{y \mid y > 1 \text{ or } y < 0\}.$$

5. Compute the limits:

   (a) $\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+5}$

   Answer We try plugging in, since this is a rational function:

   $$\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+5} = \frac{1-3+2}{1-4+5} = \frac{0}{2} = 0.$$

   (b) $\lim_{x\to -1} \frac{x^2+9x+8}{x^2-1}$

   Answer Again try plugging in:

   $$\lim_{x\to -1} \frac{x^2+9x+8}{x^2-1} = \frac{1-9+8}{1-1} = \frac{0}{0}.$$

   Thus we have to try something else:

   $$\lim_{x\to -1} \frac{x^2+9x+8}{x^2-1} = \lim_{x\to -1} \frac{(x+1)(x+8)}{(x+1)(x-1)} = \lim_{x\to -1} \frac{x+8}{x-1} = -\frac{7}{2}.$$

   (c) $\lim_{x\to -1} \frac{x^2-5x+4}{x^2+1}$

   Answer Again plug in:

   $$\lim_{x\to -1} \frac{x^2-5x+4}{x^2+1} = \frac{1+5+4}{1+1} = 5.$$

6. (a) How do you know the function $f(x) = x^4 - 17x + 3$ is continuous for all x? (One sentence only.)

   Answer It is a polynomial; all polynomials are continuous.

   (b) Find an interval of length 1 that contains a solution of $f(x) = 0$. What technique are you using?

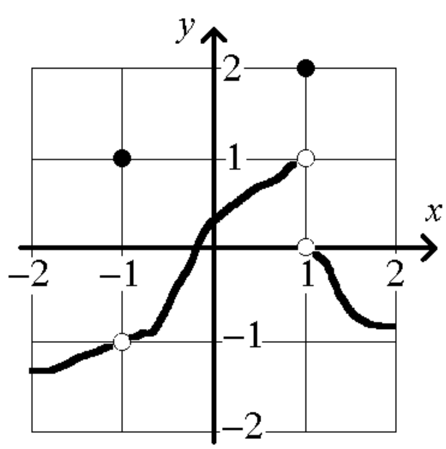
Answer We use the Intermediate Value Theorem. We have to experiment to find an interval by computing special values. $f(0) = 3$, $f(1) = -13$, $f(-1) = 21$, $f(2) = -15$, $f(3) = 30$, ...

Since f changes sign twice, there are roots between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$. (Either answer is acceptable.)

7. On the given axes, sketch the graph of a function defined at all points of $[-2, 2]$ such that

   - f is continuous at all points of $[-2, 2]$ except for $x = 1$ and $x = -1$.
   - $\lim_{x\to -1} f(x) = -1$
   - $\lim_{x\to 1^+} f(x) = 0$
   - $\lim_{x\to 1^-} f(x) = 1$
   - $f(1) = 2$

   Answer The only requirement is that $f(-1) \neq -1$, since otherwise f would be continuous at $x = -1$. A possible solution is sketched below.



8. Compute the limits.

   (a) $\lim\limits_{x\to -\infty} \sqrt{x^2 + x - 1} - x$

   Answer As $x \to -\infty$, the term $\sqrt{x^2 + x - 1} \to +\infty$ and also $-x \to +\infty$. Therefore the limit is the sum of two positive infinities and is also positive infinity:

$$\lim_{x\to -\infty} \sqrt{x^2 + x - 1} - x = +\infty + \infty = +\infty.$$

(b) $\lim_{x\to\infty} \sqrt{x^2+x-1} - x$

Answer This limit is of the form $+\infty - \infty$, which is indeterminate. Therefore we have to rationalize the difference of square roots:

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x-1} - x &= \lim_{x\to\infty} \frac{\sqrt{x^2+x-1} - x}{1} \frac{\sqrt{x^2+x-1} + x}{\sqrt{x^2+x-1} + x} \\
&= \lim_{x\to\infty} \frac{x^2+x-1-x^2}{\sqrt{x^2+x-1} + x} \\
&= \lim_{x\to\infty} \frac{x-1}{\sqrt{x^2+x-1} + x} \\
&= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{\sqrt{1+\frac{1}{x}-\frac{1}{x^2}} + 1} \\
&= \frac{1-0}{\sqrt{1}+1} \\
&= \frac{1}{2}
\end{aligned}$$

Math 131 Name:____________________

Midterm 1 - Solutions ID:____________________

February 20, 2007 Section:____________________

Calculators are not allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 1 hour and 30 minutes to do the test. When you finish, raise your hand and someone will come to collect your exam. You may then leave.

Part I consists of 10 multiple choice questions worth 5 points each. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Part II consists of 5 partial credit problems worth a total of 50 points. Write your answer and show **all** your work on the page on which the question appears.

Mark answers in 1-10 with an X.

Good Luck!

1. a **X** c d e
2. a b c **X** e
3. **X** b c d e
4. a b **X** d e
5. a b c **X** e
6. **X** b c d e
7. a **X** c d e
8. a b c d **X**
9. a **X** c d e
10. **X** b c d e

For grading use:

1-10	
11	
12	
13	
14	
15	
Total	

Part I: Multiple choice questions (5 points each)

1. Suppose a tank which holds 60 gallons of water is draining from the bottom, and

$$V(t) = 60\left(1 - \frac{t}{6}\right)^2 \qquad 0 \le t \le 6$$

is the volume of water remaining after t minutes. What is the average rate at which water is flowing out over the first 3 minutes?

(a) 10 gal/min

(b) **15 gal/min**

(c) 20 gal/min

(d) 30 gal/min

(e) 45 gal/min

$$\frac{V(3) - V(0)}{3 - 0} = \frac{60(.5)^2 - 60(1)^2}{3} = \frac{15 - 60}{3} = -15 \text{ gal/min.}$$

The volume of water in the tank is decreasing at an average rate of 15 gal/min over the first 3 minutes. The water is flowing out at an average rate of 15 gal/min over the first 3 minutes.

2. Let

$$f(x) = \begin{cases} (cx-2)^2 & x \le 1 \\ x + 3 & x > 1 \end{cases}$$

For what value(s) of c is $f(x)$ continuous everywhere?

(a) $c = 0$

(b) $c = \frac{1}{2}$

(c) $c = -2, 2$

(d) c=0,4

(e) No such c exists.

Since both parts of the piecewise function are continous, $f(x)$ is automatically continuous for $x \neq 1$. We therefore only need to impose continuity at $x = 1$.

$$\begin{aligned} f(1) &= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \\ 4 &= \lim_{x\to 1^-} x + 3 = \lim_{x\to 1^+} (cx-2)^2 \\ 4 &= 4 = (c-2)^2 \\ 4 &= c^2 - 4c + 4 \\ c^2 - 4c &= 0 \\ c(c-4) &= 0 \\ c &= 0, 4 \end{aligned}$$

3. If $g(x) = \sqrt{2}e^{2x+3}$, then the inverse function $g^{-1}(x) =$

(a) $\mathbf{\frac{1}{2}\ln x - \frac{1}{4}\ln 2 - \frac{3}{2}}$

(b) $\frac{1}{2}(\ln x)(\ln\frac{1}{\sqrt{2}}) - \frac{3}{2}$

(c) $\frac{-1}{\sqrt{2}e^{2x+3}}$

(d) $(\ln\sqrt{2})\,(2x+3)$

(e) $\frac{\sqrt{2}}{2}e^{-x} - 3$

$$\begin{aligned}
y &= \sqrt{2}e^{2x+3} \\
\frac{y}{\sqrt{2}} &= e^{2x+3} \\
\ln(\frac{y}{\sqrt{2}}) &= 2x+3 \\
x &= \frac{\ln(\frac{y}{\sqrt{2}}) - 3}{2} \\
x &= \frac{\ln y - \frac{1}{2}\ln 2 - 3}{2} \\
x &= \frac{1}{2}\ln y - \frac{1}{4}\ln 2 - \frac{3}{2} \\
f^{-1}(x) &= \frac{1}{2}\ln x - \frac{1}{4}\ln 2 - \frac{3}{2}
\end{aligned}$$

4. $\lim_{x\to 0} x^4 \cos(\frac{1}{x}) =$

   (a) $-\infty$
   (b) -1
   (c) **0**
   (d) 1
   (e) $+\infty$

   For all $x \neq 1$,

   $$-1 \leq \cos(\frac{1}{x}) \leq 1$$

   Therefore, (using the Squeeze Theorem)

   $$\begin{aligned}
   &-x^4 \leq x^4 \cos(\frac{1}{x}) \leq x^4 \\
   &\lim_{x\to 0} -x^4 \leq \lim_{x\to 0} x^4 \cos(\frac{1}{x}) \leq \lim_{x\to 0} x^4 \\
   &0 \leq \lim_{x\to 0} x^4 \cos(\frac{1}{x}) \leq 0 \\
   &\lim_{x\to 0} x^4 \cos(\frac{1}{x}) = 0
   \end{aligned}$$

5. Let

   $$f(x) = \frac{3x^3 - 3x}{2x^3 - 8x}$$

   Which of the following are horizontal assymptotes for $f(x)$?

   (a) $y = -2, 2$
   (b) $y = -1.5, 1.5$
   (c) $y = 0$
   (d) $\mathbf{y = 1.5}$
   (e) No horizontal assymptotes.

   $f(x)$ is a rational function, and the degree of the numberator is equal to the degree of the denominator. Therefore, the horizontal assymptote is given by the ratio of the leading coefficients.

   $$\lim_{x\to\pm\infty} f(x) = \frac{3}{2}.$$

   Alternatively, one can do the following algebraic manipulations:

   $$\lim_{x\to\pm\infty} \frac{3x^3 - 3x}{2x^3 - 8x} = \lim_{x\to\pm\infty} \frac{3x^3 - 3x}{2x^3 - 8x}\left(\frac{x^{-3}}{x^{-3}}\right) = \lim_{x\to\pm\infty} \frac{3 - \frac{3}{x^2}}{2 - \frac{8}{x^2}} = \frac{3}{2}$$

6. $\displaystyle\lim_{x\to 2^-} \frac{3x^3-3x}{2x^3-8x} =$

(a) $-\infty$

(b) -18

(c) 0

(d) $\dfrac{3}{2}$

(e) $+\infty$

First, notice that if we directly plug in $x = 2$ to $f(x)$, we get $\dfrac{18}{0}$. Since the denominator is zero, and the numberator is non-zero, this implies that $x = 2$ is a vertical assymptote. We now need to see if $\displaystyle\lim_{x\to 2^-} \frac{3x^3-3x}{2x^3-8x}$ goes to $+\infty$ or $-\infty$. To see this, we need to see if $f(x)$ is positive or negative for x less than, but close to, 2. This can be done by direct substitution, or looking at whether the individual factors are positive or negative. As $x \to 2^-$,

$$\frac{3x^3-3x}{2x^3-8x} = \frac{3x(x-1)(x+1)}{2x(x+2)(x-2)} = \frac{+++}{++-} = \frac{+}{-} = -$$

Therefore, $\displaystyle\lim_{x\to 2^-} \frac{3x^3-3x}{2x^3-8x} = -\infty$.

7. $\displaystyle\lim_{x\to \pi/2} \frac{\cos(x)}{1+\sin(x)} =$

(a) $-\infty$

(b) **0**

(c) $\dfrac{1}{2}$

(d) $+\infty$

(e) None of the above.

The above function is continuous at all x in the domain. The domain includes $x = \pi/2$. Therefore,

$$\lim_{x\to \pi/2} \frac{\cos(x)}{1+\sin(x)} = \frac{\cos(\pi/2)}{1+\sin(\pi/2)} = \frac{0}{1+1} = 0.$$

8. $\lim\limits_{x \to -\infty} \dfrac{x^5 - 4x^3 + 100}{10x^2 + 45} =$

(a) 0

(b) $\dfrac{1}{10}$

(c) $\dfrac{5}{2}$

(d) $+\infty$

(e) **None of the above.**

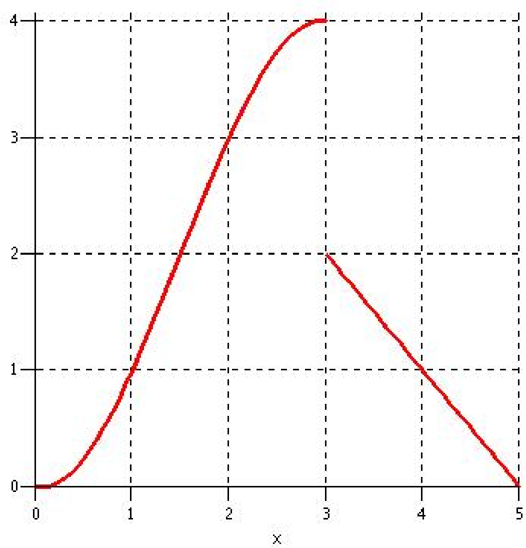
Since the function is a rational function, and the degree of the numerator is greater than the degree of the denominator, then the limit will go off to $+\infty$ or $-\infty$. We determine the sign by looking at the sign of both the numerator and denominator as $x \to -\infty$. The degree of the numerator is odd (it is a degree 5 polynomial), and therefore the numerator is negative as x goes to $-\infty$. Likewise, the denominator is a quadratic function, and hence it is positive as x goes to $-\infty$. Therefore, the quotient is negative as $x \to -\infty$.

Equivalently, one can perform the algebraic manipulation

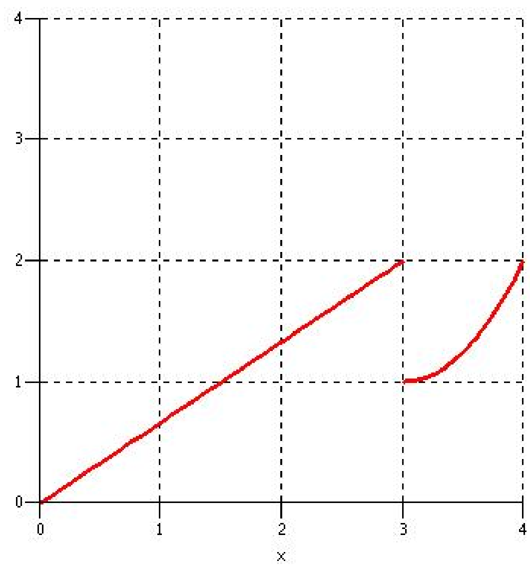
$$\lim_{x \to -\infty} \frac{x^5 - 4x^3 + 100}{10x^2 + 45} = \lim_{x \to -\infty} \frac{x^5 - 4x^3 + 100}{10x^2 + 45} \frac{x^{-2}}{x^{-2}} = \lim_{x \to -\infty} \frac{x^3 - 4x + 100x^{-1}}{10 + 45x^{-2}} = \frac{-\infty}{10} = -\infty.$$

For problems 9-10, use the graphs of $f(x)$ and $g(x)$ below.

$y = f(x)$



$y = g(x)$



9. $\displaystyle\lim_{x\to 3} \frac{f(x)}{g(x)} =$

(a) 1

(b) **2**

(c) 4

(d) $+\infty$

(e) Limit does not exist.

We analyze the limits from the left-hand and right-hand sides.

$$\lim_{x\to 3^-} \frac{f(x)}{g(x)} = \frac{4}{2} = 2$$

$$\lim_{x\to 3^+} \frac{f(x)}{g(x)} = \frac{2}{1} = 2.$$

The one-sided limits agree, and

$$\lim_{x\to 3} \frac{f(x)}{g(x)} = 2.$$

10. $\lim_{x \to 4} \frac{f(x) - 1}{x - 4} =$

(a) **-1**

(b) 0

(c) 1

(d) $-\infty$

(e) Cannot be determined

$$\lim_{x \to 4} \frac{f(x) - 1}{x - 4} = f'(4) = -1.$$

The first equality is the definition of the derivative. The value $f'(4)$ is the slope of $f(x)$ at $x = 4$, which is -1 in the graph.

Part II: Partial credit questions (10 points each)
Show all of your work!! Write clearly!

11. (a) State the Intermediate Value Theorem.

IF $f(x)$ is continuous on $[a, b]$, and L is some number in between $f(a)$ and $f(b)$, THEN there exists some c in (a, b) such that $f(c) = L$.

(b) Show that there exists a solution to the equation

$$\cos(x) + x^2 = \pi.$$

Let $f(x) = \cos x + x^2$. Note that $f(x)$ is continuous at all x.

$$f(0) = \cos 0 + 0^2 = 1 < \pi.$$

$$f(\pi) = \cos(2\pi) + (2\pi)^2 = 1 + 4\pi^2 > \pi.$$

Therefore, by the Intermediate Value Theorem, there exists some c in $(0, 2\pi)$ such that

$$f(c) = \cos(c) + c^2 = \pi.$$

(Different x-values can be chosen. The above x-values 0 and 2π were chosen to make $\cos(x)$ easy to evaluate.)

12. Let $f(x) = \sqrt{x+2}$.

(a) Using the definition of the derivative, find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+2+h} - \sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+2+h} - \sqrt{x+2}}{h} \left(\frac{\sqrt{x+2+h} + \sqrt{x+2}}{\sqrt{x+2+h} + \sqrt{x+2}} \right) \\
&= \lim_{h\to 0} \frac{x+2+h-x-2}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\
&= \lim_{h\to 0} \left(\frac{h}{h}\right) \frac{1}{\sqrt{x+2+h} + \sqrt{x+2}} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+2+h} + \sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

(b) Write the equation for the tangent line to the graph of $f(x)$ at $x = 7$.

Slope $m = f'(7) = \dfrac{1}{2\sqrt{7+2}} = \dfrac{1}{6}$.

Point $(x_0, y_0) = (7, f(7)) = (7, \sqrt{7+2}) = (7, 3)$.

$$\begin{aligned}
y - y_0 &= m(x - x_0). \\
y - 3 &= \frac{1}{6}(x - 7)
\end{aligned}$$

13. For which values of x is $w(x)$ continuous?

$$w(x) = \begin{cases} \dfrac{2}{x} & x < 0 \\ \\ \cos(x) + 1 & 0 \leq x \leq 2 \\ \\ \dfrac{1}{(x-1)^2} & x > 2 \end{cases}$$

First, note that each portion of the piecewise function is continuous on its domain. The function $\dfrac{1}{(x-1)^2}$ is only discontinuous at $x = 1$, but $w(x)$ only utilizes $\dfrac{1}{(x-1)^2}$ for $x > 2$. Therefore, we need to check for discontinuities at $x = 0, 2$, which is where the piecewise function switches between functions.

$\lim\limits_{x \to 0^-} w(x) = \lim\limits_{x \to 0^-} \dfrac{2}{x} = -\infty$. Therefore, $w(x)$ is discontinuous at $x = 0$.

$\lim\limits_{x \to 2^-} w(x) = \lim\limits_{x \to 2^-} \cos(x) + 1 = \cos(2) + 1$.

$\lim\limits_{x \to 2^+} w(x) = \lim\limits_{x \to 2^+} \dfrac{1}{(x-1)^2} = \dfrac{1}{(2-1)^2} = 1$.

$\lim\limits_{x \to 2^-} w(x) \neq \lim\limits_{x \to 2^+} w(x)$. Therefore, $w(x)$ is discontinuous at $x = 2$.

The function $w(x)$ is continuous at all x except $x = 0, 2$.

14. Suppose a tall astronaut lands on a fictional planet which happens to have a nice gravitational constant (making calculations simpler). The astronaut throws a rock vertically in the air. The rock's height in meters, t seconds after being thrown, is given by the function

$$s(t) = -5t^2 + 15t + 2.$$

(a) Find the average velocity over the first 2 seconds.

$$\frac{s(2) - s(0)}{2 - 0} = \frac{-5(4) + 15(2) + 2 - 2}{2 - 0} = \frac{10}{2} = 5\text{m/s}.$$

(b) Find the instantaneous velocity at $t = 2$ seconds. (You must use the definition of the derivative).

$$\begin{aligned}
v(2) = s'(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \\
&= \lim_{h \to 0} \frac{-5(2+h)^2 + 15(2+h) + 2 - (-5(2)^2 + 15(2) + 2)}{h} \\
&= \lim_{h \to 0} \frac{-5(4 + 4h + h^2) + 15(2) + 15h + 2 - (-5(4) + 15(2) + 2)}{h} \\
&= \lim_{h \to 0} \frac{-20h - 5h^2 + 15h}{h} \\
&= \lim_{h \to 0} \left(\frac{h}{h}\right)(-5 - 5h) \\
&= \lim_{h \to 0} -5 - 5h \\
&= -5\text{m/s}
\end{aligned}$$

15. Draw the graph of a function $h(x)$ that satisfies the following properties:

- $\lim_{x \to -\infty} h(x) = 1$
- $h(0) = 2$
- $\lim_{x \to 1^-} h(x) = +\infty$
- $\lim_{x \to 1^+} h(x) = -\infty$
- $h(3) = 2$
- $h'(3) = 0$
- $\lim_{x \to +\infty} h(x) = 0$