MAT 118: Mathematical Thinking Fall 2015

Stony Brook University

General Information

In the course we will explore various applications of mathematics. The main objective is to develop your mathematical thinking and problem solving abilities. During the semester we will work on different real-life mathematical problems such as: determining a winner in elections, finding efficient route, studying population growth etc..

Instructor:

Artem Dudko, artem.dudko@stonybrook.edu
Lectures: MWF 9:00-9:53am, Harriman Hall 137
Office hours: W 10:00-12:00am, Math Tower 3114, and W 12:00-1:00pm, Math Learning Center, Math Tower S-240A

Teaching assistants:

R01, W 5:30-6:23pm, Library N4072, Rayne Goldberg, rayne@math.stonybrook.edu
R02, M 1:00-1:53pm, Physics P127, Fangyu Zou, fangyu.zou@stonybrook.edu
R03, Th 1:00-1:53pm, Library N4072, Fangyu Zou

Textbook: Excursions in Modern Mathematics, by Peter Tannenbaum (8th edition, preferably)

Assignments: There will be weekly homework assignments (with a few exceptions) posted on the course web page due on Wednesday. You should hand in your assignments to the instructor at the end of Wednesday classes. Each homework will consist of several problems two or three of which will be graded (but you don't know which, so expected to do all of them). The first homework assignment will be due on September 7. Also, there will be also recommended problem sets. The focus of the course is on learning how to recognise, formulate and solve mathematical problems, therefore it is highly recommended that you work on recommended problems as well (even though it is not for grading).

Tests:

Midterm II: Monday, November 2, in class. TBA.
Midterm I: Monday, October 5, in class. It covers sections 1,2,3 and 5 of the course book. There will be 4 problems of the same type as homework problems and a few multiple choice type questions.
There will be a **review session** on Wednesday, September 30, 5-7pm in Melville Library W4525.
Last day of classes: Friday, December 4.

Final Exam: Wednesday, December 9, 8:30pm-11:00pm, Harriman 137 (the classroom). The **review** will be on Monday 3:30pm-5:30pm in the library building, W4550. For the final exam you need to know everything we learned after midterm 2 (6 problems + 4 multiple choice questions for this part):
Credit card debt, Installment loans (Section 10.4); Fibonacci numbers, Golden ratio, Binet's formula, sum of the first Fibonacci numbers and their squares, Gnomons (Chapter 13); Sample spaces and events, Probability rules, Permutations and combinations, Equiprobable and Non-equiprobable spaces (Sections 16.1-16.3; notice that the topics of Odds and Expectations are not included in the final); and the following topics from the material covered by the midterms:
Plurality with Elimination method, Shapley-Shubik power and Method of Sealed Bids (2 problems);
Cheapest Link Algorithm, Exponential Growth Model and Compound Interest (2 problems).

Course grade is computed by the following scheme:
Homework: 20%
Midterms: 40%
Final Exam: 40%

Letter grade cutoffs:
85-100 A
80-85 A-
75-80 B+
65-75 B
60-65 B-
55-60 C+
45-55 C
35-45 D
0-35 F

Information for students with disabilities
If you have a physical, psychological, medical, or learning disability that may impact your course work, please contact Disability Support Services at (631) 632-6748 or http://studentaffairs.stonybrook.edu/dss/. They will determine with you what accommodations are necessary and appropriate. All information and documentation is confidential.

Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.sunysb.edu/ehs/fire/disabilities.shtml

MAT 118 Homework assignments.

The exercises (unless stated otherwise) are from the course book "Excursions in modern mathematics", 8th edition, by Peter Tannenbaum. They can be found at the end of the corresponding chapter. Only 2-3 problems from each assignment will be graded, but you don't know which and are expected to do all of them. It is recommended that you read the corresponding chapters before doing the problems. Recommended problems are not for grading, but for practicing purposes.

HW1 (due on Wednesday, September 9):
Chapter 1, problems 3, 13, 23, 33, 43, 53.
Recommended problems: 7, 17, 27, 37, 47, 57.

HW2 (due on Wednesday, September 16):
Chapter 2, problems 2, 6, 12, 20 (a) and (d), 27, 29.
Recommended problems: 7, 13, 26, 30, 36.

HW3 (due on Wednesday, September 23):
Chapter 3, problems 2, 13, 15, 36, 43, 54.
Recommended problems: 4, 14, 18, 39, 44, 59.

HW4 (due on Wednesday, September 30):
Chapter 5, problems 4, 9, 14, 26, 29, 35.
Recommended problems: 8, 16, 19, 34, 37.

HW5 (due on Wednesday, October 14):
Chapter 6, problems 2, 11, 17, 19, 31, 33.
Recommended problems: 4, 6, 15, 23, 28.

HW6 (due on Wednesday, October 21):
Chapter 9, problems 2, 8, 12, 20.
Recommended problems: 5, 10, 13, 22.

HW7 (due on Friday, October 30):
Chapter 9, problems 26, 29, 38, 47, 54, 56.
Recommended problems: 24, 31, 40, 49, 58, 60.

Chapter 10 recommended problems: 3, 6, 13, 21, 26, 32, 33, 34
Remark: in problems 33 and 34 in case of investment compounded annually for a term which is not a whole number round it down, since a fraction of a year would not generate any interest.

HW8 (due on Wednesday, November 11):
Chapter 10, problems 38, 40, 51, 53.
Recommended problems: 36, 47, 52, 56.

HW9 (due on Friday, November 20):
Chapter 13, problems 4, 18, 35, 45 and the following
Problem A. Find the sum of the first 25 Fibonacci numbers.
Problem B. Draw a picture explaining why the sum of the squares of the first six Fibonacci numbers is equal to the product of the sixth and the sevenths Fibonacci numbers: $F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2 = F_6 F_7$.
Recommended problems: 3, 13, 15, 37, 44 and the following
Problem C. Find the sum of the squares of the first 25 Fibonacci numbers.

HW10 (due on Monday, November 30):
Chapter 16, problems 1, 6, 14, 39, 41, 44.
Recommended problems: 3, 7, 16, 40, 45.

MAT 118: Mathematical Thinking Fall 2015

Solutions

Midterm 1 Review
Midterm 1 Solutions
Midterm 2 Review
Midterm 2
Midterm 2 Solutions
Final Review

1)

MAT 118 Midterm 1 review.

Sections 1, 2, 3 and 5.

4 problems + 4 multiple choice questions.

1. **The Math of elections**

Three elements: candidates, voters, ballots.

Ballots can be: single-choice, preference or truncated preference.

We use <u>preference ballots</u> organized in a <u>preference schedule</u>

Four methods: plurality, Borda count, plurality-with elimination, pairwise comparison.

<u>Example</u> Using each of the methods find the outcome of the elections given a preference schedule:

Number of voters	3	5	1	4
1st	A	C	D	B
2nd	B	A	C	D
3rd	D	B	A	C
4th	C	D	B	A

2) a) plurality method (number of 1st votes)

A: 3, C: 5, D: 1, B: 4

$\boxed{\text{1st is C, 2nd is B, 3rd is A, 4th is D}}$

b) Borda Count

Number of voters	3	5	1	4
1st (4)	A (12)	C (20)	D (4)	B (16)
2nd (3)	B (9)	A (15)	C (3)	D (12)
3rd (2)	D (6)	B (10)	A (2)	C (8)
4th (1)	C (3)	D (5)	B (1)	A (4)

A: 12+15+2+4 = 33

B: 9+10+1+16 = 36

C: 20+3+3+8 = 34

D: 6+5+4+12 = 27

$\boxed{\text{1st is B, 2nd is C, 3rd is A, 4th is D}}$

c) pairwise comparison:

	Count	Winner
A v B	9:4	A
A v C	3:10	C
A v D	8:5	A
B v C	7:6	B
B v D	12:1	B
C v D	5:8	D

A: 2, B: 2, C: 1, D: 1

A, B share 1st and 2nd

C, D share 3rd and 4th

3) d) plurality with elimination.

1st round : A: 3, C: 5, D: 1, B: 4

D has fewest => gets eliminated, has 4th place

Number of voters	3	5	1	4
1st	A	C	C	B
2nd	B	A	A	C
3rd	C	B	B	A

2nd round: A: 3, C: 6, B: 4

A has fewest => gets eliminated

A has 3rd place

Number of voters	3	5	1	4
1st	B	C	C	B
2nd	C	B	B	C

B: 7, C: 6

B wins!

$\boxed{\text{1st B, 2nd C, 3rd A, 4th D}}$

4) $\boxed{\text{Weighted voting}}$

Basic elements: players, weights, quota
$P_1, P_2, \dots P_N$ $w_1, w_2, \dots w_N$ q

Weighed voting system

$[q \mid w_1, w_2, \dots w_N]$.

Terms to remember: dictator, veto power

$V = w_1 + w_2 + \dots + w_N$ the total number of votes.

Two method of computing the power of every player: Banzhaf power and Shapley - Shubik power.

1. Banzhaf power:

Important terms:

winning coalition, critical player, critical counts $B_1, B_2, \dots B_N$, total critical count $T = B_1 + B_2 + \dots + B_N$, Banzhaf power indices $\beta_1 = \frac{B_1}{T}$, $\beta_2 = \frac{B_2}{T}$, $\dots \beta_N = \frac{B_N}{T}$.

5) Example $[9 : 7, 4, 4, 1]$

Winning coalitions	Weight
$\{\underline{P_1}, \underline{P_2}\}$	11
$\{\underline{P_1}, \underline{P_3}\}$	11
$\{\underline{P_1}, P_2, P_3\}$	15
$\{\underline{P_1}, \underline{P_2}, P_4\}$	12
$\{\underline{P_1}, \underline{P_3}, P_4\}$	12
$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$	

$B_1 = 5, \quad B_2 = 3, \quad B_3 = 3, \quad B_4 = 1$

$T = 12$

$\beta_1 = \frac{5}{12}, \quad \beta_2 = \beta_3 = \frac{3}{12} = \frac{1}{4}, \quad \beta_4 = \frac{1}{12}.$

2. Shapley-Shubik power

Important terms:
sequential coalition, pivotal player, pivotal counts SS_i, Shapley-Shubik power indices $\sigma_i = \frac{SS_i}{N!}$, factorial of $N = 1 \times 2 \times 3 \times \ldots \times N$.
(total number of sequential coalitions.

6) <u>Example</u> $[9:7,4,4]$

$\langle P_1, \underline{P_2}, P_3 \rangle$, $\langle P_1, \underline{P_3}, P_2 \rangle$, $\langle P_2, \underline{P_1}, P_3 \rangle$,
$\langle P_2, P_3, \underline{P_1} \rangle$, $\langle P_3, \underline{P_1}, P_2 \rangle$, $\langle P_3, P_2, \underline{P_1} \rangle$

$SS_1 = 4$, $SS_2 = SS_3 = 1$

$\sigma_1 = \frac{4}{6} = \frac{2}{3}$, $\sigma_2 = \sigma_3 = \frac{1}{6}$

Fair-division games

<u>Basic elements</u>: the assets S, the players, the value systems, a fair division method.

<u>Important terms</u>: fair share, fair division, continuous and discrete fair division

Considered three fair-division methods: Lone-Chooser, Sealed Bids and Method of Markers.

1. <u>Lone-Chooser</u> works best for continuous fair division games.

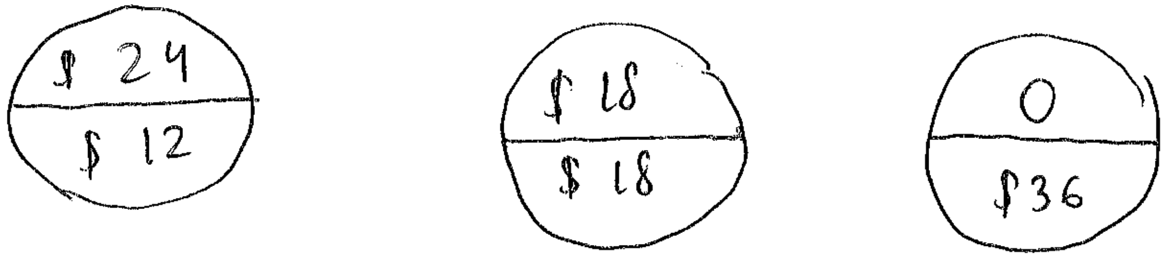
<u>example</u> peter, paul and Mary are dividing a chocolate-cheese cake worth \$36. Mary is a chooser, peter divides first

7)

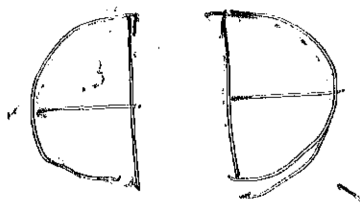
peter , paul mary

Chocolate

Cheese



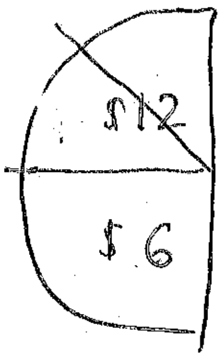
1) peter



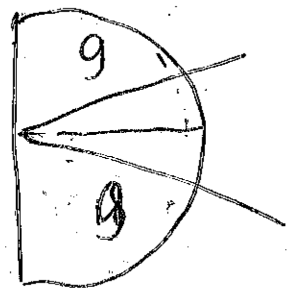
peter

paul

2) peter



paul



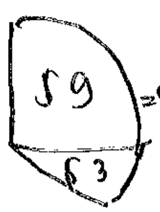
3)

mary

mary

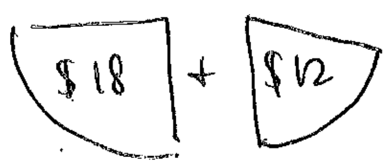
peter

paul



=$12

mary



=$30

8) 2. The method of sealed bids (discrete)

John and Ann inherited an apartment and a house

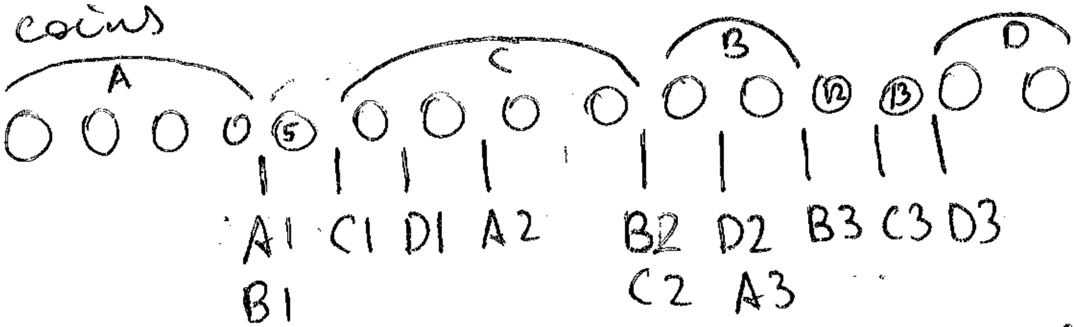
in thousands $

	John	Rita
apartment	400	300
House	300	320
Fair share	350	310
To (from) estate	50	10
Share of surplus	30	30
Total count	gets ap., pays 20	gets House and 20

surplus

$50 + 10 = 60$

3. The method of markers (discrete when there are many similar value items, or continuous). Four friend dividing 15 coins



Divide 5, 12 and 13 randomly.

9) $\boxed{\text{Street-routing problems}}$

Model using graphs

Basic elements: vertices and edges.

Important terms: degree of a vertex, adjacent vertices, connected graph, simple graph, path, circuit, bridges

(see more on p. 149)

Important methods

Euler's path, Euler circuit, Fleury's algorithm

<u>Example</u>

1. The following table shows the preference schedule for an election with four candidates (A, B, C and D). Use the pairwise comparison method to find the complete ranking of the candidates.

Number of voters	5	4	7	3	2
1st	C	A	D	D	B
2nd	B	B	B	C	C
3rd	A	D	A	B	D
4th	D	C	C	A	A

pair	Count	Winner
A v B	4 : 17	B
A v C	11 : 10	A
A v D	9 : 12	D
B v C	13 : 8	B
B v D	11 : 10	B
C v D	7 : 14	D

points: A : 1, B : 3, C : 0, D : 2

Answer: 1st is B, 2nd is D, 3rd is A, 4th is C

2. Find the Banzhaf power distribution of the weighted voting system $[11 : 6, 5, 3, 2]$. You can leave the answer in the form of a simple fraction (like $\frac{2}{7}$).

Winning coalition	Weight
$\{\underline{P_1}, \underline{P_2}\}$	11
$\{\underline{P_1}, \underline{P_2}, P_3\}$	14
$\{\underline{P_1}, \underline{P_2}, P_4\}$	13
$\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$	11
$\{\underline{P_1}, P_2, P_3, P_4\}$	16

Critical counts:

$B_1 = 5, \ B_2 = 3, \ B_3 = 1, \ B_4 = 1.$

Total count: $T = 5+3+1+1 = 10$

Answer: Banzhaf power indices:

$\beta_1 = \frac{5}{10} = 0.5, \ \beta_2 = \frac{3}{10} = 0.3, \ \beta_3 = \beta_4 = \frac{1}{10} = 0.1$

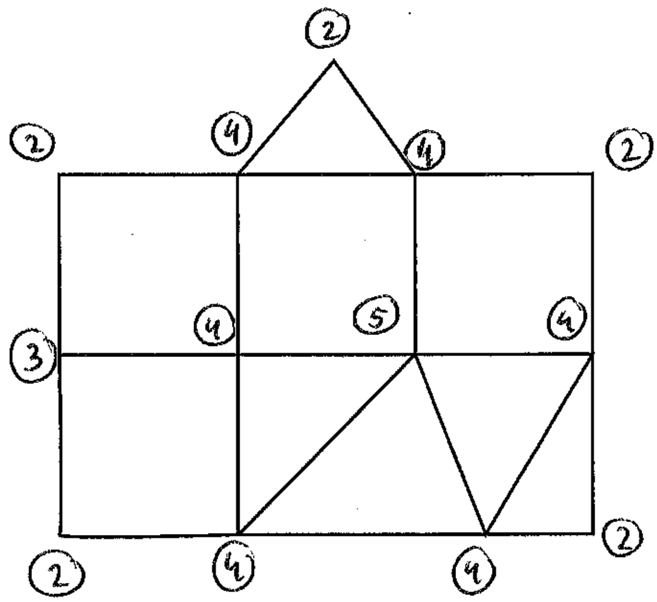
3. John and Rita are getting a divorce. They decide to split the house using the method of sealed bids. John's bid on the house is 400,000. Rita's bid is 600,000. Describe the outcome.

	John	Rita
Bid	400	600
Fair share	200	300
To(from)	(200)	300
Share of surplus	50	50
Final settlement	gets 250,000	gets house pays 250,000

surplus : 300−200=100

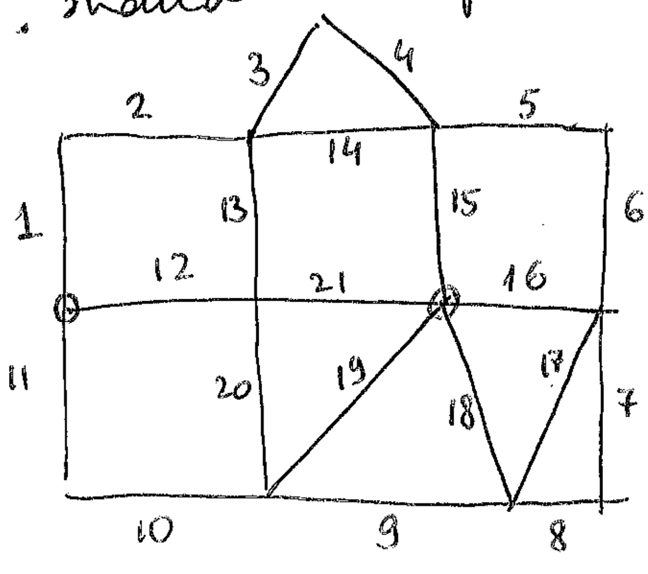
Answer John gets 250,000, Rita gets the house and pays 250,000.

4. A security guard is hired to patrol the streets of a neighborhood. A schematic picture of the neighborhood is shown below (lines represent streets). Is it possible to find a route for security guard covering each segment of the street exactly one time? The route can start and end at different points. If there is such a route then find it.



Two odd vertices, graph is connected.
By Euler's theorem, there exists an Euler's path.
We can find the path using Fleury's algorithm. Should start from an odd vertex.

Example of such path →



5. In each of the following multiple choice questions circle the correct answer.

1) The majority fairness criterion says that:

a) a candidate that beats each of the other candidates in a pairwise comparison should always be the winner;

b) a candidate with more than half of first place votes should always be the winner;

c) there is no such fairness criterion;

d) plurality method is the best method to determine a winner in an election.

2) A sequential coalition is:

a) the Shapley-Shubik power of the weighted voting system;

b) a coalition which has the total number of votes sufficient to pass a motion;

c) a coalition which has a veto power;

d) an ordered list of the players of a weighted voting system.

3) Which of the following statements is FALSE:

a) in a fair-division game with N players a share is called fair for a player P if its value for P is at least $\frac{1}{N}$th of the total value of assets;

b) if the set of assets can be divided in infinitely many ways and in arbitrarily small parts the fair division game is called indifferent;

c) a fair-division method guarantees that each player gets his/her fair share;

d) the method of markers is a discrete fair-division method.

4) Which statement is FALSE about a simple clique with 10 vertices (each of the vertices is connected by exactly one edge to each other):

a) it has no bridges;

b) it has an Euler path;

c) it is connected;

d) the degree of each vertex is 9.

1. Mat 118, Fall 2015, Midterm 2

Review

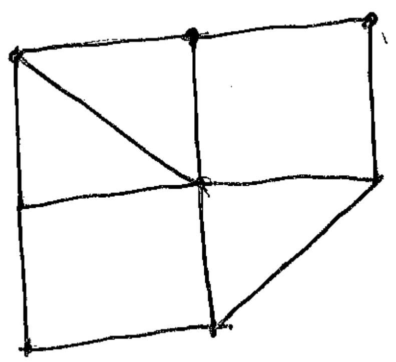
Section 5.4 Eulerizing and Semi-Eulerizing graphs

Def 1) Eulerization is the process of adding edges to a graph to make all vertices even.

2) semi-Eulerization is the process of adding edges to a graph to make all vertices except two even.

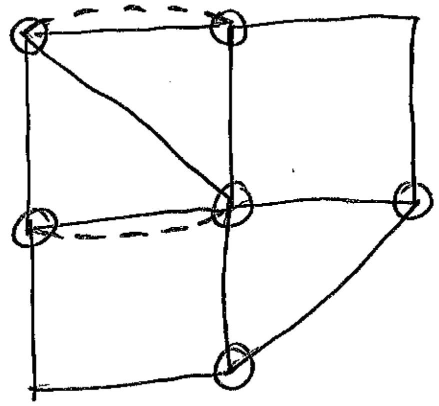
If initially the graph does not have an Euler cycle (or path) that is it has odd vertices (more than two odd vertices) to find an optimal circuit (or path) visiting all edges Eulerize (or semi-Eulerize) the graph first. Then use the Fleury's algorith ("don't born your bridges behind you").

Example Find an optimal a) path, b) circuit visiting every edge of the graph

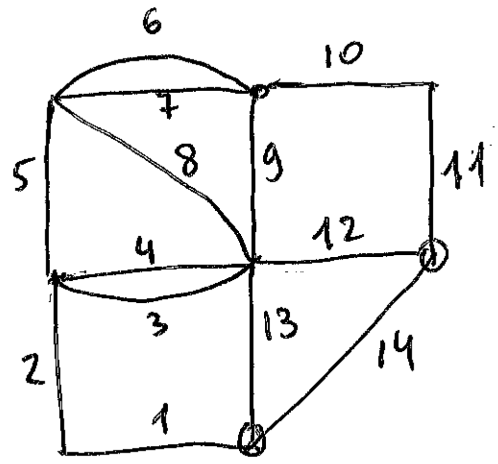


2. <u>Solution</u> there are 6 odd vertices.

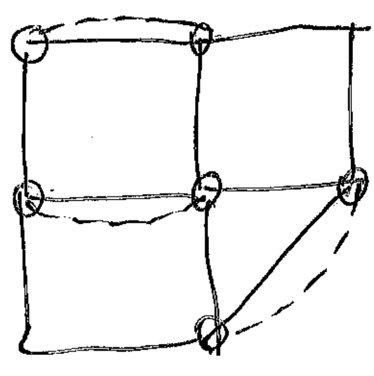
a) Need to Semi-Eulerize first. Add edges to make the number of odd vertices equal to two (as shown).

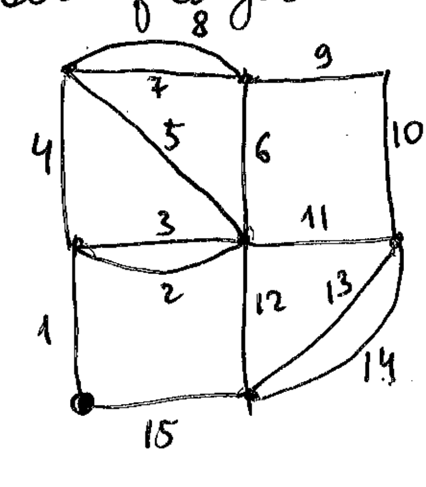


Minimal number of extra vertices is 2. On the new graph find an Euler path using the Fleury's algorithm (start at an odd vertex)



b) Need to Eulerize (make all vertices even), then find an Euler circuit. Minimal number of edges to add is 3.





start euler circuit anywhere

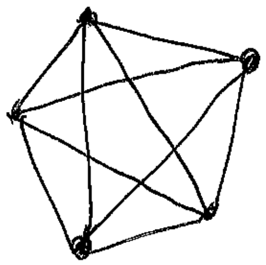
3. Section 6: Traveling Salesman Problem

Elements:
- a traveler
- a set of N sites
- a set of costs

Solution of a TSP is a tour that starts and ends at a site and visits all the other sites ones. An optimal solution is a tour of minimal total cost.

Hamilton path is a path visiting all the vertices of the graph exactly ones.
Hamilton circuit is a circuit visiting all the vertices of the graph exactly ones.
Complete graph K_N is a graph in which each two vertices are connected by an edge

K_5



properties of K_N:
- degree of every vertex is $N-1$
- number of edges is $\frac{N(N-1)}{2}$
- number of Hamilton paths is $N!$
- number of Hamilton circuits is $(N-1)!$

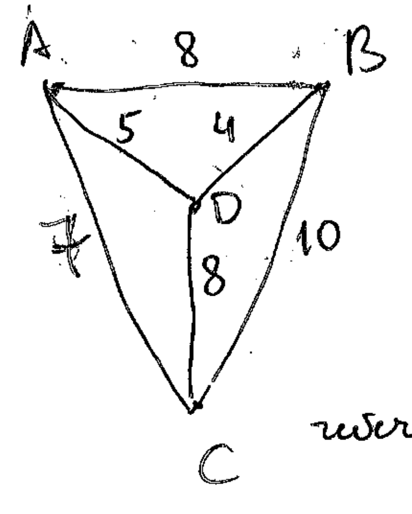
4. Algorithms for solving TSP

- Exact algorithm: Brute-force (comparing values of all possible Hamilton circuits)

Approximate algorithms:

- Nearest-neighbor algorithm (NNA) (start from any vertex, each step go to the "nearest" one)
- Repetitive nearest-neighbor algorithm (RNNA) (use NNA starting from each vertex, choose the cheapest route)
- Cheapest-link algorithm (CLA) (construct the path from cheapest links not violating the partial-circuit or three-edge rule).

Example Use the Brute-force algorithm to find the shortest route visiting 4 cities (A, B, C, D) starting and ending at B.



Solution List all paths, calculate the lengths.

B, A, C, D, B : $8+7+8+4=27$
B, A, D, C, B : $8+5+8+10=31$
B, C, A, D, B : $10+7+5+4=26$
B, C, D, A, B : 31
B, D, A, C, B : 26
B, D, C, A, B : 27

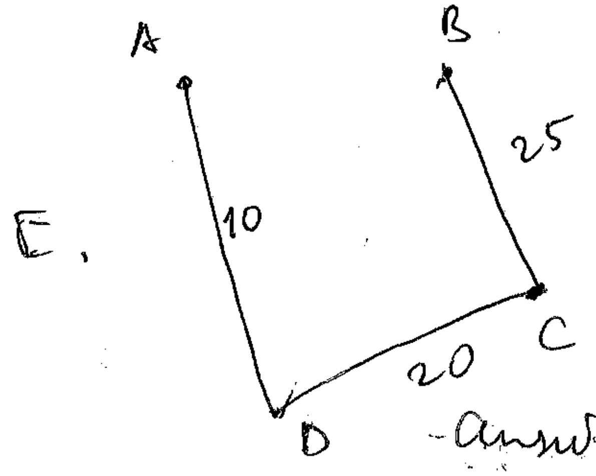
reversals

5. <u>answer</u>: B, C, A, D, B (or B, D, A, C, B), total length 26.

<u>Example</u> given prices of getting between cities A, B, C, D, E find a cheap route starting and ending at D and visiting all cities using: a) NNA starting at A b) RNNA, c) CLA.

	A	B	C	D	E
A		20	35	10	60
B	20		25	30	50
C	35	25		20	45
D	10	30	20		50
E	60	50	45	50	

<u>Solution</u> a) From A the cheapest is getting to D (10), from D to C (20), from C to B (25), from B we can go only to E (50) since all other cities are visited, and then come back to A (60).



A, D, C, B, E, A, total cost 165.
Rewrite starting at D:
answer: D, C, B, E, A, D, total cost 165

6. b) Hamilton circuit | Total cost

$A \xrightarrow{10} D \xrightarrow{20} C \xrightarrow{25} B \xrightarrow{50} E \xrightarrow{60} A$: 165

$B \xrightarrow{20} A \xrightarrow{10} D \xrightarrow{20} C \xrightarrow{45} E \xrightarrow{50} B$: 145

$C \xrightarrow{20} D \xrightarrow{10} A \xrightarrow{20} B \xrightarrow{50} E \xrightarrow{45} C$: 145

$D \xrightarrow{10} A \xrightarrow{20} B \xrightarrow{25} C \xrightarrow{45} E \xrightarrow{50} D$: 150

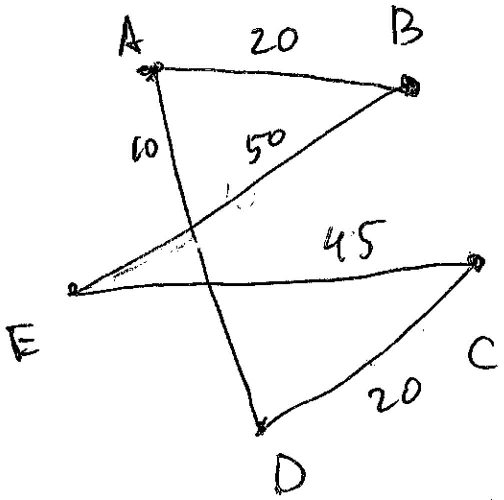
$E \xrightarrow{45} C \xrightarrow{20} D \xrightarrow{10} A \xrightarrow{20} B \xrightarrow{50} E$: 145

B, A, D, C, E, B is the least expensive: 145.

Rewrite starting at D:

answer: D, C, E, B, A, D. Total cost 145.

c) The cheapest link is AD (10). Next cheapest are AB and CD (both 20). Pick one, then another (as soon as three-edge and partial circuit rules are not violated). Next cheapest is BC (25), but we can't pick it since this would create a partial circuit. In fact, there is only one way to complete the circuit: by adding edges EB and EC (50 and 45).



7. Writing this path starting at D gives D, A, B, E, C, D (or its reversal D, C, E, B, A, D) of total cost 145.

answer: D, C, E, B, A, D. Total cost 145

<u>Relative error of a tour</u>

If C is the total cost of a given tour and Opt is the total cost of the optimal tour the relative error is $\varepsilon = \dfrac{C - Opt}{Opt}$

In the example above in fact the optimal tour has cost $Opt = 145$. The RNNA and CLA give optimal tours (relative error is zero). But NNA starting at A gives a tour which costs 165 and has a relative error

$$\varepsilon = \frac{165-145}{145} \simeq 0.14 = 14\%$$

8.

Section 9: population growth models

<u>Sequences</u>

$A_1, A_2, A_3, A_4, \dots$

Can be described by:

- explanation in words
- several terms $A_1, A_2, A_3, A_4, \dots$
- recursive formula
- general formula

<u>Example</u>

$A_n = \frac{1}{n}$ is the general formula describing the sequence of inverse positive integers.

First several terms:

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

Recursive formula:

$$A_{n+1} = \frac{1}{\frac{1}{A_n}+1} = \frac{A_n}{A_n+1}, \quad A_1 = 1.$$

<u>Linear growth model</u> means that in each generation the population changes by a constant amount d.

- $P_0, P_0+d, P_0+2d, P_0+3d, \dots$
- recursive formula: $P_{n+1} = P_n + d$
- general formula: $\boxed{P_n = P_0 + n \cdot d}$

d is called a common difference

9. The sequence P_n is called an arithmetic sequence.

Example The population of the town of Smallville was 50,000 in 1990 and reached 110,000 in 2002. Assuming linear growth what was the population of Smallville in 1995?

Solution Let P_n be the population in the year of $1990+n$. Then

in $1990: n=0, P_0 = 50,000$

in $2002; n=12, P_{12} = 110,000$

Want to find: in $1995; n=5, P_5 = ?$

We know: $P_n = P_0 + n \cdot d$

$$P_{12} = P_0 + 12 \cdot d$$

$$110,000 = 50,000 + 12d$$

$$60,000 = 12d$$

$$5,000 = d.$$

Thus, $P_5 = P_0 + 5d = 50,000 + 5 \cdot 5,000 =$

75,000.

Answer 55,000

arithmetic sum formula:

$$P_0 + P_1 + \ldots + P_{n-1} = \frac{P_0 + P_{n-1}}{2} \cdot n$$

10 Example Each resident of smallville pays fixed \$2000 tax per year for town renovation. How much tax the town collected from 1990 to 2002 inclusively?

Solution 1990: $2000 P_0$

1991: $2000 P_1$

...

2002: $2000 P_{12}$.

In total: $2000 \cdot (P_0 + P_1 + P_2 + \ldots + P_{12})$.

To us the arithmetic sum formula!

$P_{n-1} = P_{12}$, $n-1 = 12$, $n = 13$.

$$P_0 + P_1 + \ldots + P_{12} = \frac{P_0 + P_{12}}{2} \cdot 13 =$$

$$\frac{50{,}000 + 110{,}000}{2} \cdot 13 = 1{,}040{,}000$$

Total tax: $2{,}000 \cdot 1{,}040{,}000 =$

\$2,080 mln

Answer \$2,080 mln

11. <u>Exponential growth model</u> means that in each generation the population growth by the same constant factor R.

- $P_0, RP_0, R^2 P_0, R^3 P_0, \ldots$
- Recursive formula: $P_n = R P_{n-1}$
- general formula: $\boxed{P_n = R^n P_0}$

R is called the common ration.

The sequence is called a geometric sequence

<u>Growth rate</u> $r = \frac{Y-X}{X}$

end value

baseline

population sequence is exponential if the growth rate is constant. Then $r = R - 1$ and $\boxed{P_n = (r+1)^n P_0}$

<u>Example</u> An epidemic of new disease started in 1950. In February there were 50 infected individuals. In March the number of new infected was 75. assuming exponential growth approximate the number of new infected in November of 1950.

<u>Solution</u> Let P_n be the number of new infected n months after February 1950.

12. Then $P_0 = 50$, $P_1 = 75$. Exponential growth means $P_n = R^n P_0$, $n=1$:

$P_1 = R P_0$,

$75 = 50R$, $R = \frac{75}{50} = 1.5$.

Thus, $P_n = (1.5)^n \cdot 50$.

In November, 9 months after February, the number of new infected is

$$P_9 = (1.5)^9 \cdot 50 \approx 1922$$

answer: 1922 new infected

Geometric sum formula:

$$\boxed{P_0 + P_0 R + P_0 R^2 + \dots + P_0 R^{n-1} = \frac{R^n - 1}{R - 1} P_0}$$

In the previous example, what's the total number of infected from January to November?

Solution $P_0 = 50$, $R = 1.5$, $P_0 R^{n-1} = (1.5)^9 \cdot 50$

thus, $n - 1 = 9$, $n = 10$

$$50 + 50 \cdot 1.5 + 50 \cdot (1.5)^2 + \dots + 50 \cdot (1.5)^9 = \frac{1.5^{10} - 1}{1.5 - 1} \cdot 50 \approx$$

5,667.

answer 5,667

13. <u>Logistic growth model</u>

Elements: maximal carrying capacity C, p-value of the population $p_n = \frac{P_n}{C}$, growth parameter R.

For logistic growth model in n-th generation the growth ratio is proportional to R and the "elbow room" $1-p_n$.

- Recursive formula: $\boxed{p_{n+1} = R(1-p_n) \cdot p_n}$ is called the logistic equation.

<u>Example</u> N 59 p. 289. Find $p_1, p_2, \dots p_{10}$, $R = 2.8$, $p_0 = 0.15$. Describe the behavior.

<u>Solution</u> $p_1 = R(1-p_0)p_0 = 2.8 \cdot (1-0.15) \cdot 0.15 = 0.357$

$p_2 = R(1-p_1)p_1 = 2.8 \cdot (1-0.357) \cdot 0.357 \approx 0.6427$,

...

p-values sequence:

0.15, 0.357, 0.6427, 0.6429, 0.6428, 0.6429, 0.6428, 0.6429, 0.6428, 0.6429, 0.6428, 0.6429, 0.6428.

The switching between 0.6428 and 0.6429. The two numbers are very close. Likely, the p-value stabilizes at some value between 0.6428 and 0.6429.

Answer: stabilizes at some number close to 0.6429.

14. Other types of behavior possible for Logistic model:

- two-cycle behavior (switching between two values starting from some generation)
- four-cycle behavior (" " 4 values " ")
- random behavior (no pattern)

MAT 118 FALL 2015 MIDTERM II

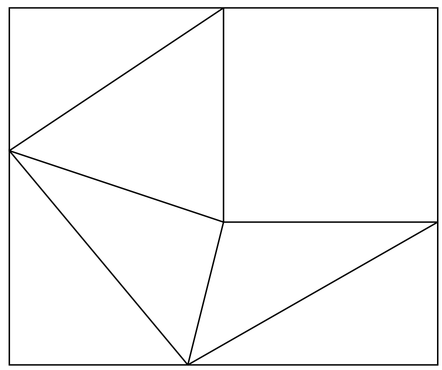
NAME : ID :

RECITATION : (M, W or Th)

THERE ARE 4 PROBLEMS, 16 POINTS EACH
AND 4 MULTIPLE CHOICE QUESTION, 9 POINTS EACH
SHOW YOUR WORK
DO NOT TEAR-OFF ANY PAGE
NO NOTES NO CELLS ETC.
ON YOUR DESK: ONLY test, pen, pencil, calculator, eraser and student ID

1		16pts
2		16pts
3		16pts
4		16pts
5		36pts
Total		100pts

1. Eulerize the following graph and find an optimal circuit covering each edge of the initial graph at least once.



2. Distances between 5 villages (A, B, C, D and E) are given in a table. Find an effective route visiting all the villages and coming back to the initial city using the Nearest Neighbor Algorithm starting at A. Find the total length of this route.

	A	B	C	D	E
A	x	7	5	3	4
B	7	x	6	6	5
C	5	6	x	2	4
D	3	6	2	x	3
E	4	5	4	3	x

3. Consider a population of rabbits in a forest that grows according to a linear growth model. If there were 400 rabbits in the beginning of 2000 and 500 rabbits in the beginning of 2004 how many rabbits were there in the beginning of 2015?

4. Suppose you purchase a 5 year U.S. savings bond with an APR of 5%. The face value of the bond is $6,000. Find the purchase price of the bond.

5. In each of the following multiple choice questions circle the correct answer.

1) Semi-eulerization is:

a) the process of adding additional vertices to the graph so that all the edges except two are even;

b) finding the shortest route visiting at least half of all edges;

c) the process of adding additional edges to the graph so that all the vertices except two are even;

d) a method of solving the Traveler Salesman Problem.

2) Which of the following is FALSE about Hamilton paths and circuits:

a) any Hamilton circuit is a Hamilton path;

b) a complete graph with N vertices has $N!$ Hamilton paths;

c) disconnected graphs do not have Hamilton paths;

d) any Hamilton path is a Hamilton circuit.

3) Which of the following is TRUE about population growth models:

a) in the logistic growth model animal population may alternate cyclically between two different levels of population;

b) in the exponential growth model the population is always growing;

c) in the linear growth model the population is always decreasing;

d) in the logistic growth model the growth rate is constant (does not depend on the generation).

4) Among the following statements choose the one which describes simple interest most accurately:

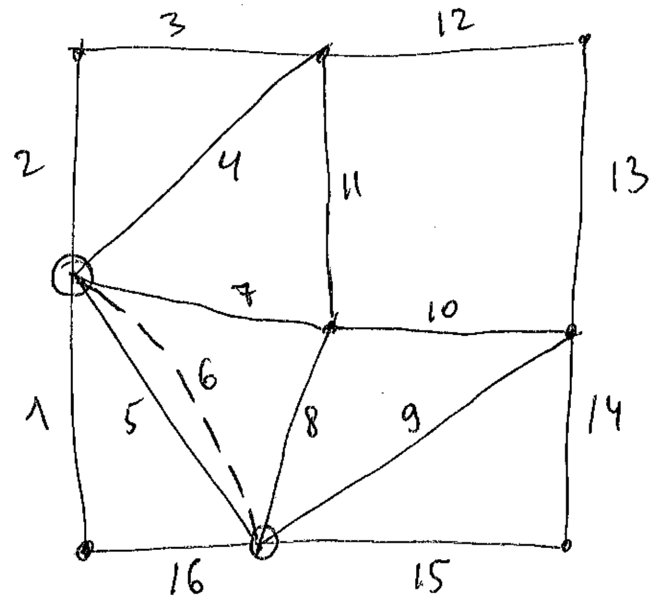
a) the interest rate is applied both to the principal value P and to the previously accumulated interest;

b) it is always applied once per year;

c) this is the only type of interest used in savings accounts;

d) the interest rate is applied only to the principal value P.

N 1.



Eulerizing means making all vertices even. There are two odd vertices. Sufficient to add one edge joining them. Then an Euler circuit on a new graph will be an optimal circuit for the initial one. Such circuit can be found using Fleury's algorithm. Start at any vertex.

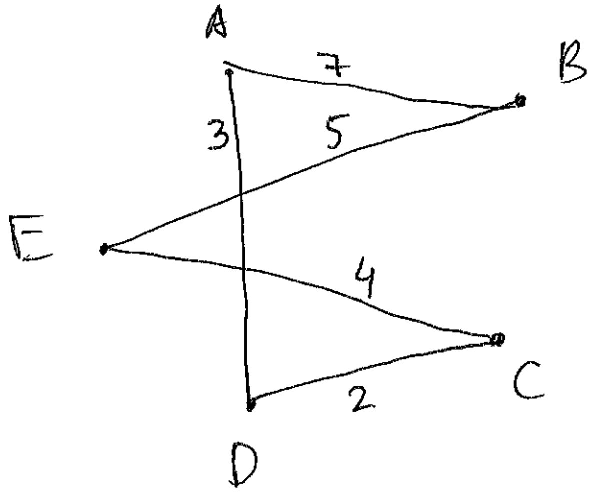
N2. From A go to the closest: D

From D to next closest: C

From C to next closest: E.

From E it remains to visit only B.

From B back to A



A, D, C, E, B, A

Total length $3+2+4+5+7 = 21$

Answer: A, D, C, E, B, A, length 21

N3 In 2000: $P_0 = 400$

In 2004: $P_4 = 500$

Linear growth means

$$P_n = P_0 + nd.$$

When $n = 4$:

$$P_4 = P_0 + 4d$$

$$500 = 400 + 4d$$

$$4d = 100$$

$$d = \frac{100}{4} = 25.$$

Thus, $P_n = P_0 + n \cdot d = 400 + n \cdot 25$

In 2015: $n = 15$

$$P_{15} = 400 + 25 \cdot 15 = 775$$

Answer: 775

N4. Bonds use simple interest. If P is the purchase price, F is the face value, r is the interest rate and t is the term then $F = P(1 + r \cdot t)$. The decimal value of 5% is $r = \frac{5}{100} = 0.05$. Thus

$$6000 = P \cdot (1 + 0.05 \cdot 5) = P \cdot 1.25$$

$$P = \frac{6000}{1.25} = 4{,}800$$

Answer: \$4,800

N5
1) c
2) a or d
3) a or d
4) d

Fibonacci numbers and the golden ratio

Basic facts

- 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- $F_1 = F_2 = 1$, $\boxed{F_{n+1} = F_n + F_{n-1}}$ ← recursive formula
- $\boxed{F_n = [[\left(\frac{\sqrt{5}+1}{2}\right)^n / \sqrt{5}]]}$ ← Binet's formula
- $\frac{F_{n+1}}{F_n}$ approaches the golden ratio

φ $\frac{\sqrt{5}+1}{2}$ when n grows.

Example Find F_{13} a) using the recursive formula; b) using Binet's formula.

Solution a) 1, 1, 2, 3, 5, 8, 13, 21, 34

F_1 F_2 F_3 ... F_8 F_9

$F_{10} = F_9 + F_8 = 21 + 34 = 55$

$F_{11} = F_{10} + F_9 = 55 + 34 = 89$

$F_{12} = F_{11} + F_{10} = 89 + 55 = 144$

$F_{13} = F_{12} + F_{11} = 144 + 89 = 233.$

b) $F_{13} = [[\left(\frac{\sqrt{5}+1}{2}\right)^{13} / \sqrt{5}]]$

$\left(\frac{\sqrt{5}+1}{2}\right)^{13} / \sqrt{5} = 232.9991402...$

Round to the nearest integer: $F_{13} = 233$

2. <u>Sum of the first n Fibonacci numbers:</u>

$$\boxed{F_1 + F_2 + \ldots + F_n = F_{n+2} - 1}$$

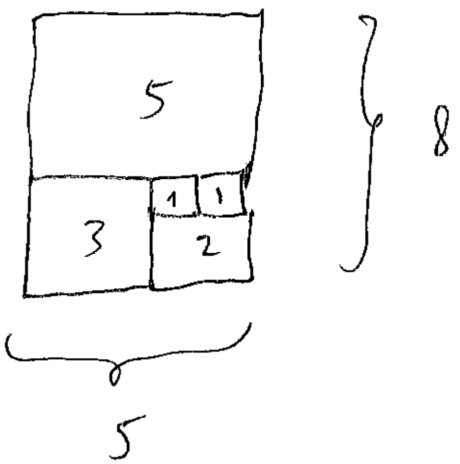
<u>Sum of the squares of the first n Fibonacci numbers:</u>

$$\boxed{F_1^2 + F_2^2 + \ldots + F_n^2 = F_n \cdot F_{n+1}}$$

Illustration:

$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 = F_5 \cdot F_6$$

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 5 \cdot 8$$



<u>Example</u> a) Find the sum of the first 17 Fibonacci numbers; b) find the sum of the squares of the first 17 Fibonacci numbers.

3- <u>Solution</u> a) $F_1 + F_2 + \dots + F_{17} = F_{19} - 1$

$$F_{19} = \left[\!\left[\left(\frac{\sqrt{5}+1}{2}\right)^{19} / \sqrt{5} \right]\!\right] = 4181$$

Thus, $F_1 + F_2 + \dots + F_{17} = 4180$

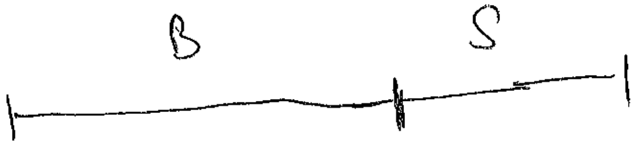
b) $F_1^2 + F_2^2 + \dots + F_{17}^2 = F_{17} \cdot F_{18}$

$$F_{17} = \left[\!\left[\left(\frac{\sqrt{5}+1}{2}\right)^{17} / \sqrt{5} \right]\!\right] = 1597$$

$$F_{18} = \left[\!\left[\left(\frac{\sqrt{5}+1}{2}\right)^{18} / \sqrt{5} \right]\!\right] = 2584$$

$F_1^2 + F_2^2 + \dots + F_{17}^2 = 1597 \cdot 2584 =$
4126648

<u>golden ratio</u>



$$\frac{B}{S} = \frac{B+S}{B}, \quad \varphi = \frac{B}{S} \text{ is}$$

the golden ratio. It satisfies

$$\boxed{\varphi^2 = \varphi + 1}$$

golden ratio value: $\boxed{\varphi = \frac{\sqrt{5}+1}{2}}$

approximate value: $\boxed{\varphi \approx 1.618}$

4. Example: Show that $\varphi^4 = 3\varphi + 2$.

Solution: $\varphi^4 = \varphi \cdot \varphi \cdot \varphi \cdot \varphi = \varphi^2 \cdot \varphi^2$

$\varphi^4 = (\varphi+1)(\varphi+1) = \varphi^2 + \varphi + \varphi + 1$

$\varphi^4 = \varphi^2 + 2\varphi + 1 = (\varphi+1) + (2\varphi+1)$

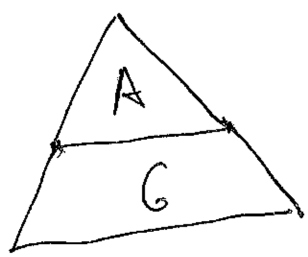
$\varphi^4 = 3\varphi + 2$.

Gnomons

If a figure G suitably attached to figure A produces a new figure similar to A then G called a gnomon to A.

Examples

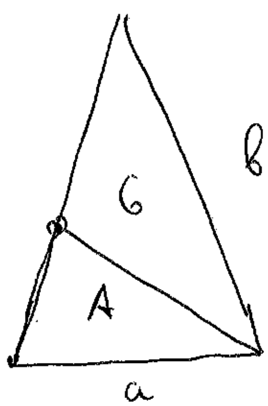
1)



2)

Related to golden ratio:

3) Triangle with angles $36°, 72°, 72°$

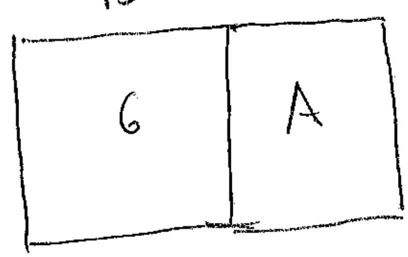


Ratio of sides is

$$\frac{b}{a} = \varphi$$

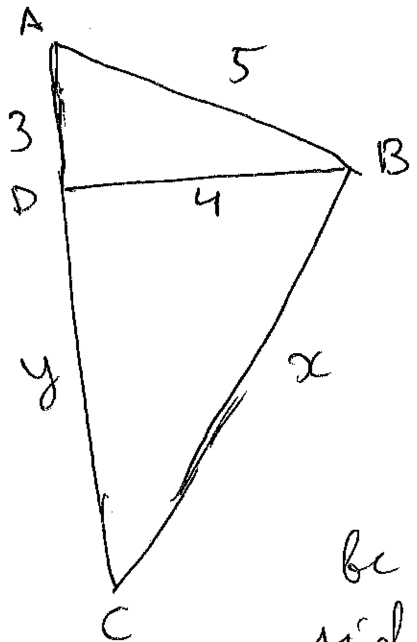
5,

4) Golden rectangle (ratio of sides is the golden ratio).



$$\frac{B}{S} = \varphi$$

<u>Example</u> Find the values of x and y so that 6 is a gnomon to the smaller triangle



<u>Solution</u> The largest triangle ABC should be similar to the smallest ABD. Sides of the largest $3+y$, x and 5 should be proportional to the corresponding sides of the smallest: 5, 4, 3.

So $\frac{3+y}{5} = \frac{x}{4} = \frac{5}{3}$. Thus,

$$x = 4 \cdot \frac{5}{3} = \frac{20}{3}$$

$$3+y = 5 \cdot \frac{5}{3} = \frac{25}{3}$$

$$y = \frac{25}{3} - 3 = \frac{25-9}{3} = \frac{16}{3}$$

<u>answer</u> $x = \frac{20}{3}$, $y = \frac{16}{3}$.

6. $\boxed{\text{probability theory}}$

Basic elements!

- Random experiment
- Sample space = set of possible outcomes
- Events = subsets of the sample space

<u>Example</u>: Soccer penalties.
Four players shooting after match penalties.
Describe the sample space and the event that
a) no one scored, b) at least three players scored.

<u>Solution</u> G for goal, M for missed. Then
an outcome can be written as for symbols G or M

Sample space = {GGGG, GGGM GGMG, GGMM, ...
MMMM } = all possible 4-tuples of G and M.

In total $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$ outcomes

$N = 16$

events

E_1 = no one scored = {MMMMM} $k_1 = 1$

E_2 = at least three goals =
{GGGM, GGMG, GMGG, MGGG, GGGG}

$k_2 = 5$

7. Equiprobable space probabilities of all outcomes are equal. $\boxed{Pr(E) = \frac{k}{N}}$

Example Assume that in the previous example probability that each players scores is 50%. What is the probability that a) no one scores; b) at least three players out of 4 score.

Solution a) $Pr(E_1) = \frac{k_1}{N} = \frac{1}{16}$

b) $Pr(E_2) = \frac{k_2}{N} = \frac{5}{16}$

Rules of probability:

- Complement If E and F are complementary events then $Pr(E) = 1 - Pr(F)$
- Additivity If E and F are disjoint events then $Pr(E \text{ or } F) = Pr(E) + Pr(F)$
- Multiplication If E and F are independent events then $Pr(E \text{ and } F) = Pr(E) \cdot Pr(F)$

Example In the previous example what is the probability that at least two players will miss?

Solution F = at least two players will miss is the opposit to E_2

8. F and E_2 are complementary

$$Pr(F) = 1 - Pr(E_2)$$

$$Pr(F) = 1 - \frac{5}{16} = \frac{11}{16}$$

<u>Example</u> Assume that each of the players in fact has 70% chance to score. What is the probability that, at least three of four will score in this case?

<u>Solution</u> Scoring or missing for different players are independent events. $Pr(G) = 0.7$, $Pr(M) = 0.3$

$$E_2 = \{GGGM, GGMG, GMGG, MGGG, GGGG\}$$

$$Pr(E_2) = Pr(GGGM) + Pr(GGMG) + \ldots + Pr(GGGG)$$

additivity rule

By multiplication rule

$$Pr(GGGM) = Pr(G) \cdot Pr(G) \cdot Pr(G) \cdot Pr(M) =$$

$$0.7 \cdot 0.7 \cdot 0.7 \cdot 0.3 = 0.1029$$

Similarly, $Pr(GGMG) = Pr(GMGG) =$

$Pr(MGGG) = 0.1029$.

But $Pr(GGGG) = 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 = 0.2401$

$$Pr(E_2) = 4 \cdot 0.1029 + 0.2401 = 0.6517$$

9. Combinations and permutations

Permutation is an ordered selection of r objects from a set of n objects

Combination is an unordered selection of r objects from a set of n objects.

Number of permutations:

$$_nP_r = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+1) = \frac{n!}{(n-r)!}$$

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Example: A lottery ticket has a random 7-digit number (which is allowed to start from 0). You win if all digits are different (e.g. 5817632 or 0192587 are winning, but 2910713 and 6722298 are not). What are the chances to win?

Solution Sample space S = all seven digit numbers from 0000000 to 9999999

$N = 10^7$.

Event E = numbers with different digits

E = permutations of 7 digits from a total of 10.

10. $k = {}_{10}P_7 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 604800$

$$Pr(E) = \frac{k}{N} = \frac{604800}{10000000} = 0.06048$$

11. Installment loans:

Amortization formula for monthly payment M on a loan with principle P, annual percentage rate r paid over T monthly installments

$$\boxed{M = P \cdot \frac{p(1+p)^T}{(1+p)^T - 1}},$$

where $p = \frac{r}{12}$ is the monthly interest rate

Example Buying a house for \$500,000 financing for 27 years at 4% APR. How much will be paid in total?

Solution $r = 4\% = 0.04$

$P = 500{,}000$, $p = \frac{r}{12} = \frac{0.04}{12} \approx 0.00333$

$T = 27 \cdot 12 = 324$.

$$M = 500\,000 \cdot \frac{0.00333 \cdot 1.00333^{324}}{1.00333^{324} - 1} \approx 2562.38$$

Total payment

$T \cdot M = 324 \cdot 2562.38 \approx 830211$