EXERCISES FROM THE PROBLEM SESSIONS FOR THE CONFERENCE, "NEW TECHNIQUES IN BIRATIONAL GEOMETRY", 7-11 APRIL 2015

Alexander Kuznetsov. Exercises on Exceptional Collections. Let P be a k-point of $\mathbb{P}^2_k$. Let $\pi : X \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2_k$ at P. Let $i : E \to X$ be the inclusion of the exceptional divisor, $E \cong \mathbb{P}^1_k$. Let $f_P : X \to \mathbb{P}^1$ be the linear projection away from P.

Associated to the blowing up π there is a full exceptional collection in $D^b(X)$,

$$\mathcal{E}_1 = \langle i_*\mathcal{O}_E(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \pi^*\mathcal{O}_{\mathbb{P}^2}(2)\rangle.$$

Associated to the $\mathbb{P}^1$-bundle f there is another full exceptional collection in $D^b(X)$,

$$\mathcal{E}_2 = \langle f^*\mathcal{O}_{\mathbb{P}^1}, f^*\mathcal{O}_{\mathbb{P}^1}(1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_{\mathbb{P}^1}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_{\mathbb{P}^1}(1)\rangle.$$

Problem 0.1. Find a sequence of mutations from $\mathcal{E}_1$ to $\mathcal{E}_2$.

Next, let P_1 and P_2 be two distinct k-points of $\mathbb{P}^2_k$. Let $\rho : Y \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2_k$ at P_1 and P_2. The exceptional locus F has two disjoint connected components, F_1 and F_2, each isomorphic to $\mathbb{P}^1_k$. Denote by $j : F \to X$ the inclusion. The linear projections f_{P_1} and f_{P_2} induce a morphism,

$$(f_{P_1}, f_{P_2}) : Y \to \mathbb{P}^1_k \times_{\mathrm{Spec}(k)} \mathbb{P}^1_k.$$

This morphism is a blowing up at the point $Q = (f_{P_1}(P_2), f_{P_2}(P_1))$. Denote by $h : G \to Y$ the exceptional divisor of this blowing up, $G \cong \mathbb{P}^1_k$.

Associated to the blowing up ρ there is a full exceptional collection in $D^b(Y)$,

$$\mathcal{F}_1 = \langle j_*\mathcal{O}_{F_1}(-1), j_*\mathcal{O}_{F_2}(-1), \rho^*\mathcal{O}_{\mathbb{P}^2}, \rho^*\mathcal{O}_{\mathbb{P}^2}(1), \rho^*\mathcal{O}_{\mathbb{P}^2}(2)\rangle.$$

Associated to the blowing up (f_{P_1}, f_{P_2}) there is a full exceptional collection in $D^b(Y)$,

$$\begin{aligned}\mathcal{F}_2 = \langle h_*\mathcal{O}_G(-1), f^*_{P_1}\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathcal{O}_Y} f^*_{P_2}\mathcal{O}_{\mathbb{P}^1}, f^*_{P_1}\mathcal{O}_{\mathbb{P}^1}(1) \otimes_{\mathcal{O}_Y} f^*_{P_2}\mathcal{O}_{\mathbb{P}^1},\\ f^*_{P_1}\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathcal{O}_Y} f^*_{P_2}\mathcal{O}_{\mathbb{P}^1}(1), f^*_{P_1}\mathcal{O}_{\mathbb{P}^1}(1) \otimes_{\mathcal{O}_Y} f^*_{P_2}\mathcal{O}_{\mathbb{P}^1}(1)\rangle.\end{aligned}$$

Problem 0.2. Find a sequence of mutations from $\mathcal{F}_1$ to $\mathcal{F}_2$.

Burt Totaro. Exercises on Base Change Homomorphisms. Let k be a field. For every k-variety X_k and for every integer q, there is the free Abelian group of all q-cycles,

$$Z_q(X_k) = \langle [V_k] | V_k \subset X_k, \text{ closed, integral, } \dim(V_k) = q\rangle.$$

Inside $Z_q(X_k)$ there is the subgroup $\mathrm{Rat}_q(X_k)$ generated by all q-cycles of the form $u_*\mathrm{div}(f)$, where $u : W \to X_k$ is any proper morphism from any normal k-variety of

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dimension $q+1$, where f is any nonzero rational function on W, and where $\mathrm{div}(f)$ is the principal (Weil) divisor on W of f. The quotient group is the **Chow group**,

$$\mathrm{CH}_q(X_k) = Z_q(X_k)/\mathrm{Rat}_q(X_k).$$

For every field extension F/k, denote $X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F)$ by X_F. For every $V_k \subset X_k$ as above, the base change $V_F \subset X_F$ is a closed subscheme of pure dimension q that gives a cycle $[V_F] \in Z_q(X_F)$ (note that V_F may not be integral). The induced base change homomorphism,

$$u_{F/k,X_k,Z^q} : Z_q(X_k) \to Z_q(X_F), \quad [V_k] \mapsto [V_F],$$

maps $\mathrm{Rat}_q(X_k)$ to $\mathrm{Rat}_q(X_F)$. Thus, there is a well-defined homomorphism of Chow groups,

$$u_{F/k,X_k,\mathrm{CH}_q} : \mathrm{CH}_q(X_k) \to \mathrm{CH}_q(X_F).$$

There is also an induced homomorphism of $\mathbb{Q}$-vector spaces,

$$u_{F/k,X_k,\mathrm{CH}_q} \otimes \mathbb{Q} : \mathrm{CH}_q(X_k) \otimes \mathbb{Q} \to \mathrm{CH}_q(X_F) \otimes \mathbb{Q}.$$

Problem 0.3. Find an example of a field k, a smooth, projective k-variety X_k, field extension F/k, and an integer q such that the induced homomorphism $u_{F/k,X_k,\mathrm{CH}_q}$ is not surjective. In fact, find an example such that $u_{F/k,X_k,\mathrm{CH}_q} \otimes \mathbb{Q}$ is not surjective.

Problem 0.4. Find an example of a field k, a smooth, quasi-projective k-variety X_k, field extension F/k, and an integer q such that the induced homomorphism $u_{F/k,X_k,\mathrm{CH}_q}$ is not injective. For a challenge, find an example where X_k is projective.

Problem 0.5. Prove that for every field k, for every quasi-projective k-variety X_k, for every field extension F/k, and for every integer q the induced homomorphism $u_{F/k,X_k,\mathrm{CH}_q} \otimes \mathbb{Q}$ is injective.

Claire Voisin. Exercises on Torsion Cohomology, Griffiths Groups and Decompositions of the Diagonal.

Problem 0.6. For every (second countable, Hausdorff) topological manifold M, prove that the singular cohomology group $H^1(M;\mathbb{Z})$ is torsion-free.

Problem 0.7. For every smooth, projective, complex variety X, for the subgroups $\mathrm{CH}_1(X)^{\mathrm{alg}} \subset \mathrm{CH}_1(X)^{\mathrm{hom}} \subset \mathrm{CH}_1(X)$ of cycles that are algebraically equivalent to zero, resp. homologically equivalent to zero, prove that the quotient **Griffiths group**, $\mathrm{CH}_1(X)^{\mathrm{hom}}/\mathrm{CH}_1(X)^{\mathrm{alg}}$ is a birational invariant.

Problem 0.8. For every smooth, projective, complex variety X, for every element $\alpha \in H^0(X, \Omega^q_X)$, if there exists a dense, Zariski open subset $U \subset X$ such that $\alpha|_U$ is exact, then prove that α equals 0. Please do this without using mixed Hodge structures.

Problem 0.9. For every (second countable, Hausdorff) topological manifold M that is connected and oriented, prove that the Künneth homomorphism,

$$\bigoplus_{0\leq p,q} H^p(M;\mathbb{Z})\otimes H^q(M;\mathbb{Z}) \to \bigoplus_{0\leq r} H^r(M\times M;\mathbb{Z}),$$

is an isomorphism if and only if $\bigoplus_r H^r(M;\mathbb{Z})$ is torsion-free.

Problem 0.10. For every smooth, projective, complex variety X of dimension n, for the diagonal class $[\Delta_X]\in \mathrm{CH}_n(X\times X)$, there exists a decomposition

$$[\Delta_X] = a_1[Z_1\times T_1]+\cdots+a_r[Z_r\times T_r]$$

for integers $a_1,\dots,a_r$, and for integral subvarieties of X, $Z_1,\dots,Z_r,T_1,\dots,T_r$ if and only if numerical equivalence of cycles in X equals rational equivalence.

Problem 0.11. For every smooth, projective, complex surface X, for the cohomological diagonal class $[\Delta_X]$ in $H^*(X^{\text{an}}\times X^{\text{an}};\mathbb{Q})$, resp. in $H^*(X^{\text{an}}\times X^{\text{an}};\mathbb{Z})$, there exists a decomposition

$$[\Delta_X] = a_1[Z_1\times T_1]+\cdots+a_r[Z_r\times T_r]$$

for integers $a_1,\dots,a_r$, and for integral subvarieties of X, $Z_1,\dots,Z_r,T_1,\dots,T_r$ if and only if $q(X)=p_g(X)=0$, resp. if and only if both $q(X)=p_g(X)=0$ and $H^*(X^{\text{an}};\mathbb{Z})$ is torsion-free.

Lev Borisov. Degrees of Calabi-Yaus. Recall that a smooth, projective variety X is a **Calabi-Yau variety** if $\omega_X\cong\mathcal{O}_X$, if X is simply connected, and if $h^0(X,\Omega^q_X)$ vanishes for $0<q<\dim(X)$.

Problem 0.12. Using computer code, using (skew-symmetric) Thom-Porteous, and using Schubert calculus, compute the degrees of Pfaffian Calabi-Yau varieties, respectively Grassmannian Calabi-Yau varieties.

Alena Pirutka. Problems on Rationality.

Problem 0.13. For a smooth quadric hypersurface $X_k\subset\mathbb{P}^n_k$, prove that X is rational if and only if X has a k-point.

Problem 0.14. Let k be an algebraically closed field. For every k-variety X_k and for every field extension K/k, prove that X_k is k-rational if and only if the base change X_K is K-rational.

For the next sequence of exercises, let k be a field (not necessarily algebraically closed nor even infinite). Let X_k be a k-variety of dimension m. Let $\phi:\mathbb{A}^n_k \dashrightarrow X_k$ be a dominant rational transformation. Necessarily $n\geq m$, and these exercises investigate whether there exists ϕ with $n=m$.

Problem 0.15. Prove that there exists a dense, Zariski open $U\subset\mathbb{A}^n_k$ such that $\phi|_U$ is a morphism whose (nonempty) fibers are pure-dimensional of dimension $d=n-m$.

Problem 0.16. Assume now that k is infinite. Prove that there exists a k-point u of U and a hyperplane $H \subset \mathbb{A}_k^n$ containing u such that the restriction of ϕ to $U \cap H$ is dominant. Use induction on n to prove that there exists a dominant rational transformation from $\mathbb{A}_k^m$ to X_k.

Problem 0.17. Finally, assume that k is a finite field. Let ℓ be an integer different from the characteristic. Let $K \subset \overline{k}$ be the union of all extension fields of k of degree ℓ^s, $s > 0$.

(a) First prove that $U(K)$ is not empty.
(b) For an arbitrary point $u = (u_1, \ldots, u_n)$ in $\mathbb{A}_k^n(K)$, use the Primitive Element Theorem to prove that, up to a permutation, $k(u_n) \subset k(u_{n-1}) \subset \cdots \subset k(u_1)$. Use this to prove that the ideal $\mathfrak{m}_u \subset k[x_1, \ldots, x_{n-1}, x_n]$ is generated by elements in $k[x_1, \ldots, x_{n-1}]$ and elements of the form $x_n - P(x_1, \ldots, x_{n-1})$.
(c) Finally, prove that there exists an affine hypersurface $Z = \mathrm{Zero}(x_n - P(x_1, \ldots, x_{n-1}))$ in $\mathbb{A}_k^n$ containing u such that the restriction of ϕ to $Z \cap U$ is dominant. Again use induction to prove that there exists a dominant rational transformation from $\mathbb{A}_k^m$ to X_k.