NOTES FROM THE DISCUSSION SESSION FOR THE CONFERENCE, "NEW TECHNIQUES IN BIRATIONAL GEOMETRY", 10 APRIL 2015

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Topic 1. Claire Voisin. One-cycles on Rationally Connected Varieties. Let $(X, \mathcal{O}_X)$ be a smooth, projective, complex variety of dimension n with associated complex manifold $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. For every integer d, denote by $\mathrm{Hdg}_{2d}(X^{\mathrm{an}}, \mathbb{Z}) = \mathrm{Hdg}^{2n-2d}(X^{\mathrm{an}}, \mathbb{Z})$ the inverse image of the Hodge summand $H^{n-d,n-d}(X^{\mathrm{an}})$ under the change-of-coefficients homomorphism,

$$H^{2n-2d}(X^{\mathrm{an}}, \mathbb{Z}) \to H^{2n-2d}(X^{\mathrm{an}}, \mathbb{C}).$$

This is the *integral Hodge group*. There is a cycle class homomorphism,

$$\mathrm{cycle}_d : CH_d(X) \to \mathrm{Hdg}_{2d}(X^{\mathrm{an}}, \mathbb{Z}).$$

According to the Hodge conjecture, the image has finite index. However, even in those cases where this conjecture is proved, typically the image is not the entire Hodge group.

Denote the cokernel of the cycle class map by

$$Z_d(X) := \mathrm{Hdg}_{2d}(X^{\mathrm{an}}, \mathbb{Z})/CH_d(X).$$

By the same argument as used in the proof of the Artin-Mumford theorem, the group $Z_2(X)$ is a birational invariant of X. Typically $Z_2(X)$ is nonzero. For instance, Koll'ar's "Trento examples" prove that $Z_2(X)$ is nonzero for X a very general hypersurface in $\mathbb{P}^4$ of sufficiently large degree.

Question 0.1. If X is rationally connected, is $Z_2(X)$ trivial? In other words, is the cycle class map to $\mathrm{Hdg}_2(X^{\mathrm{an}}, \mathbb{Z})$ surjective?

Remark 0.2. There are many partial results.

(1) For rationally connected varieties, $Z_2(X)$ is locally a deformation invariant via the "comb-smoothing" technique. Also the group is stable for several elementary operations, such as passing to the total space of a projective bundle, Grassmannian bundle, etc. (which proves that this birational invariant is a stable birational invariant).

(2) For $\mathfrak{o} \subset \mathbb{C}$ a normal, finitely generated $\mathbb{Z}$-algebra, and for a smooth, projective $\mathfrak{o}$-scheme, $X_{\mathfrak{o}}$, for the base change $X = X_{\mathfrak{o}} \otimes_{\mathfrak{o}} \mathbb{C}$, in an appropriate sense the group $Z_2(X)$ is invariant under specializing to closed fibers $X_F = X_{\mathfrak{o}} \otimes_{\mathfrak{o}} F$, for finite residue fields $F = \mathfrak{o}/\mathfrak{m}$ of $\mathfrak{o}$. Using work of Chad Schoen, it is known that if the Tate conjecture is true for all divisor classes on all surfaces over all finite fields, then for every rationally connected X over $\mathbb{C}$, the group $Z_2(X)$ is trivial.

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(3) Voisin proved that $Z_2(X)$ is trivial for every rationally connected threefold.
(4) Höring and Voisin proved that $Z_2(X)$ is trivial for every Fano fourfold.

There is another stable birational invariant defined in terms of one-cycles. Denote by $\mathrm{CH}_1(X)^{\mathrm{hom}} \subset \mathrm{CH}_1(X)$ the kernel of the cycle class homomorphism. This contains the subgroup $\mathrm{CH}_1(X)^{\mathrm{alg}}$ of one-cycles that are algebraically equivalent to zero. The **Griffiths group** is the quotient,

$$\mathrm{Griff}_1(X) := \mathrm{CH}_1(X)^{\mathrm{hom}}/\mathrm{CH}_1(X)^{\mathrm{alg}}.$$

For instance, for a very general quintic threefold X, Clemens proves that $\mathrm{Griff}_1(X) \otimes \mathbb{Q}$ has *infinite* dimension as a $\mathbb{Q}$-vector space. On the other hand, Bloch and Srinivas proved that $\mathrm{Griff}_1(X)$ is a torsion group whenever X is rationally connected.

Question 0.3. For every smooth, projective, rationally connected variety X over $\mathbb{C}$, is $\mathrm{Griff}_1(X)$ trivial?

Remark 0.4. There are a number of partial results.

(1) Zhiyu Tian and Runhong Zong prove that $\mathrm{Griff}_1(X)$ is generated by classes of rational curves. Via the "bend-and-break" technique, the group is even generated by classes of curves with bounded anticanonical degree. In particular, this implies *a priori* bounds for Fano manifolds.
(2) For every Fano complete intersection of index ≥ 2, Tian-Zong prove that $\mathrm{Griff}_1(X)$ is trivial.
(3) For rationally connected threefolds, Block and Srinivas prove that $\mathrm{Griff}_1(X)$ is trivial.

One particularly interesting test case for both questions are total spaces of conic bundles over a rationally connected bases when the discriminant locus is reducible.

Topic 2. Olivier Debarre. Hodge Classes on Principally Polarized Abelian Varieties. Let (A, Θ) be an n-dimensional, prinicipally polarized Abelian variety over $\mathbb{C}$. Denote by $\theta \in \mathrm{Hdg}^2(A^{\mathrm{an}}, \mathbb{Z})$ the cycle class of Θ. Then the class

$$\theta_{n-1} := \frac{\theta^{n-1}}{(n-1)!}$$

is an element in $\mathrm{Hdg}_2(A^{\mathrm{an}}, \mathbb{Z})$. If (A, Θ) is sufficiently generic, this class should even generate $\mathrm{Hdg}_2(A^{\mathrm{an}}, \mathbb{Z})$.

Question 0.5. For which principally polarized Abelian varieties is θ_{n-1} trivial in $Z_2(X)$, i.e., when is θ_{n-1} algebraic?

Remark 0.6. First of all, whenever (A, Θ) is the polarized Jacobian of a curve, then θ_{n-1} is algebraic. Also, since $Z_2(X)$ specializes, this problem also specializes. Thus θ_{n-1} is algebraic if $n \leq 3$.

Voisin points out that for the 5-dimensional polarized intemediate Jacobian of a cubic threefold X, θ_4 is algebraic if and only if $\mathrm{CH}_0(X)$ is universally trivial.

Topic 3. Moduli Spaces of Fano Manifolds (and their Specializations). This is a somewhat open-ended problem: what is known about the existence of quasi-projective moduli spaces of Fano manifolds and their specializations? Totaro points out that there is recent progress on this problem for Fano manifolds that support a Kähler-Einstein manifold, at least those that also have discrete automorphism group (existence of a Kähler-Einstein metric is not always stable under small deformation if the automorphism group has positive dimension).