The Projective Geometry of the Gale Transform

by

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The Gale transform, an involution on sets of points in projective space, appears in a multitude of guises, in subjects as diverse as optimization, coding theory, theta-functions, and recently in our proof that certain general sets of points fail to satisfy the minimal free resolution conjecture (see Eisenbud-Popescu [1996]). In this paper we reexamine the Gale transform in the light of modern algebraic geometry. We give a more general definition, in the context of finite (locally) Gorenstein subschemes. We put in modern form a number of the more remarkable examples discovered in the past, and we add new constructions and connections to other areas of algebraic geometry. We generalize Goppa's theorem in coding theory and we give new applications to Castelnuovo theory. We give references to classical and modern sources.

Contents

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It was in joint work with David Buchsbaum that the first author first became familiar with the notion of a Gorenstein ring. A large part of this paper ("self-associated sets") is concerned, from an algebraic point of view, with the classification and study of a special type of Gorenstein ring, generalizing some of the examples found in that joint work. It is with especial pleasure that we dedicate this paper to David.

Let r, s be positive integers, and let $\gamma = r + s + 2$. The classical *Gale transform* is an involution that takes a (reasonably general) set $\Gamma \subset \mathbf{P}^r$ of γ labeled points in a projective space $\mathbf{P}^r$ to a set Γ' of γ labeled points in $\mathbf{P}^s$, defined up to a linear transformation of $\mathbf{P}^s$. Perhaps the simplest (though least geometric) of many equivalent definitions is this: if we choose homogeneous coordinates so that the points of $\Gamma \subset \mathbf{P}^r$ have as coordinates the rows of the matrix

$$\left(\begin{array}{c} I_{r+1} \\ \hline A \end{array}\right),$$

where I_{r+1} is an $(r+1) \times (r+1)$ identity matrix and A is a matrix of size $(s+1) \times (r+1)$, then the Gale transform of Γ is the set of points Γ' whose homogeneous coordinates in $\mathbf{P}^s$ are the rows of the matrix

$$\left(\begin{array}{c} A^T \\ \hline I_{s+1} \end{array}\right),$$

where A^T is the transpose of A.

It is not obvious from this definition that the Gale transform has any "geometry" in the classical projective sense. Here are some examples that suggest it has:

a) $r = 1$: The Gale transform of a set of $s+3$ points in $\mathbf{P}^1$ is the corresponding set of $s+3$ points on the rational normal curve that is the s-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^s$. Conversely, the Gale transform of any $s+3$ points in linearly general position in $\mathbf{P}^s$ is the same set in the $\mathbf{P}^1$ that is the unique rational normal curve through the original points. See Corollary 3.2 and the examples following it.

b) $r = 2, s = 2$: There are two main cases: a complete intersection of a conic and a cubic is its own Gale transform (a "self-associated set"). On the other hand, if Γ consists of 6 points not on a conic, then the Gale transform of Γ is the image of the 5 conics through 5 of the 6 points of Γ via the Cremona transform that blows up the 6 points and then blows down the conics. See Example 5.12.

c) $r = 2, s = 3$: A set Γ of 7 general points in $\mathbf{P}^3$ lies on 3 quadrics, which intersect in 8 points. The Gale transform of Γ is the projection of Γ from the eighth point. Again, see the examples following Corollary 3.2.

Section 1 contains what we know of the history of the Gale transform, including work of Pascal, Hesse, Castelnuovo, Coble, Dolgachev-Ortland, and Kapranov.

In Section 2 we introduce a general definition of the Gale Transform as an involution, induced by Serre duality, on the set of linear series on a finite Gorenstein scheme over a field (here as always in this paper, Gorenstein means locally Gorenstein). This language turns out to be very convenient even in the classical (reduced) case. The main result of this section is an extension of the Cayley-Bacharach

Theorem for finite complete intersections (or, more generally, finite arithmetically Gorenstein schemes) to finite locally Gorenstein schemes. It interprets the failure of a set of points to impose independent conditions on a linear series as a condition on the Gale transform.

The next sections treat basic properties of the Gale transform, and examples derived from them, such as the ones above. Section 3 is devoted to an extension of a famous theorem of Goppa in coding theory. A linear code is essentially a set of points in projective space, and the dual code is its Gale transform. Goppa's theorem asserts that if a linear code comes from a set of points on a smooth linearly normal curve, then the dual code lies on another image of the same curve. Examples a) and c) above are special cases. We show how to extend this theorem (and its scheme-theoretic generalization) to sets of points contained in certain other varieties, such as ruled varieties over a curve. Using these results we exhibit some of the classical examples of the Gale transform, and provide some new ones as well; for instance 9 general points of $\mathbf{P}^3$ lie on a smooth quadric surface, which is a ruled variety in two different ways. It follows from our theory that the Gale transform, which will be 9 general points of $\mathbf{P}^4$, lies on two different cubic ruled surfaces. In fact, we show that the 9 general points are the complete intersection of these two surfaces.

In Section 4 we use the Gale transform to give a simple proof of Eisenbud-Harris' generalization to schemes of Castelnuovo's lemma that $r+3$ points in linearly general position in $\mathbf{P}^r$ lie on a unique rational normal curve in $\mathbf{P}^r$. We also prove a similar result on when finite schemes in linearly general position lie on higher dimensional rational normal scrolls, and when these scrolls may be taken to be smooth. Our method provides a simple proof (in many cases) for a result of Cavaliere-Rossi-Valla [1995].

In Section 5 we show that if $\Gamma \subset \mathbf{P}^r$ and $\Gamma' \subset \mathbf{P}^s$ are related by the Gale transform, then the canonical modules ω_Γ and $\omega_{\Gamma'}$ are related in a simple way. This is the idea exploited in Eisenbud-Popescu [1996] to study the minimal free resolutions associated to general point sets $\Gamma \subset \mathbf{P}^r$ and in particular to disprove the Minimal Resolution Conjecture. As an application we exhibit an example due to Coble, connecting the Gale transform of 6 points in the plane with the Clebsch transform (blow up the six points, blow down the proper transforms of the conics through five of the six.)

One family of examples that does not seem to have been considered before are the determinantal sets of points. In Section 6 we describe a novel relationship, expressed in terms of the Gale transform of Veronese re-embeddings, between the zero-dimensional determinantal varieties defined by certain "adjoint" pairs of matrices of linear forms.

A major preoccupation of the early work on the Gale transform was the study of "self-associated" sets of points, that is, sets of points $\Gamma \subset \mathbf{P}^r$ that are equal to their own Gale transforms (up to projective equivalence, of course). This notion only applies to sets of $2r+2$ points in $\mathbf{P}^r$, since otherwise the Gale transform doesn't even lie in the same space. For example, 6 points in the plane are self-associated iff they are complete intersection of a conic and a cubic (and this is the essential content of Pascal's "Mystic Hexagram"). It turns out that this is indeed a natural notion: under mild non-degeneracy assumptions Γ is self-associated iff its homogeneous coordinate ring is Gorenstein! Section 7 is devoted to a study of self-association and a generalization: Again under mild extra hypotheses, a Gorenstein scheme $\Gamma \subset \mathbf{P}^r$ has Gorenstein homogeneous coordinate ring iff the Gale transform of Γ is equal to a Veronese transform of Γ. We review the known geometric constructions of self-associated sets, and add a few new ones. It would be interesting to know whether the list contains any families of Gorenstein ideals not yet investigated by the algebraists.

In Section 8 we continue the study of self-associated sets of points, showing how they are related to nonsingular bilinear forms on the underlying vector space of $\mathbf{P}^r$. A classical result states that self-associated sets correspond to pairs of orthogonal bases of such a form. We say what it means for a non-reduced scheme to be the "union of two orthogonal bases", and generalize the result correspondingly. We also reprove and generalize some of the other classic results on self-associated sets, showing for example that the variety of self-associated sets of labeled points in $\mathbf{P}^r$ is isomorphic to an open set of the variety of complete flags in $\mathbf{P}^r$, a result of Coble and Dolgachev-Ortland (see the references below).

It is interesting to ask, given a set of γ points in $\mathbf{P}^r$ with $\gamma < 2r+2$, whether it can be extended to a set of points of degree $2r+2$ with Gorenstein homogeneous coordinate ring—indeed, such questions arise implicitly in our work on free resolutions [1996]. From the theory developed in Section 8 we are able to give interesting information in some cases. For example, we show that a set of 11 general points in $\mathbf{P}^6$ can be completed to a set of 14 points with arithmetically Gorenstein homogeneous coordinate ring. We show that (although the extension is not unique) the three points added span a plane that is uniquely determined. This plane appears as the "obstruction" to the truth of the minimal resolution conjecture for 11 points in $\mathbf{P}^6$, as treated in our paper [1996].

In Section 9 we continue with self-associated sets, and describe what is known about the classification of small dimensional projective spaces, up to $\mathbf{P}^5$.

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1 History

Perhaps the first result that belongs to the development of the Gale transform is the theorem of Pascal (from his "Essay Pour Les Coniques" from 1640, reproduced in Struik [1969]) that the vertices of two triangles circumscribed around the same conic lie on another conic. As we shall see in Section 8, this is a typical result about sets of points that are Gale transforms of themselves ("self-associated sets"). Hesse, in his Dissertation and Habilitationschrift in Königsberg [1840] (see the paper in Crelle's Journal [1840] and the reprinted Dissertation in Werke [1897]), found an analogue of Pascal's result (see also Zeuthen [1889, p. 363]) that held for 8 points in three dimensional space and gave various applications. Some of Hesse's results were made clearer and also extended by von Staudt [1860], Weddle [1850], Zeuthen [1889], and Dobriner [1889] (see also p.152, Tome 3, of the Encyclopedie [1992])

The step to defining the Gale transform itself in the corresponding cases of 6 points in $\mathbf{P}^2$ was taken by Sturm [1877], and extended by Rosanes [1880, 1881]. More important is the realization by Castelnuovo in [1889] that one could do the same sort of thing for $2r+2$ points in $\mathbf{P}^r$ in general. He called two sets of $2r+2$ points that are Gale transforms of one another "gruppi associati di punti". Castelnuovo, who refers to Sturm and Rosanes but seems unaware of Hesse's work, gives the following geometric definition:

Two sets Γ and Γ', each of $2r+2$ labelled points in $\mathbf{P}^r$ and $(\mathbf{P}^r)^*$ respectively, are defined to be associated when there exist two simplices Δ and Δ' in $\mathbf{P}^r$ such that the points of Γ are projective with the $2r+2$ vertices of Δ and Δ', while the points of Γ' are projective with the $2r+2$ facets of the two simplices (each facet being labeled by the opposite vertex).

Castelnuovo was primarily interested in the case when a set of points is self-associated; this is the case of Pascal's 6 points on a conic, for example. As we shall see, a general set of $2r+2$ points is associated to itself iff its homogeneous coordinate ring is Gorenstein. It is interesting that the stream of work that lead Castelnuovo in this direction has the same source in Pascal's theorem as the stream that lead to the Cayley-Bacharach Theorem and its ramifications, another early manifestation of the Gorenstein property (see Eisenbud-Green-Harris [1996] for a discussion).

The first one to have studied the Gale transform of a set of γ points in $\mathbf{P}^r$ without assuming $\gamma = 2r+2$ seems to have been Coble, and we begin with his definition. As before, we may represent an ordered set of γ points $\Gamma \subset \mathbf{P}^r$ by a $\gamma \times (r+1)$ matrix of homogeneous coordinates, though this involves some choices. (To make the symmetry of the relation of Γ and its Gale transform better visible, we will no longer insist, as above, that the first part of the matrix is the identity.)

Definition 1.1 *Let k be a field, and let $r, s \geq 1$ be integers. Set $\gamma = r+s+2$, and let $\Gamma \subset \mathbf{P}^r$, $\Gamma' \subset \mathbf{P}^s$ be ordered nondegenerate sets of γ points represented by $\gamma \times (r+1)$ and $\gamma \times (s+1)$ matrices G and G', respectively. We say that Γ' is the Gale transform of Γ if there exists a nonsingular diagonal $\gamma \times \gamma$ matrix D such that $G^{\mathrm{T}} \cdot D \cdot G' = 0$*

Put more simply, the Gale transform of a set of points represented by a matrix of homogeneous coordinates G is the set of points represented by the kernel of G^{T} (the diagonal matrix above is necessary to avoid the dependence on the choices of homogeneous coordinates). Note that the Gale transform is really only defined up to automorphisms of the projective space. Since we are going to give a more general modern definition in the next section, we will not pause to analyze this version further.

This definition is related to the one given in the introduction by the identity

$$(A \quad | \quad I_{s+1}) \begin{pmatrix} -I_{r+1} & 0 \\ 0 & I_{s+1} \end{pmatrix} \left(\frac{I_{r+1}}{A} \right) = 0.$$

In a remarkable series of papers ([1915, 1916, 1917, 1922]) in the early part of this century Coble gave the definition above, gave applications to theta functions and Jacobians of curves, and described many amazing examples. Although Coble used the same term for the "associated sets" as Castelnuovo, he doesn't mention Castelnuovo or any of the other references given above, leaving us to wonder how exactly he came to the idea. (The related paper of Conner [1911], often quoted by Coble, doesn't mention Castelnuovo either.)

Castelnuovo's work on self-associated sets of points, on the other hand, was continued by Bath [1938], Ramamurti [1942] and Babbage [1948], but they seem ignorant of Coble; perhaps the old and new worlds were too far apart.

Similar ideas in the affine case were developed, apparently without any knowledge of this earlier work, by Whitney [1940] and Gale [1963], the latter in the study of polytopes. As a duality theory for polytopes, and in linear programming, it has had a multitude of applications. The names "Gale Transform" and "Gale diagram" are well-established in these fields, and in the absence of a more descriptive term than "associated points" in algebraic geometry we have adopted them.

Another important group of applications was initiated by Goppa [1970,1984]. In coding theory the Gale transform is the passage from a code to its dual, and Goppa proved that a code defined by a set of rational points on an algebraic curve was dual to another such code. See Corollary 3.2 for a generalization.

Dolgachev and Ortland [1988] give a modern exposition of the geometric theory of the Gale transform. These authors treat many topics covered by Coble. Their main new contribution is the use of geometric invariant theory to extend the definition of the Gale transform to a partial compactification of the "configuration space" of sets of general points. In a similar vein, Kapranov [1993] shows that the Gale transform extends from general sets of points to the Chow compactification. (A similar result can be deduced from our description of the Gale transform as an operation on linear series, since the Chow compactification is an image of the set of linear series on a reduced set of points containing a given generator of the line bundle.)

Although we will not pursue it here, there is a possible extension of the theory, suggested to us by Rahul Pandharipande, which deserves mention. One may easily extend Definition 1.1 to ordered collections of linear subspaces: the Gale transform of a collection of γ linear spaces of dimension d in $\mathbf{P}^r$ will be a collection of γ linear spaces of dimension also d in $\mathbf{P}^{\gamma(d+1)-r-2}$. (For each space, choose an independent set of points spanning it. Take the Gale transform of the union of these collections of points. The spaces spanned by the subsets corresponding to the original spaces make up the Gale transform.) Thus the Gale transform of a set of 4 lines in $\mathbf{P}^3$ would be again a set of 4 lines in $\mathbf{P}^3$. This definition can be shown to be independent of the choice of frames and generalized to finite Gorenstein subschemes of Grassmannians in the spirit of Definition 2.1 bellow. It might be interesting to understand its geometric significance, at least in such simple examples as that of the four lines above.

2 The Scheme-theoretic definition

Throughout this paper Γ will denote a Gorenstein scheme, finite over a field k (we will usually just say a *finite Gorenstein scheme*.) We recall that a subscheme of $\mathbf{P}^n$ is *nondegenerate* if it is not contained in any hyperplane.

The Gale transform is the involution on the set of linear series on Γ induced by Serre duality. In more detail:

Let Γ be a finite Gorenstein scheme. let L be a line bundle on Γ, and consider the canonical "trace" map $\tau:\ \mathrm{H}^0(K_\Gamma) \xrightarrow{\tau} k$ provided by Serre duality. The composition of τ with the multiplication map

$$\mathrm{H}^0(L) \otimes_k \mathrm{H}^0(K_\Gamma \otimes L^{-1}) \longrightarrow \mathrm{H}^0(K_\Gamma) \longrightarrow k,$$

is a perfect pairing between $\mathrm{H}^0(L)$ and $\mathrm{H}^0(K_\Gamma \otimes L^{-1})$. For any subspace $V \subset \mathrm{H}^0(L)$ we write $V^\perp \subset \mathrm{H}^0(K_\Gamma \otimes L^{-1})$ for the annihilator with respect to this pairing.

Definition 2.1 *Let Γ be a finite Gorenstein scheme, let L be a line bundle on Γ, and let $V \subset \mathrm{H}^0(L)$ be a subspace. The Gale Transform of the linear series (V, L) is the linear series $(V^\perp, K_\Gamma \otimes L^{-1})$.*

In Coble's work [1915, 1922] the following observation, always in the reduced case and with $\deg \Gamma_1 = r+1$, is used as the foundation of the theory:

Proposition 2.2 *Let Γ be a finite Gorenstein scheme of degree $r+s+2$. Let $\Gamma_1 \subset \Gamma$ be a subscheme, and let Γ_2 be the residual scheme to Γ_1. Let (V, L) be a linear series of (projective) dimension r on Γ, and suppose that $W \subset \mathrm{H}^0(K_\Gamma \otimes L^{-1})$ is its Gale transform. The failure of Γ_1 to span $\mathbf{P}(V)$ (that is, the codimension of the linear span of the image of Γ_1 in $\mathbf{P}(V)$) is equal to the failure of Γ_2 to impose independent conditions on W.*

Proof. Consider the diagram with short exact row and column

$$\begin{tikzcd}
 & \mathrm{H}^0(L_{|\Gamma_1}) & \\
V \arrow[r] \arrow[ur, "a"] & \mathrm{H}^0(L) \arrow[u] \arrow[r] & W^* \\
 & \mathrm{H}^0(\mathcal{I}_{\Gamma_1} L). \arrow[u] \arrow[ur, "b"'] &
\end{tikzcd}$$

A diagram chase shows that $\ker(a) \cong \ker(b)$. But the failure of Γ_1 to span $\mathbf{P}(V)$ is the dimension of the kernel of a, and since $\mathrm{H}^0(\mathcal{I}_{\Gamma_1} L)^* = \mathrm{H}^0(L\mid_{\Gamma_2})$, the failure of Γ_2 to impose independent conditions on W is the dimension of the cokernel of b^*. ∎

Problem 2.3 Coble often uses this result as follows: he gives some transformation taking a general set of $r+s+2$ labeled points in $\mathbf{P}^r$ to a general set of $r+s+2$ labeled points in $\mathbf{P}^s$, definable by rational functions. He then proves that it takes a set of points whose first $r+1$ elements are dependent to a set of points whose last $s+1$ elements are dependent. He then claims that in consequence this transformation must agree with the Gale transformation. Coble establishes many of the examples described below in this way. Can this argument be made rigorous?

As a corollary of Proposition 2.2 we deduce a characterization of base-point-free and very ample linear series in terms of their Gale transforms. It will usually be applied to the Gale transform of a known linear series V, so we formulate it for $W = V^\perp$.

Corollary 2.4 *Let Γ be a finite Gorenstein scheme over a field k with algebraic closure $\overline{k}$, let L be a line bundle on Γ, and let $V \subset \mathrm{H}^0(L)$ be a linear series. Let $W = V^\perp \subset \mathrm{H}^0(K_\Gamma \otimes L^{-1})$ be the Gale transform of (V, L).*

a) The series W is base point free iff no element of $\overline{k} \otimes V$ vanishes on a codegree 1 subscheme of $\overline{k} \otimes \Gamma$.

b) The series W is very ample iff no element of $\overline{k} \otimes V$ vanishes on a codegree 2 subscheme of $\overline{k} \otimes \Gamma$.

For example, suppose that Γ is reduced over $k = \overline{k}$ and Γ is embedded by V into $\mathbf{P}(V)$. The series W is base point free iff no hyperplane in $\mathbf{P}(V)$ contains all but one point of Γ; and W is very ample iff no hyperplane in $\mathbf{P}(V)$ contains all but two points of Γ.

In the case of basepoint freeness, a criterion can be given which does not invoke $\overline{k}$. The reader may check that condition a) is equivalent to the statement that no proper subscheme Γ' of degree γ' in Γ imposes only $\gamma' - \dim(W)$ conditions, the smallest possible number, on $W^\perp$.

Proof. Both statements reduce immediately to the case $k = \overline{k}$. Then W is basepoint free iff every degree 1 subscheme of Γ imposes one condition on W. By Proposition 2.2, this occurs iff every codegree 1 subscheme of Γ spans $\mathbf{P}(V)$, that is, iff no element of V vanishes on a codegree 1 subscheme. Similarly, W is very ample iff every degree 2 subscheme imposes 2 conditions on W, and again the result follows from Proposition 2.2. ∎

Proposition 2.2 may be seen as a generalization of the Cayley-Bacharach theorem from the case of arithmetically Gorenstein schemes to the case of (locally) Gorenstein schemes. (See Eisenbud-Green-Harris [1996] for historical remarks on this other forerunner of the Gorenstein notion.) To exhibit this aspect we first explain how to find the Gale transforms of series cut out by hypersurfaces of given degree.

Recall that if $\Gamma \subset \mathbf{P}(U)$ is any finite scheme, with homogeneous coordinate ring S_Γ, then the *canonical module* of Γ is the S_Γ-module

$$\omega_\Gamma = \operatorname{Ext}_S^{\dim(U)-1}(S_\Gamma, S(-\dim(U))),$$

where S denotes the symmetric algebra of U, the homogeneous coordinate ring of $\mathbf{P}(U)$. The sheaf on Γ associated to ω_Γ is the dualizing sheaf K_Γ.

Proposition 2.5 *Let $\Gamma \subset \mathbf{P}(U)$ be a finite Gorenstein scheme, and let U^d be the series cut out on Γ by the hypersurfaces of degree d in $\mathbf{P}(U)$. The Gale transform of $U^d \subset \mathrm{H}^0(\mathcal{O}_\Gamma(d))$ is the image of $(\omega_\Gamma)_{-d}$ in $\mathrm{H}^0(K_\Gamma(-d))$.*

Proof. From the exact sequence $0 \longrightarrow \mathcal{I}_\Gamma \longrightarrow \mathcal{O}_{\mathbf{P}(U)} \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0$ we get the sequence

$$\ldots \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbf{P}(U)}(d)) \xrightarrow{\alpha} \mathrm{H}^0(\mathcal{O}_\Gamma(d)) \longrightarrow \mathrm{H}^1(\mathcal{I}_\Gamma(d)) \longrightarrow 0$$

which identifies $(U^d)^\perp$ with $\mathrm{H}^1(\mathcal{I}_\Gamma(d))^*$. By duality, ω_Γ is the module

$$\bigoplus_{e=-\infty}^{\infty} \mathrm{H}^1(\mathcal{I}_\Gamma(-e))^*,$$

with degree $-d$ part equal to $\mathrm{H}^1(\mathcal{I}_\Gamma(d))^*$, so we are done. ∎

For any finite scheme $\Gamma \subset \mathbf{P}(V)$, let $a(\Gamma)$ be the largest integer a such that Γ fails to impose independent conditions on forms of degree a. For example, if Γ is a complete intersection of forms of degrees $d_1, \ldots, d_c$, then $a(\Gamma) = \sum(d_i - 1) - 1$. The scheme Γ is *arithmetically Gorenstein*—that is, the homogeneous coordinate ring S_Γ is Gorenstein—iff there is an isomorphism $S_\Gamma(a) \cong \omega_\Gamma$ for some a, and it is easy to see that then $a = a(\Gamma)$. See for example Bruns-Herzog [1993], Proposition 3.6.11 and Definition 3.6.13.

Corollary 2.6 *Suppose that the finite scheme $\Gamma \subset \mathbf{P}(U)$ is arithmetically Gorenstein. For any integer $d \leq a(\Gamma)$, the linear series of forms of degree d on Γ is the Gale transform of the linear series of forms of degree $a(\Gamma) - d$.*

Proof. Use Proposition 2.5 and the fact that $\omega_\Gamma = S_\Gamma(a)$. ∎

The Cayley-Bacharach theorem is now the special case of Proposition 2.2 in which Γ is arithmetically Gorenstein in some embedding in a $\mathbf{P}^n$, and the linear series involved is induced by forms of small degree on this projective space.

Corollary 2.7 (Cayley-Bacharach for arithmetically Gorenstein schemes) *Suppose that $\Gamma \subset \mathbf{P}(U)$ is a finite arithmetically Gorenstein scheme, and let Γ_1, Γ_2 be mutually residual subschemes of Γ. For any integer $d < a(\Gamma)$, the failure of Γ_1 to impose independent conditions on forms of degree d is equal to the number of forms of degree $a(\Gamma) - d$ vanishing on Γ_2.*

Proof. Apply Proposition 2.2 and Corollary 2.6. ∎

This includes the classic version:

Corollary 2.8 (**Chasles**) *If a set Γ_1 of 8 points in $\mathbf{P}^2$ lies in the complete intersection Γ of two cubics, then any cubic vanishing on Γ_1 vanishes on Γ.*

Proof. In this case $a(\Gamma) = 3$. Since the number of forms of degree 0 vanishing on the empty set (respectively any one-point set) is 1 (respectively 0), the nine points of Γ impose dependent conditions on cubics, while any 8-point subset imposes independent conditions on cubics. ∎

3 A Generalized Goppa Theorem

This section is devoted to an extension of Goppa's classical result on the duality of algebro-geometric codes.

Theorem 3.1 *Let Γ be a zero-dimensional Gorenstein scheme with a finite map to a locally Gorenstein base scheme B of dimension c, and let $\mathcal{O}_\Gamma(1)$ be a line bundle on Γ. Suppose that*

$$0 \longrightarrow \mathcal{E}_c \longrightarrow \mathcal{E}_{c-1} \longrightarrow \dots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{O}_\Gamma(1) \longrightarrow 0$$

is a resolution of $\mathcal{O}_\Gamma(1)$ by locally free sheaves on B, and hence that

$$K_\Gamma(-1) \;=\; \operatorname{coker}\bigl[\mathcal{H}om(\mathcal{E}_{c-1}, K_B) \longrightarrow \mathcal{H}om(\mathcal{E}_c, K_B)\bigr].$$

If

$$\mathrm{H}^{i+1}(\mathcal{E}_i) = \mathrm{H}^{i+1}(\mathcal{E}_{i+1}) = 0, \quad \textit{for all} \quad 0 \le i \le c-2,$$

then the induced sequence

$$\begin{aligned} \mathrm{H}^0(\mathcal{E}_1) \longrightarrow \mathrm{H}^0(\mathcal{E}_0) \longrightarrow \bigl[\mathrm{H}^0(\mathcal{O}_\Gamma(1)) = \mathrm{H}^0(K_\Gamma(-1))^*\bigr] \longrightarrow \\ \mathrm{H}^0(\mathcal{H}om(\mathcal{E}_c, K_B))^* \longrightarrow \mathrm{H}^0(\mathcal{H}om(\mathcal{E}_{c-1}, K_B))^* \end{aligned}$$

is exact. In particular, if Γ is embedded in $\mathbf{P}^r$ by the linear series which is the image of $\mathrm{H}^0(\mathcal{E}_0) \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(1))$, so that Γ lies on the image of $\mathbf{P}(\mathcal{E}_0)$ embedded by the complete linear series $|\mathcal{O}_{\mathbf{P}(\mathcal{E}_0)}(1)|$, then the Gale transform of Γ is defined by the image of $\mathrm{H}^0(\mathcal{E}_c^* \otimes K_B) \longrightarrow \mathrm{H}^0(K_\Gamma(-1))$, and lies on the image of $\mathbf{P}(\mathcal{E}_c^* \otimes K_B)$ mapped by the complete linear series $|\mathcal{O}_{\mathbf{P}(\mathcal{E}_c^* \otimes K_B)}(1)|$.

Proof. Break up the given resolution into short exact sequences

$$0 \longrightarrow \mathcal{K}_{i+1} \longrightarrow \mathcal{E}_i \longrightarrow \mathcal{K}_i \longrightarrow 0, \quad 0 \le i \le c-1,$$

and then use the vanishings in the hypothesis to obtain

$$\begin{aligned} \operatorname{coker}(\mathrm{H}^0(\mathcal{E}_0) \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(1))) &= \mathrm{H}^1(\mathcal{K}_1) = \dots \\ &= \mathrm{H}^{c-1}(\mathcal{K}_{c-1}) = \ker(\mathrm{H}^c(\mathcal{K}_c) \longrightarrow \mathrm{H}^c(\mathcal{E}_{c-1})). \end{aligned}$$

Identifying $\mathcal{K}_c$ with $\mathcal{E}_c$ and using Serre duality, we get the asserted result. ∎

In the special case when $\Gamma \subset B = \mathbf{P}^r$, Theorem 3.1 is the special case $d = 1$ of Proposition 2.5.

Corollary 3.2 *(Goppa Duality) Let B be a locally Gorenstein curve, embedded in $\mathbf{P}^r$ by the complete linear series associated to a line bundle $\mathcal{O}_B(H)$, and let $\Gamma \subset B \subset \mathbf{P}^r$ be a Cartier divisor on the curve B. The Gale transform of Γ lies on the image of B under the complete linear series associated to $\mathcal{O}_B(K_B - H + \Gamma)$.*

Proof. Apply Theorem 3.1 to the resolution

$$0 \longrightarrow \mathcal{O}_B(H - \Gamma) \longrightarrow \mathcal{O}_B(H) \longrightarrow \mathcal{O}_\Gamma(1) \longrightarrow 0.$$

∎

This result is essentially due to Goppa [1970], [1984], and expresses the duality among the algebro-geometric codes bearing his name (see e.g. van Lint-van der Geer [1988] for more details). From Corollary 3.2 we may immediately derive the following consequences:

- If a curvilinear finite scheme of degree $\gamma = r + s + 2$ lies on a rational normal curve in $\mathbf{P}^r$, then its Gale transform lies also on a rational normal curve in $\mathbf{P}^s$. The analogous statement also holds for finite subschemes of an elliptic normal curve. See Coble [1922] for the statement in the reduced case.
- A set of $\gamma = 2g-2$ points in $\mathbf{P}^{g-2}$ which is the hyperplane section of a canonical curve of genus g is its own Gale transform.
- Let Γ be a set of seven general points in $\mathbf{P}^2$, and let $E \subset \mathbf{P}^2$ be a smooth plane cubic curve passing through Γ. Write h for the hyperplane divisor of $E \subset \mathbf{P}^2$. By Corollary 3.2, the Gale transform Γ' of Γ is the image of Γ via the re-embedding of E as an elliptic normal quartic curve $E' \subset \mathbf{P}^3$ with hyperplane divisor $H = \Gamma - h$. Similarly, Γ is obtained from Γ' by the linear series $\Gamma' - H$. If we write $\Gamma' = 2H - p$, so that the 3 quadrics containing Γ' intersect in $\Gamma' + p$, then we see that $\Gamma' - H = H - p$, so that Γ is obtained from Γ' by projection from p.

Corollary 3.3 *Let B be a locally Gorenstein curve, and let $\mathcal{E}$ be a vector bundle over B. Let Γ be a zero dimensional Gorenstein subscheme of the ruled variety $X = \mathbf{P}(\mathcal{E})$, and assume that Γ is embedded in $\mathbf{P}^r$ by the restriction of the complete series $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$. Assume that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|_\Gamma$ is very ample. Then the Gale transform of Γ lies on the image of the ruled variety $X' = \mathbf{P}((\mathcal{E}')^* \otimes K_B)$, mapped by $|\mathcal{O}_{\mathbf{P}((\mathcal{E}')^* \otimes K_B)}(1)|$, where the vector bundle $\mathcal{E}'$ is defined as the kernel of the natural epimorphism*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_\Gamma(1) \longrightarrow 0.$$

If X is a ruled surface, that is, $\mathrm{rank}(\mathcal{E}) = 2$*, then $(\mathcal{E}')^* \otimes K_B \cong \mathcal{E}' \otimes \det(\mathcal{E}')^* \otimes K_B$, and hence X' is the elementary transform of X along the scheme Γ.*

Proof. Apply Theorem 3.1 to the resolution

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_\Gamma(1) \longrightarrow 0,$$

∎

Example 3.4 A smooth quadric surface Q in $\mathbf{P}^3$ can be regarded in two ways as a ruled surface over $\mathbf{P}^1$, hence we deduce that 9 general points in $\mathbf{P}^4$ lie in the intersection of two rational cubic scrolls in $\mathbf{P}^4$. The 9 points are actually the complete intersection of the two scrolls.

To see this, let $\Gamma \subset Q \subset \mathbf{P}^3$ be a set of 9 general points. The ideal I_Γ is 3-regular so by Bertini the general cubic through Γ cuts out on Q a general canonically embedded smooth curve of genus 4. Such a curve has exactly two g^1_3s, namely those cut out by the two rulings of the smooth quadric Q. By Corollary 3.2, the Gale transform of Γ is a hyperplane section of the re-embedding of C in $\mathbf{P}^5$ via the linear system $|\Gamma|$. In this embedding, each g^1_3 of C sweeps out a rational cubic threefold scroll X_i, $i = 1, 2$ (isomorphic to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^2$ into $\mathbf{P}^5$). Each X_i is determinantal, cut out be the 2×2 minors of a 2×3 matrix with linear entries. Since $C \subset \mathbf{P}^5$ is a general curve of degree 9 and genus 4 in $\mathbf{P}^5$, and since the complete intersection of two general cubic scrolls is of this type, $C = X_1 \cap X_2$, and Γ' is correspondingly the complete intersection of the two scrolls in $\mathbf{P}^4$.

If the points Γ lie on a complete intersection of a cubic with a singular quadric, then the two scrolls in $\mathbf{P}^4$ coincide, so Γ' is not a complete intersection as above. This leads us to the following formulation.

Problem 3.5 Suppose that $\Gamma \subset \mathbf{P}^3$ is a finite Gorenstein scheme of degree 9 that lies on a unique quadric surface, and suppose that this quadric is smooth. Is the Gale transform of Γ the intersection of the corresponding pair of rational cubic scrolls in $\mathbf{P}^4$?

More generally the Gale transform of $5 + r$ general points on a smooth quadric surface in $\mathbf{P}^3$ is contained in the intersection of two rational normal scrolls in $\mathbf{P}^r$.

We refer the reader to Section 4 for other applications of Corollary 3.2 and Corollary 3.3.

4 Castelnuovo's $r + 3$ Theorem

A scheme-theoretic version of Castelnuovo's Lemma for $r + 3$ points in linearly general position (over an algebraically closed field) was proved by Eisenbud and Harris [1992]:

A finite subscheme $\Gamma \subset \mathbf{P}^r_k$ of degree $r + 3$ in linearly general position over an algebraically closed field k lies on a unique rational normal curve.

As an application of the Gale transform we give here a simpler direct proof of this result.

It is well-known that any finite subscheme Γ of a rational normal curve over an arbitrary field k is in *linearly general position*, in the sense that any subscheme of Γ that lies on a d-dimensional linear subspace has degree $\leq d + 1$. It is also *curvilinear* (each local ring $\mathcal{O}_{\Gamma,p}$ is isomorphic to $F[x]/(x^n)$ for some n, and some field extension F of k) and *unramified* (if V is the vector space of linear forms on

$\mathbf{P}^r$, then the natural map $V \longrightarrow \mathcal{O}_{\Gamma,p}$ has image the complement of an ideal). Thus the following result characterizes subschemes of a rational normal curve:

Theorem 4.1 *Let Γ be a finite scheme geometrically in linearly general position in $\mathbf{P}^r_k$.*
a) If $\deg \Gamma = r+3$, then Γ lies on a rational normal curve iff Γ is Gorenstein.
b) If $\deg \Gamma \geq r+3$ and Γ is Gorenstein, then Γ is curvilinear and unramified.

Proof. *a)* If Γ lies on a rational normal curve, then Γ is curvilinear, thus Gorenstein. Conversely, suppose that Γ is Gorenstein and geometrically in linearly general position. By Corollary 2.4, the Gale transform of $\Gamma \subset \mathbf{P}^r_k$ is an embedding of Γ as a subscheme $\Gamma' \subset \mathbf{P}^1_k$. By Corollary 3.2, Γ the Gale transform of Γ', lies on a rational normal curve. (One could also use Theorem 7.2: since any subscheme of $\mathbf{P}^1_k$ is arithmetically Gorenstein, its Gale transform is equal to its Veronese transform.)

b) By part *a)*, any subscheme of degree $r+3$ of Γ is curvilinear and unramified. It follows that every component of Γ is too, and this implies the desired result. ∎

Remark 4.2 The condition of being geometrically in linearly general position cannot be replaced by the condition of being in linearly general position. Let F be a field of characteristic p, and let $k = F(s,t)$. Consider the Gorenstein scheme $\Gamma = \operatorname{Spec} k(s^{1/p}, t^{1/p})$. Set $r = 2p-3$, and let $V \subset \mathcal{O}_\Gamma = k(s^{1/p}, t^{1/p})$ be a k-subspace of dimension $r+1 = 2p-2$. If $p \geq 3$, then V has dimension more than half the dimension of $\mathcal{O}_\Gamma$, and thus V is very ample. As $\Gamma \subset \mathbf{P}^r_k = \mathbf{P}(V)$ has no proper subschemes at all, it is in linearly general position. But it does NOT lie on any rational normal curve, since after tensoring with $\overline{k}$ the local ring $\overline{k} \otimes \mathcal{O}_\Gamma \cong \overline{k}[x,y]/(x^p, y^p)$ is not generated by one element over $\overline{k}$.

To connect this result with the result for algebraically closed ground fields proved by Eisenbud and Harris we use:

Theorem 4.3 *Let Γ be a finite scheme in linearly general position in $\mathbf{P}^r_k$. If k is algebraically closed and $\deg \Gamma \geq r+2$, then Γ is Gorenstein.*

Example 4.4 The following shows that the condition of algebraic closure cannot be dropped in Theorem 4.3. Let Γ be the scheme in $\mathbf{P}^2$ defined by the 2×2 minors of the matrix

$$\begin{pmatrix} x & y & 0 \\ -y & x & x^2 \end{pmatrix}.$$

Γ is a finite scheme of degree 5, concentrated at the point p defined by $x = y = 0$. It is in linearly general position over $\mathbf{R}$ (but not after base change to $\mathbf{C}$.) It is not Gorenstein since the matrix above gives a minimal set of syzygies locally at p. In particular, it does not lie on a rational normal curve.

Proof of Theorem 4.3. The result amounts to a very special case of Theorem 1.2 of Eisenbud-Harris [1992]. Here is a greatly simplified version of the proof given there. See the original for further classification, examples, and remarks.

Let $V \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(1))$ be the map defining the embedding of $\mathcal{O}_\Gamma$ in $\mathbf{P}^r_k = \mathbf{P}(V)$. By choosing a generator of $\mathcal{O}_\Gamma(1)$ we may identify V with a subspace of $\mathcal{O}_\Gamma$. Let Γ' be a subscheme of Γ, and let $I \subset \mathcal{O}_\Gamma$ be its defining ideal. Since Γ is in linearly general position the composite map $V \longrightarrow \mathcal{O}_\Gamma \longrightarrow \mathcal{O}_\Gamma/I = \mathcal{O}_{\Gamma'}$ is either a monomorphism (Γ' is nondegenerate), or an epimorphism, as one sees directly by comparing the dimension of the linear space defined by the image of V with the degree of Γ'.

Now suppose that Γ_1 is a component of Γ of degree δ which is not Gorenstein, so that $\dim(\operatorname{socle} \mathcal{O}_{\Gamma_1}) > 1$. We will derive a contradiction. We divide the argument into cases according to the value of δ.

First suppose $\delta \leq r+1$ so that $V \longrightarrow \mathcal{O}_{\Gamma_1}$ is a surjection. It follows that the preimage of the socle in V contains a $\mathbf{P}^{r+2-\delta}$ of hyperplanes, each of which meets Γ_1 in a subscheme of degree at least $\delta - 1$. Since Γ must have also some other components, there is a hyperplane in the family meeting Γ in a scheme of degree $\geq \delta - 1 + (r+2-\delta) = r+1$, contradicting our assumption of linearly general position.

Suppose now $\delta = r+2$, so that $V \subset \mathcal{O}_{\Gamma_1}$. If $\dim(\operatorname{socle} \mathcal{O}_{\Gamma_1}) > 1$, then V meets the socle; thus there is a linear form x on $\mathbf{P}^r$ that meets Γ_1 in $r+1$ points, again contradicting our hypothesis.

If $\delta = r+3$, then again we have $V \subset \mathcal{O}_{\Gamma_1}$, and we again get a contradiction as above if V meets the socle. Thus we may suppose $\mathcal{O}_{\Gamma_1} = V \oplus \operatorname{socle} \mathcal{O}_{\Gamma_1}$, and that the dimension of the socle is 2.

Set $\mathfrak{m} = \mathfrak{m}_{\Gamma_1}$. We see that $V \cap \mathfrak{m}$ projects isomorphically onto $\mathfrak{m}/\mathfrak{m}^2$, while $\operatorname{socle} \mathcal{O}_{\Gamma_1} = \mathfrak{m}^2$. The multiplication on $\mathcal{O}_{\Gamma_1}$ induces a map

$$V \cap \mathfrak{m} \longrightarrow \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, \mathfrak{m}^2)$$

Counting dimensions, and using the algebraic closure of k, we see that for some $x \in V$ the transformation induced by x has rank at most 1. Thus $\mathcal{O}_{\Gamma_1} x$ has dimension at most 2, and the hyperplane defined by $x = 0$ meets Γ_1 in at least $r+1$ points, a contradiction.

Finally, suppose that $\delta > r+3$. By the previous case and Theorem 4.1, every subscheme of Γ of degree $< r+3$ is curvilinear; it follows that every component is curvilinear, and thus Gorenstein. ∎

We prove now a higher dimensional version of Theorem 4.1. (See also Cavaliere-Rossi-Valla [1995, Theorem 3.2] for another proof in a reduced case.)

Theorem 4.5 *Let $\Gamma \subset \mathbf{P}^r_k$, $r \geq 3$, be a finite Gorenstein scheme of degree γ which is in linearly general position, and let s be an integer with $1 \leq s \leq r-2$.*

a) *If $\gamma \leq r+s+2$, then $\Gamma \subset \mathbf{P}^r_k$ lies on an s-dimensional rational normal scroll (possibly singular).*

b) *If moreover $\gamma \leq r+s+2$ and $s \leq \frac{r+1}{2}$, then Γ lies on a smooth s-dimensional rational normal scroll in $\mathbf{P}^r_k$.*

Proof. *a)* The case $s = 1$ was proved in part $a)$ of Theorem 4.1 so we may assume in the sequel that $s \geq 2$. It is also enough to prove the assertion when $\gamma = r + s + 2$. Then the Gale transform of Γ is a finite scheme $\Gamma' \subset \mathbf{P}^s_k$ of degree γ which is also in linearly general position, by Proposition 2.2. Regarding now $\mathbf{P}^s_k$ as a "cone" over $\mathbf{P}^1_k$ with "vertex" $\Pi \cong \mathbf{P}^{s-2}_k$ we may resolve $\mathcal{O}_{\Gamma'}(1)$ as an $\mathcal{O}_{\mathbf{P}^1_k}$-module:

$$0 \longrightarrow \oplus_{i=1}^{s} \mathcal{O}_{\mathbf{P}^1}(-a_i) \longrightarrow \oplus_{i=1}^{s-1} \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1) \longrightarrow \mathcal{O}_{\Gamma'}(1) \longrightarrow 0,$$

where $\sum_{i=1}^{s} a_i = \gamma - 1 = r + s + 1$.

Let $\mathcal{E} = \oplus_{i=1}^{s-1}\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1)$, and let $\mathcal{F} = \oplus_{i=1}^{s}\mathcal{O}_{\mathbf{P}^1}(-a_i)$. Since Γ' is nondegenerate, there are no sections in $\mathrm{H}^0(\mathcal{E}) = \mathrm{H}^0(\mathcal{O}_{\mathbf{P}^s}(1))$ vanishing identically on Γ' and thus $a_i \geq 1$ for all i.

To prove the claim we need to check that $a_i \geq 2$ for all i, since then we may use Corollary 3.3 to deduce that Γ lies on the birational image of $\mathbf{P}(\mathcal{F}^* \otimes \mathcal{O}_{\mathbf{P}^1}(-2))$ in $\mathbf{P}^r$ (since $r + 1 = \sum_{i=1}^{s}(a_i - 1)$). Twisting by $\mathcal{O}_{\mathbf{P}^1}(1)$ and taking cohomology in the above short exact sequence we see that $a_i \geq 2$ for all i iff there are no sections in $\mathrm{H}^0(\mathcal{E}(1)) = \mathrm{H}^0(\oplus_{i=1}^{s-1}\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$ vanishing identically on Γ'. Such sections correspond to hyperquadrics in $\mathbf{P}^s$ containing Π, the "vertex of the cone", and since $\gamma > 2s + 1 = \mathrm{h}^0(\mathcal{E}(1))$, this means that, for a general choice of Π, $a_i \geq 2$ for all i iff there are no hyperquadrics containing both Π and Γ. The scheme Γ' is in linearly general position so we may conclude by the following lemma:

Lemma 4.6 *Let $\Gamma \subset \mathbf{P}^s_k$, $s \geq 2$, be a finite scheme of degree $\gamma \geq s + 3$ which is in linearly general position, and let $\Pi \subset \mathbf{P}^s_k$ be a general codimension 2 linear subspace. If $d \leq 3$, then Γ imposes $\min(\gamma, ds + 1)$ independent conditions on hypersurfaces of degree d vanishing to order $d - 1$ on Π.*

Conjecture 4.7 *Castelnuovo's classic result, as generalized to schemes by Eisenbud-Harris [1992], says that Γ imposes independent conditions on forms of degree d. The Lemma above represents a strengthening, in that the points impose independent conditions on a smaller subsystem. We conjecture that the Lemma remains true for every d.*

Proof of Lemma 4.6 (This is Conjecture 4.7 in the case $d \leq 3$.) Since any scheme of length $\geq s+3$ in linearly general position can be extended by the addition of general points (see Eisenbud-Harris [1992, Theorem 1.3]), we may assume that $\gamma = ds + 1$ and we must show that there are no d-ics F containing Γ and having multiplicity $d - 1$ along Π. We suppose we have such an F and argue by contradiction.

We first assume $d = 2$. Choose a hyperplane H containing a degree s subscheme Γ_1 of Γ, and let $\Gamma_2 = \Gamma \setminus \Gamma_1$ be the residual, a scheme of degree $s+1$. Specialize Π to be a general $(s-2)$-plane contained in H. If F contains H and Γ, then $F = H \cup H'$, where H' is another hyperplane, containing Γ_2. This contradicts the assumption that Γ is in linearly general position. Thus we may assume that F does not contain H, and so $F \cap H = \Pi \cup \Pi'$ for some $(s - 2)$-plane Π'. As Π contains no points of

Γ, Π' must contain all of $\Gamma \cap H$, again contradicting the linearly general position hypothesis.

Now suppose $d = 3$. We choose two hyperplanes H_1, H_2 as follows: Order the components of Γ by decreasing degree, and suppose the degrees are $\gamma_1 \geq \gamma_2 \geq \ldots$. If $\gamma_1 \geq 2s$, then we choose $H_1 = H_2$ to be the hyperplane meeting the first component of Γ in s points. Since Γ is in linearly general position and of degree $\geq s+3$ the components of Γ are curvilinear, so H_2 meets $\Gamma \setminus (H_1 \cap \Gamma)$ in exactly s points as well.

If on the contrary $\gamma_1 < 2s$, we can divide Γ into two disjoint subschemes each of degree at least s (Reason: Since no component has degree greater than $2s-1$, the smallest group of components with total degree $\geq s$ has total degree $\leq 2s-2$, and the remainder thus has degree $\geq s+3$). Choose H_1 containing a degree s subscheme of one of the two subschemes, and H_2 containing a degree s subscheme of the other.

In the first of these two cases, we may choose an $(s-2)$-plane Π in $H_1 \cap H_2$ that does not meet Γ; in the second case the intersection $\Pi = H_1 \cap H_2$ automatically misses Γ because Γ is in linearly general position. If F contains both H_1 and H_2 then after removing the two hyperplanes we get a hyperplane containing $s+1$ points of Γ, a contradiction as before. If on the other hand F fails to contain H_1, we restrict to H_1. As 2Π is contained in $F \cap H_1$, the scheme $H_1 \cap \Gamma$ is contained in the remaining $(s-2)$-plane in F (restricted to H_1). Once again, this is a contradiction. ∎

Proof of Theorem 4.5 continued: b) Again we may assume that $\gamma = r + s + 2$, and thus that the Gale transform Γ' is a finite scheme of degree γ in $\mathbf{P}^s$. In the notation of *a)*, we need to check that $a_i \geq 3$ for all i. Twisting the above short exact sequence by $\mathcal{O}_{\mathbf{P}^1}(2)$, and taking cohomology this amounts to the fact that no section in $\mathrm{H}^0(\mathcal{E}(2))$ vanishes identically on Γ'. The $3s+1$ sections in $\mathrm{H}^0(\mathcal{E}(2))$ correspond now to the cubics in $\mathbf{P}^s$ vanishing to second order on Π, so the claim follows again from Lemma 4.6. ∎

Remark 4.8 The result in Theorem 4.5 is not always sharp, see for instance Example 3.4. In case $r \leq 2s$, one may slightly improve the statement of *a)* by showing that the Gale transform lies on a rank 4 quadric scroll. ∎

5 The Gale Transform and Canonical Modules

In this section we consider a fundamental relation between the presentations of the canonical modules of a finite Gorenstein subscheme of projective space and its Gale transform.

As in Definition 2.1 we consider a Gorenstein scheme Γ finite of degree γ over k. Suppose that Γ is embedded in $\mathbf{P}^r_k = \mathbf{P}(V)$ by a linear series $(V, \mathcal{O}_\Gamma(1))$. We write $W = V^\perp \subset \mathrm{H}^0(K_\Gamma(-1))$ for the linear series corresponding to the Gale transform. Supposing that W is base-point-free, we write $\Gamma' \subset \mathbf{P}^s_k = \mathbf{P}(W)$ for the image of Γ under the corresponding map. Writing S for the homogeneous coordinate ring of

$\mathbf{P}^r$ and S_Γ for the homogeneous coordinate ring of Γ, we will study the canonical module $\omega_\Gamma = \mathrm{Ext}^r_S(S_\Gamma, S(-r-1))$.

We will also use the notion of adjoint of a matrix of linear forms. If V_1, V_2, and V_3 are vector spaces over k and $\phi \in V_1 \otimes V_2 \otimes V_3$ is a trilinear form, then ϕ can be regarded as a homomorphism of graded free modules over the polynomial ring $k[V_1]$:

$$\phi_{V_1} : V_2^* \otimes k[V_1] \longrightarrow V_3 \otimes k[V_1](1),$$

and in two other ways corresponding to the permutations of $\{1,2,3\}$. We call these three linear maps (which may be viewed as matrices of linear forms, once bases are chosen) *adjoints* of one another.

Proposition 5.1 *If $\Gamma' \subset \mathbf{P}(W)$ is the Gale transform of $\Gamma \subset \mathbf{P}(V)$, then the linear part of the presentation matrix of $(\omega_{\Gamma'})_{\geq -1}$, as a $k[W]$-module, is adjoint to the linear part of the presentation matrix of $(\omega_\Gamma)_{\geq -1}$ as a $k[V]$-module.*

Proof. Consider the multiplication map $V \otimes W \longrightarrow \mathrm{H}^0(K_\Gamma)$, and let N be its kernel. From Corollary 1.3 and Theorem 1.4 of Eisenbud-Popescu [1996] we see that N may be regarded as either the space of linear relations on the degree -1 elements of ω_Γ considered as a module over $k[V]$, *or* as the space of linear relations on the degree -1 elements of $\omega_{\Gamma'}$ regarded as a module over $k[W]$, which is exactly the meaning of adjointness. ∎

To exploit this result we need to know when the linear part of the presentation matrix of $(\omega_\Gamma)_{\geq -1}$ actually is the presentation matrix. This occurs when ω_Γ is generated in degree ≤ -1 and its relations are generated in degree ≤ 0. The first condition is easy to characterize completely:

Proposition 5.2 *Suppose the field k is algebraically closed, and let Γ be a finite (not necessarily Gorenstein) scheme in $\mathbf{P}^r = \mathbf{P}^r_k$, not contained in any hyperplane. Then ω_Γ is generated in degrees ≤ 0, and it fails to be generated in degrees ≤ -1 iff the homogeneous ideal I_Γ contains (after a possible change of variables) the ideal of 2×2 minors of a matrix of the form*

$$\begin{pmatrix} x_0 & \dots & x_t & x_{t+1} & \dots & x_r \\ 0 & \dots & 0 & l_{t+1} & \dots & l_r \end{pmatrix},$$

where $0 \leq t < r$ and the l_i are linearly independent linear forms.

Proof. Since I_Γ is generated in degree ≥ 2, its $(r-1)^{st}$ syzygies are generated in degree $\geq r$, which yields the first statement. A standard Koszul homology argument (see Green [1984] for the source, or for example Cavaliere-Rossi-Valla [1994], or Eisenbud-Popescu [1997, Proposition 4.2] for this particular result) shows that the $(r-1)^{st}$ syzygies are generated in degree $\geq r+1$ unless I_Γ contains the ideal of 2×2 minors of a matrix of the form

$$(*) \qquad \begin{pmatrix} x_0 & \dots & x_t & x_{t+1} & \dots & x_r \\ l_0 & \dots & l_t & l_{t+1} & \dots & l_r \end{pmatrix},$$

where the l_i are linear forms, and the row of l_i is not a scalar multiple of the first row. Because the number of variables is only $r+1$, this $2\times(r+1)$ matrix must have a "generalized zero" (Eisenbud [1988]) and thus may be transformed as in the claim of the proposition. ∎

Corollary 5.3 *If Γ is a finite Gorenstein scheme of degree $\geq r+2$ in linearly general position in $\mathbf{P}^r$, then ω_Γ is generated in degrees ≤ -1.*

Proof. Suppose not. By Proposition 5.2, I_Γ contains the ideal of minors of a matrix with linear entries as in the statement of Proposition 5.2. By our general position hypothesis, the degree of the subscheme Γ_1 of Γ contained in $V(x_0,\ldots,x_t)$ is at most $r-t$, while the degree of the subscheme Γ_2 of Γ contained in $V(l_{t+1},\ldots,l_r)$ is at most $t+1$. Since I_Γ contains the product of the ideals of these linear spaces, and Γ is Gorenstein, $\deg\Gamma_1+\deg\Gamma_2\geq\deg\Gamma$, a contradiction. ∎

In the reduced case, or more generally in the case when Γ_{red} is nondegenerate, we can give a geometric necessary and sufficient condition:

Definition 5.4 *A finite scheme $\Gamma\subset\mathbf{P}^r$ is decomposable if it can be written as the union of two subschemes contained in disjoint linear subspaces L_1, and L_2, in which case we say that $\Gamma\subset\mathbf{P}^r$ is the direct sum of its summands $\Gamma\cap L_1$ and $\Gamma\cap L_2$.*

Proposition 5.5 *Let $\Gamma\subset\mathbf{P}^r$ be a finite scheme such that Γ_{red} is nondegenerate. The module ω_Γ is generated in degrees ≤ -1 iff Γ is indecomposable.*

Proof. First suppose that ω_Γ is not generated in degree ≤ -1. By Proposition 5.2, I_Γ contains an ideal of the form $(x_0,\ldots,x_t)\cdot(l_{t+1},\ldots,l_r)$, where the l_i are independent linear forms. If the linear span of $x_0,\ldots,x_t$ in the space of linear forms meets that of $l_{t+1},\ldots,l_r$, then I_Γ would contain the square of a linear form, and thus Γ_{red} would be degenerate. It follows that

$$(x_0,\ldots,x_t,l_{t+1},\ldots,l_r)=(x_0,\ldots,x_r),$$

so

$$(x_0,\ldots,x_t)\cdot(l_{t+1},\ldots,l_r)=(x_0,\ldots,x_t)\cap(l_{t+1},\ldots,l_r)$$

is the ideal of the union of two disjoint linear spaces.

Conversely, if I_Γ contains the ideal of the disjoint union of two linear spaces, then (after a possible change of variables) it contains an ideal of the form

$$(x_0,\ldots,x_t)\cap(x_{t+1},\ldots,x_r)=(x_0,\ldots,x_t)\cdot(x_{t+1},\ldots,x_r)$$

which may be written as the ideal of minors of the matrix

$$\begin{pmatrix} x_0 & \ldots & x_t & x_{t+1} & \ldots & x_r \\ 0 & \ldots & 0 & x_{t+1} & \ldots & x_r \end{pmatrix},$$

as required. (The resolution of the product is also easy to compute directly.) ∎

The condition that the relations on ω_Γ are generated in degree 0 is deeper, and is expressed in the third part of the proposition below (we include the first two parts because of the nice pattern of results):

Proposition 5.6 *Suppose $\Lambda \subset \mathbf{P}^r_k = \mathbf{P}(V)$ is a finite Gorenstein subscheme over a field k with algebraic closure $\overline{k}$.*

a) If $\overline{k} \otimes \Lambda$ contains a subscheme of degree $r+1$ in linearly general position, then the $k[V]$-module ω_Λ is generated in degree ≤ 0 and its relations are generated in degree ≤ 1.

b) If $\overline{k} \otimes \Lambda$ contains a subscheme of degree $r+2$ in linearly general position, then ω_Λ is generated in degree ≤ -1.

c) If $\overline{k} \otimes \Lambda$ contains a subscheme of degree $r+3$ in linearly general position and $\overline{k} \otimes \Lambda$ does not lie on a rational normal curve, then the relations on ω_Λ are generated in degree ≤ 0. Thus the presentation matrix of $(\omega_\Lambda)_{\geq -1}$ is linear.

Proof. The conclusion of each part may be checked after tensoring with $\overline{k}$, so we may assume that $k = \overline{k}$ from the outset. The condition of part *a)* means that the homogeneous ideal I_Λ contains no linear form, and hence $\beta_{i,j}(I_\Lambda) = 0$ for $j < i+2$. As mentioned above, part *b)* follows from Proposition 5.2 applied to the subscheme of degree $r+2$, while *c)* is the "Strong Castelnuovo Lemma" of Green [1984] and Yanagawa [1994]; see Eisenbud-Popescu [1997] for a proof of *c)* involving syzygy ideals for the Eagon-Northcott complex. See also Ehbauer [1994] and Cavaliere-Rossi-Valla [1994] for related results. ∎

Returning to the case of the finite Gorenstein subscheme Γ we have:

Corollary 5.7 *Suppose that Γ is a Gorenstein scheme, finite over a field k with algebraic closure $\overline{k}$. Let (V, L) be a linear series that embeds Γ in $\mathbf{P}^r_k = \mathbf{P}(V)$, and set $(W, K_\Gamma \otimes L^{-1})$ be the Gale transform. Let also $s+1 = \dim_k(W)$.*

If $\overline{k} \otimes \Gamma$ contains a subscheme of degree $\geq r+3$ in linearly general position and Γ does not lie on a rational normal curve in $\mathbf{P}(V)$, then the vector space $K := \ker(V \otimes W \longrightarrow \mathrm{H}^0(K_\Gamma))$ has dimension rs, and the corresponding matrix with linear entries $K \otimes k[V] \longrightarrow W \otimes k[V](1)$ is a presentation matrix for the $k[V]$-module $(\omega_\Gamma)_{\geq -1}$.

Corollary 5.8 *Suppose that $\Gamma \subset \mathbf{P}^r_k$ is a finite nondegenerate Gorenstein subscheme of degree $\gamma = r+s+2$, with $r, s \geq 1$, and let $\Gamma' \subset \mathbf{P}^s_k$ be its Gale transform. The following conditions are equivalent, and are all satisfied if $\overline{k} \otimes \Gamma$ contains a subscheme of degree $r+2$ in linearly general position:*

a) ω_Γ is generated in degree ≤ -1 as a $k[V]$-module.

b) $\omega_{\Gamma'}$ is generated in degree ≤ -1 as a $k[W]$-module.

c) The multiplication map $V \otimes W \longrightarrow \ker(\tau) \subset \mathrm{H}^0(K_\Gamma)$ is surjective.

When these conditions are satisfied, both $(\omega_\Gamma)_{\geq -1}$ and $(\omega_{\Gamma'})_{\geq -1}$ have precisely rs linearly independent linear relations.

Proof. $a) \Rightarrow c)$: The part of $(\omega_\Gamma)_0 = \ker(\tau)$ generated by $(\omega_\Gamma)_{-1}$ is $V \cdot W$.
$c) \Rightarrow a)$: Because Γ is nondegenerate, $\overline{k} \otimes \Gamma$ contains a subscheme of length $r+1$ in linearly general position. By Proposition 5.6, part *a)*, ω_Γ is generated in degree ≤ 0, so it suffices to show that $(\omega_\Gamma)_0$ is the image of $V \otimes (\omega_\Gamma)_{-1} = V \otimes W$. As $(\omega_\Gamma)_0 = \ker(\tau)$ we are done.

The symmetry of *c)* completes the proof of the equivalences. By Proposition 5.6, the condition of *a)* follows if $\overline{k} \otimes \Gamma$ contains a subscheme of degree at least $r+2$ in linearly general position. If the conditions in *a)*, *b)*, *c)* hold, then we can compute the number of linear relations in the last statement as $\dim(V) \cdot \dim(W) - \dim(\ker(\tau)) = (r+1)(s+1) - (r+s+2-1) = rs$. ∎

In the simplest case we can say something about ω_Γ itself:

Corollary 5.9 *Suppose that $\Gamma \subset \mathbf{P}^r_k$ contains a subscheme of degree $r+3$ in linearly general position over $\overline{k}$, Γ imposes independent conditions on quadrics (this occurs for example when $\gamma \leq 2r+1$ and Γ is in linearly general position) and that Γ does not lie on a rational normal curve in $\mathbf{P}^r_k$. Then the module ω_Γ has a free presentation by the $(s+1) \times rs$ matrix of linear forms given in Corollary 5.7.*

Proof. Since Γ imposes independent conditions on quadrics it is 3-regular, whence $(\omega_\Gamma)_{-2} = 0$. ∎

In the situation of Corollary 5.9, it is useful to ask about the adjoint to the presentation matrix of ω_Γ, which is the linear part of the presentation matrix of $(\omega_{\Gamma'})_{\geq -1}$. In general we have:

Proposition 5.10 *Let $\varphi : N \longrightarrow V \otimes W$ be a map of k-vector spaces, with k algebraically closed, and let $\varphi_V : N \otimes k[V](-1) \longrightarrow W \otimes k[V]$ and $\varphi_W : N \otimes k[W](-1) \longrightarrow V \otimes k[W]$ be the corresponding adjoint matrices of linear forms.*

If the sheafification L of $\operatorname{coker}(\varphi_V)$ *is a line bundle on its support $X \subset \mathbf{P}(V)$, then the maximal minors of φ_W generate, up to radical, the ideal of the image of X under the map defined by the linear series $W \subset \mathrm{H}^0(L)$.*

Proof. Let $\widetilde{X}$ be the subscheme of $\mathbf{P}(V) \times \mathbf{P}(W)$ defined by the $(1,1)$-forms in the image of φ, and let p be the projection of $\mathbf{P}(V) \times \mathbf{P}(W)$ on the first factor. On $\mathbf{P}(V) \times \mathbf{P}(W)$ the map φ corresponds to a morphism $N \otimes \mathcal{O}_{\mathbf{P}(V)\times\mathbf{P}(W)}(-1,-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbf{P}(V)\times\mathbf{P}(W)}$ and $\varphi_V = p_*(\varphi(0,1))$. Since $R^1p_*(\mathcal{O}_{\mathbf{P}(V)\times\mathbf{P}(W)}(-1,0)) = 0$, it follows that $p_*(\mathcal{O}_{\widetilde{X}}(0,1)) = \operatorname{coker}(\varphi_V) = L$, so that the push-forward of a line bundle by p is a line bundle, which implies that $p_{|\widetilde{X}}$ is an isomorphism from $\widetilde{X}$ onto X.

An element of $\mathbf{P}(W)$ is a functional $x : W \longrightarrow k$. It is in the support of $\operatorname{coker}(\varphi_W)$ iff φ_W drops rank when its entries (elements of W) are replaced by their images under x; that is, if there is a functional $y : V \longrightarrow k$ such that $y \otimes x : V \otimes W \longrightarrow k$ annihilates the image of N via φ. The projection $p_{|\widetilde{X}}$ is an isomorphism from $\widetilde{X}$ onto X, thus the projection of $\widetilde{X}$ to $\mathbf{P}(W)$ is the image of the map defined by the sections W in L, and the maximal minors of φ_W generate, up to radical, the ideal of this image. ∎

In an important special case we can do better:

Corollary 5.11 *If Γ is a finite Gorenstein scheme and both $\Gamma \subset \mathbf{P}^r$ and its Gale transform $\Gamma' \subset \mathbf{P}^s$ satisfy the hypotheses of Corollary 5.9, then Γ' has its homogeneous ideal generated, by the $(r+1)\times(r+1)$ minors of the adjoint matrix of the presentation matrix of ω_Γ.*

Proof. Set as above $N = \ker(V \otimes W \longrightarrow \mathrm{H}^0(K_\Gamma))$ and let φ denote the inclusion of N in $V \otimes W$; the matrices φ_V and φ_W defined in Proposition 5.10 are, by virtue of Corollary 5.9, presentations of modules whose sheafifications are the canonical line bundles on Γ and Γ', respectively. Thus their minors generate the homogeneous ideals of Γ and Γ', respectively. ∎

Example 5.12 (**The Clebsch transform**) Let $\Gamma \subset \mathbf{P}^2$ be a set of six sufficiently general points. A familiar transformation in the plane, called the *Clebsch transform*, can be constructed from these points: Blow up the six points, and then blow down the six (-1)-curves in the blowup which are the proper transforms of the 6 conics through five of the six original points. The images of the six conics are 6 new points, each associated to one of the original points (the one through which the corresponding conic did not pass.) The new set of six points is the Clebsch transform of the original set.

Coble [1922] showed that the Clebsch transform of the six points is the same as the Gale transform. This follows from Corollary 5.11. The set Γ is cut out by the maximal minors of a 3×4 matrix M whose cokernel is the canonical module. By Corollary 5.11, the maximal minors of the 3×4 matrix M' which is adjoint to M (with respect to the rows) define the Gale transform Γ' of Γ. This example also illustrates the case $r = s = 2$ of Theorem 6.1.

The "third' adjoint matrix gives the connection of these ideas to the cubic surface. The determinant of the 3×3 matrix M'' in four variables which is adjoint to both M and M' (with respect to their columns) is the equation of the cubic surface in $\mathbf{P}^3$, image of $\mathbf{P}^2$ via the linear system of cubics through Γ (see also Gimigliano [1989]). The two linear series on the cubic surface blowing down the two systems of 6 lines described above correspond to the line bundles on the surface obtained from the cokernel of the 3×3 matrix (restricted to the surface, where it has constant rank 2) and from the cokernel of the transpose matrix. ∎

6 The Gale transform of Determinantal Schemes

Let $\Gamma \subset \mathbf{P}^r$ be a set of points defined by the maximal minors of a matrix with linear entries vanishing in the generic codimension; that is, suppose that there is an $(s+1)\times(r+s)$ matrix M such that the ideal $I_{s+1}(M)$ generated by the $(s+1)\times(s+1)$

minors of M defines a set of points ($r = (r+s) - (s+1) + 1$). It follows that the degree of Γ is $\gamma = \binom{r+s}{s}$.

In this section we will see that (in a sufficiently general case) the $(s-1)^{\text{st}}$ Veronese embedding of this set of points is the Gale transform of the $(r-1)^{\text{st}}$ Veronese embedding of a set of points defined by the adjoint matrix to M. We denote the d^{th} Veronese map by $\nu_d : \mathbf{P}^r \longrightarrow \mathbf{P}^N$ (where $N = \binom{r+d}{r} - 1$.)

Theorem 6.1 *Let V and W be k-vector spaces of dimension $r+1$ and $s+1$ respectively. Let $\phi : F \longrightarrow V \otimes W$ be a map of vector spaces with $\dim_k F = r+s$, and let $\phi_V : F \otimes k[V] \longrightarrow W \otimes k[V](1)$ be the corresponding map of free modules over the polynomial ring $k[V]$. Let ϕ_W be the analogous map over $k[W]$, and let $\Gamma_V \subset \mathbf{P}^r$ and $\Gamma_W \subset \mathbf{P}^s$ be the schemes defined by the ideals of minors $I_{s+1}(\phi_V) \subset k[V]$ and $I_{r+1}(\phi_W) \subset k[W]$, respectively.*

If Γ_V and Γ_W are both zero-dimensional then they are both Gorenstein, there is a natural isomorphism between them, and

$$\nu_{s-1}(\Gamma_V) \quad \textit{is the Gale transform of} \quad \nu_{r-1}(\Gamma_W).$$

The proof will have several steps. We first take care of the Gorenstein condition:

Proposition 6.2 *With notation as in the theorem, the following are equivalent:*
a) *Both Γ_V and Γ_W are zero-dimensional schemes.*
b) $\operatorname{codim} I_{s+1}(\phi_V) = r$ *and* $\operatorname{codim} I_s(\phi_V) = r+1$.
c) *Γ_V is zero-dimensional and Gorenstein.*

The proof can be analyzed to show that when these conditions are satisfied, Γ_V and Γ_W are in fact local complete intersections.

Proof. By definition, Γ_V is zero-dimensional iff $\operatorname{codim} I_{s+1}(\phi_V) = r$. Thus to prove the equivalence of *a)* and *b)* we suppose that Γ_V is zero-dimensional and we must show that Γ_W is zero-dimensional iff ϕ_V never drops rank by more than 1. Let $\phi_F : V^* \otimes k[F^*] \longrightarrow W \otimes k[F^*](1)$ be the third map induced by ϕ. The generalized zeros (in the sense of Eisenbud [1988]) of ϕ_F in a generalized row indexed by an element $\alpha \in W^*$ correspond to elements of the kernel of $\phi_{W|\alpha} : F \longrightarrow V$. Thus to say that Γ_W is zero-dimensional is equivalent to saying that only finitely many generalized rows of ϕ_F have generalized zeros. On the other hand the assumption that Γ_V is finite means that only finitely many generalized columns of ϕ_F have generalized zeros, so that the finiteness of Γ_W amounts to saying that no generalized column can have 2, and thus infinitely many, generalized zeros. That is, Γ_W is finite iff ϕ_V never drops rank by more than 1.

To prove the equivalence of *b)* and *c)* we may again assume that Γ_V is zero-dimensional. Since the $(s+1) \times (s+1)$ minors of ϕ_V have generic codimension, we may use the Eagon-Northcott complex to compute

$$\omega_{\Gamma_V} = (\operatorname{Sym}_{r-1}(\operatorname{coker} \phi_V))(s-1).$$

Now the scheme Γ_V is Gorenstein iff ω_{Γ_V} is locally principal. On the other hand, the ideal $I_s(\phi_V)$ has codimension $r+1$ iff it defines the empty set iff $\operatorname{coker}\phi_V$ is locally principal on Γ_V iff $(\mathrm{Sym}_{r-1}(\operatorname{coker}\phi_V))(s-1)$ is locally principal on Γ_V, as required. ∎

Proof of Theorem 6.1. We begin with the identification of Γ_V and Γ_W. Working on $\mathbf{P} := \mathbf{P}(V) \times \mathbf{P}(W)$ the map ϕ corresponds to a map of sheaves $\phi_{VW} : \mathcal{O}_{\mathbf{P}}(-1,-1)^{r+s} \longrightarrow \mathcal{O}_{\mathbf{P}}$. We define a subscheme $\Gamma \subset \mathbf{P}$ by setting $\mathcal{O}_\Gamma := \operatorname{coker}\phi_{VW}$. Let p be the projection on the first factor $p : \mathbf{P} \longrightarrow \mathbf{P}(V)$. We claim that p induces an isomorphism from Γ to Γ_V. Since the construction is symmetric in V and W, it will follow that Γ is naturally isomorphic to Γ_W too, as required.

We have $\phi_V = p_*\phi_{VW}(0,1)$. Thus $p_*\mathcal{O}_\Gamma(0,1) = \operatorname{coker}\phi_V$. By hypothesis $I_s(\phi_V)$ defines the empty set, so $\operatorname{coker}\phi_V$ is a line bundle on Γ_V. In particular, $p(\Gamma) = \Gamma_V$. By symmetry, the projection of Γ onto the other factor $\mathbf{P}(W)$ is Γ_W. Since the fibers of p project isomorphically to $\mathbf{P}(W)$, and Γ_W is zero-dimensional, this shows in particular that the map $p_{|\Gamma} : \Gamma \longrightarrow \Gamma_V$ is a finite map. The fact that the push-forward by p of a line bundle is a line bundle now implies that $p_{|\Gamma}$ is an isomorphism.

As noted in the proof of Proposition 6.2 the presentation over $k[V]$ of ω_{Γ_V} is given by the Eagon-Northcott complex. It has the form

$$F \otimes \mathrm{Sym}_{r-2} W \otimes k[V](s-2) \longrightarrow \mathrm{Sym}_{r-1} W \otimes k[V](s-1) \longrightarrow \omega_{\Gamma_V} \longrightarrow 0,$$

where the twists by $(s-2)$ and $(s-1)$ indicate as usual that we regard the first two terms of this sequence as free modules over $k[V]$ generated in degrees $(-s+2)$ and $(-s+1)$, respectively. Taking the $(s-1)^{\mathrm{st}}$ Veronese embedding, we see that $\omega_{\nu_{s-1}(\Gamma_V)}$ is generated in degree -1 with relations generated in degree 0, corresponding to the following right-exact sequence:

$$F \otimes \mathrm{Sym}_{r-2} W \otimes \mathrm{Sym}_{s-2} V \xrightarrow{\psi} \mathrm{Sym}_{r-1} W \otimes \mathrm{Sym}_{s-1} V \longrightarrow (\omega_{\Gamma_V})_0 \longrightarrow 0.$$

On the other hand, we would obtain the same map ψ if we had started instead from the scheme $\Gamma_W \subset \mathbf{P}(W)$. In other words, the presentation of $\omega_{\nu_{r-1}(\Gamma_W)}$ is adjoint to the presentation of $\omega_{\nu_{s-1}(\Gamma_V)}$, and by Proposition 5.1 and Corollary 5.11 this means that the two finite schemes are related by the Gale transform. ∎

Example 6.3 Let $\Gamma'' \subset \mathbf{P}^2$ be a locally Gorenstein scheme of degree ten, not contained in any plane cubic curve. It follows from the Hilbert-Burch theorem (see for example Eisenbud [1995]) that the ideal of Γ'' is generated by the maximal minors of a 4×5 matrix M'' with linear entries, and by Proposition 6.2 its 2×2 minors generate an irrelevant ideal. As above, by Theorem 6.1, the maximal minors of the 3×5-matrix M which is adjoint to M'' (with respect to the rows) define a set $\Gamma \subset \mathbf{P}^3$ of ten points, whose Gale transform $\Gamma' \subset \mathbf{P}^5$ coincides with the second Veronese embedding of Γ''. In this case the maximal minors of the “third” adjoint

3×4-matrix M' in five variables (which is adjoint to both M and M'' with respect to their columns) define a Bordiga surface in $\mathbf{P}^4$, image of $\mathbf{P}^2$ via the linear system of quartics through the set of ten points Γ'' — see Gimigliano [1989] and Room [1938].

A special case was described by Coble [1922], who refers to Conner [1915] for connections with the geometry of the "Cayley symmetroid": Let $C \subset \mathbf{P}^6$ be a rational normal sextic curve, and let $X = \mathrm{Sec}(C) \subset \mathbf{P}^6$ be the secant variety to the curve C. X has degree 10, since this is the number of nodes of a general projection of C to a plane. The homogeneous ideal of C is generated by the 2×2-minors of either a 3×5 or a 4×4 catalecticant matrix with linear entries, induced by splittings of $\mathcal{O}_{\mathbf{P}^1}(6)$ as a tensor product of two line bundles of strictly positive degree. Furthermore, the homogeneous ideal of $X = \mathrm{Sec}(C)$ is generated by the 3×3 minors of either of the above two catalecticant matrices (this is a classical result, see for example Gruson-Peskine [1982], or Eisenbud-Koh-Stillman [1988] for a modern reference). Let now $\Pi = \mathbf{P}^3 \subset \mathbf{P}^6$ be a general 3-dimensional linear subspace, and let $\Gamma \subset \mathbf{P}^3$ be a set of ten points defined by $\Gamma := \mathrm{Sec}(C) \cap \Pi$. Then the 2×2 minors of the restriction of the 3×5 catalecticant matrix generate an irrelevant ideal. So as above, by Theorem 6.1, Γ' the Gale transform of Γ lies on a quadratic Veronese surface in $\mathbf{P}^5$.

In this case, the maximal minors of the "third" adjoint matrix cut out a special Bordiga surface in $\mathbf{P}^4$, image of $\mathbf{P}^2$ via the linear system of quartics through the ten nodes of the rational plane sextic curve obtained by projecting the rational normal curve above from the $\mathbf{P}^3$. (See also Room [1938], 14.21, p. 391 and ff, Hulek-Okonek-van de Ven [1985], and Rathmann [1989].)

Problem 6.4 Can the previous method be used to tell exactly when a set of 10 points in $\mathbf{P}^3$ is determinantal?

For geometry related to the Gale transform of $\gamma \geq 11$ points on the Veronese surface in $\mathbf{P}^5$ see also Davide [1997].

7 Gorenstein and Self-associated Schemes

Castelnuovo's and Coble's original interest in associated sets of points centered on those sets that are "self-associated", that is equal to their own Gale transform up to projective equivalence.

Dolgachev and Ortland [1988] posed the problem of giving a "clear-cut geometrical statement" equivalent to self-association (Remark 3, p. 47). The following is our solution to this problem:

Theorem 7.1 *Let $\Gamma \subset \mathbf{P}^r = \mathbf{P}(V)$ be a finite Gorenstein scheme of degree $2r+2$ over an algebraically closed field k. The following are equivalent:*

a) Γ is self-associated.

b) Each of the (finitely many) subschemes of degree $2r+1$ of Γ imposes the same number of conditions on quadrics as Γ does.

c) If we choose a generator of $\mathcal{O}_\Gamma(1)$, and thus identify V with a subspace of $\mathcal{O}_\Gamma$, there is a linear form $\phi : \mathcal{O}_\Gamma \longrightarrow k$ which vanishes on V^2 and which generates $\mathrm{Hom}_k(\mathcal{O}_\Gamma, k)$ as an $\mathcal{O}_\Gamma$-module.

We include c) because it represents the most efficient way we know to check the property of self-association computationally. Namely, representing the multiplication table of the ring $\mathcal{O}_\Gamma$ as a matrix with linear entries over $\mathrm{Sym}(\mathcal{O}_\Gamma)$, we may identify V^2 with a vector space of linear forms in $\mathrm{Sym}(\mathcal{O}_\Gamma)$. Then part c) in Theorem 7.1 can be reformulated as: Γ is self-associated iff the matrix of the multiplication table reduced modulo the linear forms in V^2 has maximal rank. This test can be implemented in Macaulay/Macaulay2.

Proof. The subscheme Γ is self-associated iff there is an isomorphism of $\mathcal{O}_\Gamma$-modules $\varphi : \mathcal{O}_\Gamma(1) \longrightarrow \mathcal{O}_\Gamma(1)^*$ such that the composite of natural maps

$$V \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(1)) \xrightarrow{\varphi} \mathrm{H}^0(\mathcal{O}_\Gamma(1)^*) \longrightarrow V^*$$

is zero.

Giving a morphism of $\mathcal{O}_\Gamma$-modules $\varphi : \mathcal{O}_\Gamma(1) \longrightarrow \mathcal{O}_\Gamma(1)^*$ is the same as giving a map of vector spaces $\overline{\varphi} : \mathcal{O}_\Gamma(2) = \mathcal{O}_\Gamma(1) \otimes \mathcal{O}_\Gamma(1) \longrightarrow k$. Since $\mathcal{O}_\Gamma$ is Gorenstein, the modules $\mathcal{O}_\Gamma(1)$ and $\mathcal{O}_\Gamma(1)^*$ are isomorphic, and φ is an isomorphism iff $\overline{\varphi}$ generates $\mathrm{Hom}_k(\mathcal{O}_\Gamma(2), k)$ as an $\mathcal{O}_\Gamma$-module. We may write $\mathcal{O}_\Gamma = \prod \mathcal{O}_{\Gamma_i}$, where the Γ_i are the connected components of Γ, and the ideals $J_i = \mathrm{socle}(\mathcal{O}_{\Gamma_i})$ are all 1-dimensional. With this notation, φ is an isomorphism iff $\overline{\varphi}$ does not annihilate any of the one-dimensional submodules $J_i\mathcal{O}_\Gamma(2)$.

On the other hand, φ makes the composite map displayed above zero iff $\overline{\varphi}$ annihilates V^2, the image of $V \otimes V$ in $\mathcal{O}_\Gamma(2)$. Thus Γ is self-associated iff there is a map $\overline{\varphi}$ that annihilates V^2 but not any of the $J_i\mathcal{O}_\Gamma(2)$ iff $V^2 \cap J_i\mathcal{O}_\Gamma(2) = 0$ iff each codegree 1 subscheme of Γ imposes the same number of conditions on quadrics as Γ, thus proving that a) and b) are equivalent.

Once we choose an identification of $\mathcal{O}_\Gamma$ and $\mathcal{O}_\Gamma(1)$, part c) is a reformulation of this condition. ∎

A classical theorem of Pascal says that given a conic in the plane and two triangles circumscribing it, (algebraically this means that the vertices of each triangle are apolar to the conic), then the six vertices of the two triangles all lie on another conic. In other words the six points form a set of self-associated points in the plane, as one sees from Theorem 7.1.

Coble [1929] generalized this statement to say that for sufficiently general sets of $2r+2$ points in $\mathbf{P}^r$, self-association is the same as failing to impose independent conditions on quadrics. We next characterize arithmetically Gorenstein schemes in terms of the Gale transform, and we will see in a somewhat more precise way that a self-associated scheme in $\mathbf{P}^r$ is the same as an arithmetically Gorenstein scheme of degree $2r+2$ except in degenerate circumstances:

Theorem 7.2 *If $\Gamma \subset \mathbf{P}^r$ is a finite nondegenerate Gorenstein scheme, then Γ is arithmetically Gorenstein iff ω_Γ is generated in degrees ≤ -1 and the Gale transform of Γ is the d^{th} Veronese embedding of Γ for some $d \geq 0$. In particular if* $\deg(\Gamma) = 2r+2$*, then Γ is arithmetically Gorenstein iff ω_Γ is generated in degrees ≤ -1 and Γ is self-associated.*

The case $d = 0$ occurs only for $r+2$ points in $\mathbf{P}^r$; such a scheme is arithmetically Gorenstein iff it is in linearly general position.

Proof. If Γ is nondegenerate and arithmetically Gorenstein, then the symmetry of the free resolution of the homogeneous coordinate ring S_Γ shows that $\omega_\Gamma = S_\Gamma(d+1)$ for some $d \geq 0$, and ω_Γ is generated in degree $-d-1 \leq -1$. By Proposition 2.5 the Gale transform is given by the image of $(\omega_\Gamma)_{-1} = (S_\Gamma)_d$ in $\mathrm{H}^0(K_\Gamma(-1))$, so the Gale transform is the d^{th} Veronese embedding.

Conversely, suppose ω_Γ is generated in degrees ≤ -1 and the Gale transform coincides with the d^{th} Veronese embedding for some $d \geq 0$. Since $S_\Gamma(d+1)$ is also generated in degrees ≤ -1, and both ω_Γ and $S_\Gamma(d+1)$ are Cohen-Macaulay modules, they are isomorphic iff $(\omega_\Gamma)_{\geq -1} \cong (S_\Gamma(d+1))_{\geq -1}$, and this occurs precisely when there is an isomorphism of sheaves of $\mathcal{O}_\Gamma$-modules $K_\Gamma(-1) \cong \mathcal{O}_\Gamma(d)$, which maps $V^\perp$ to $(S_\Gamma)_d$. This last is the condition that the Gale transform of Γ is the d^{th} Veronese embedding of Γ. ∎

Perhaps the best characterization of this kind is

Theorem 7.3 *If $\Gamma \subset \mathbf{P}^r_k$ is a nondegenerate finite scheme of degree $2r+2$ over an algebraically closed field k, then S_Γ is Gorenstein if and only if Γ is self-associated and fails by 1 to impose independent conditions on quadrics.*

By Theorem 7.1, we could restate the condition of the Theorem by saying that Γ fails to impose independent conditions on quadrics but that every maximal proper subscheme of Γ (equivalently every proper subscheme of Γ) imposes independent conditions on quadrics. This is also a consequence of Kreuzer [1992], Theorem 1.1, which generalizes the main result of Davis-Geramita-Orecchia [1985] to the non-reduced case.

A result of Dolgachev-Ortland [1988] (Lemma 3, p. 45) and Shokurov [1971] shows that every proper subscheme does impose independent conditions if Γ is reduced and, for every $s < r$, no subset of $2s+2$ points of Γ is contained in a $\mathbf{P}^s$, which is the same as saying that Γ is *stable* in this case. (See Proposition 8.10 for the general stability test.) Dolgachev and Ortland [1988] use this to prove that a reduced set of stable points is self-associated if and only if it fails to impose independent conditions on quadrics, generalizing a result of Coble [1929].

Proof of Theorem 7.3. If S_Γ is Gorenstein then Γ is self-associated by Theorem 7.2, and fails by just 1 to impose independent conditions on quadrics since $(S_\Gamma)_0 = (\omega_\Gamma)_{-2} = ((S_\Gamma)_2)^\perp$ is 1-dimensional.

Conversely, suppose that Γ is self-associated. By definition there is an isomorphism of $\mathcal{O}_\Gamma$-modules $\mathcal{O}_\Gamma(1) \longrightarrow K_\Gamma(-1)$ carrying $V = (S_\Gamma)_1$ to $(\omega_\Gamma)_{-1}$. This defines a map of modules $(S_\Gamma(2))_{\geq 1} \longrightarrow \omega_\Gamma$. Since $\mathrm{Ext}^1_{S_\Gamma}(k, \omega_\Gamma) = k$, concentrated in degree 0, this map lifts to a map $\alpha : (S_\Gamma(2)) \longrightarrow \omega_\Gamma$, necessarily a monomorphism. As Γ is nondegenerate, ω_Γ is generated in degree ≤ 0. We have $(\omega_\Gamma)_0 = ((S_\Gamma)_0)^\perp$, so it has dimension just one less than $\deg \Gamma$.

If further Γ fails by 1 to impose independent conditions on quadrics, then $(S_\Gamma)_2$ has the same dimension, and we see that α is an isomorphism in all degrees ≥ -2. Further, if every subscheme of Γ imposes independent conditions on quadrics then Γ imposes independent conditions on cubics (Proof: Find a quadric vanishing on a codegree 2 subscheme. It does not generate a minimal submodule of $\mathcal{O}_\Gamma(2)$, so we can multiply by a linear form to get a cubic vanishing precisely on a codegree 1 subscheme.) Thus $(\omega_\Gamma)_{-d} = 0$ for $d \geq 3$, and α is an isomorphism. Thus Γ is arithmetically Gorenstein. ∎

As mentioned above, if we restrict to the case of stable sets of points, there is a particularly simple characterization of self-association, due to Coble [1929].

Corollary 7.4 *(Coble, Dolgachev-Ortland) A stable set of $2r + 2$ distinct k-rational points in $\mathbf{P}^r_k$ is self-associated iff it fails to impose independent conditions on quadrics.*

Proof. Any self-associated scheme of degree $2r + 2$ fails to impose independent conditions on quadrics, for example by Theorem 7.1. Conversely, assume that Γ is stable and fails to impose independent conditions on quadrics. Any subset of $2r + 1$ points of Γ imposes independent conditions on quadrics (see Dolgachev-Ortland [1986], Lemma 3, p.45 and Shokurov [1971]), and so the result follows from Theorem 7.1. ∎

As a corollary of Theorem 7.3 we can exhibit an interesting class of examples. To simplify the notation we systematically identify effective divisors on a smooth curve with the schemes they represent.

Corollary 7.5 *Let C be a reduced irreducible canonically embedded curve in $\mathbf{P}^n$, and let $\Gamma \subset C$ be a Cartier divisor in the class $K_C + D$, where D is an effective divisor of degree 2, so that the degree of Γ is $2n+2$. The scheme Γ is arithmetically Gorenstein iff Γ does not contain D.*

Proof. We check the conditions of Theorem 7.3, noting that by Riemann-Roch an effective Cartier divisor E fails to impose independent conditions on quadrics iff

$$\mathrm{h}^1(2K_C - E) = \mathrm{h}^0(E - K_C) \neq 0.$$

In particular, any effective divisor in the class $K_C + D$ fails to impose independent conditions on quadrics because D imposes just one condition. Further, if $\Gamma \supset D$ and $p \in D$, then $\mathrm{h}^0(\Gamma - p - K_C) = \mathrm{h}^0(D - p) \neq 0$, and we see that Γ is not arithmetically Gorenstein.

Now suppose that Γ does not contain D, that is, $\Gamma - D$ is ineffective. As $\mathrm{h}^0(K_C) = \mathrm{h}^0(K_C + D) - 1$, we see Γ cannot even contain a point of D. Thus, for any $p \in \Gamma$, $\mathrm{h}^0(\Gamma - p - K_C) = \mathrm{h}^0(D - p) = 0$, so $\Gamma - p$ imposes independent conditions on quadrics, and Γ is arithmetically Gorenstein by Theorem 7.3. ∎

We conclude with a remark that will be used for the classifications in Section 9: Self-associated schemes can be direct sums (in the sense of Section 5 above):

Proposition 7.6 *If $\Gamma \subset \mathbf{P}^r$ is a decomposable finite scheme, then Γ is self-associated iff each of its summands is self-associated.*

Proof. If Γ decomposes with summands $\Gamma_1 \subset L_1$ and $\Gamma_2 \subset L_2$, then the natural map $V \longrightarrow \mathrm{H}^0(\mathcal{O}_\Gamma(1))$ splits as the direct sum of maps $V_i \longrightarrow \mathrm{H}^0(\mathcal{O}_{\Gamma_i}(1))$, where $V_i = \mathrm{H}^0(\mathcal{O}_{L_i}(1))$. Thus the Gale transform $V^\perp \subset \mathrm{H}^0(K_\Gamma(-1))$ splits too, and the proposition follows. ∎

Corollary 7.7 *A finite locally Gorenstein scheme $\Gamma \subset \mathbf{P}^r$ of degree $2r+2$ which is a direct sum of arithmetically Gorenstein subschemes is self-associated. The number of summands is exactly the amount by which Γ fails to impose independent conditions on quadrics.* ∎

8 Linear Algebra and Self-association

Let $\Gamma \subset \mathbf{P}^r_k = \mathbf{P}(V)$ be a nondegenerate set of $2r+2$ distinct points. Choose $\Gamma_1 \subset \Gamma$ a subset of $r+1$ points that spans $\mathbf{P}^r$, and let Γ_2 be the complementary set. If Γ is self-associated, then by Proposition 2.2 the set Γ_2 must also span $\mathbf{P}^r$.

Babbage [1948] pointed out that Γ is self-associated iff in addition there is a nonsingular quadric $Q \subset \mathbf{P}^r$ such that each of Γ_1 and Γ_2 are *apolar* (self-conjugated simplexes) with respect to Q. In modern language (and replacing quadratic forms with symmetric bilinear forms to avoid problems in characteristic 2):

Theorem 8.1 *Let $\Gamma \subset \mathbf{P}^r_k$ be a nondegenerate set of $2r+2$ distinct points. The set Γ is self-associated if and only if it can be decomposed into a disjoint union $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 correspond to orthogonal bases for the same nonsingular symmetric bilinear form on V.*

Proof. The result follows immediately from Castelnuovo's definition (in Section 1) which amounts to saying that set of points is self-associated iff there is a nonsingular bilinear form B on V and a decomposition of Γ into two disjoint bases $\{e_i\}$ and $\{f_i\}$ for V (the vertices of the two simplices) such that the orthogonal complement of any e_i is the span of $\{e_j\}_{j \neq i}$, and similarly for the f_i. Since the form B has orthogonal bases, it is symmetric. ∎

The bilinear form really does depend on the choice of splitting $\Gamma = \Gamma_1 \cup \Gamma_2$, since otherwise each vector in Γ would be orthogonal to all the other vectors in Γ, and these other vectors span V. Babbage also asserts that Q is unique, which, as we shall see, is false in general.

To generalize Babbage's result to schemes, we first extend the notion of an orthogonal basis.

Definition 8.2 *As above, let V be a k-vector space of dimension $\dim_k(V) = r+1$.*

a) *A scheme $\Lambda \subset \mathbf{P}(V)$ is a basis for V if it is nondegenerate and of degree $r+1$ over k; that is, if the natural map $V \longrightarrow \mathrm{H}^0(\mathcal{O}_\Lambda(1)) = \mathcal{O}_\Lambda(1)$ is an isomorphism.*
b) *If $B : V \longrightarrow V^*$ is a k-linear map, then we say Λ is an orthogonal basis with respect to the bilinear form (or quadratic form) corresponding to B if Λ is a basis and the composition*

$$\mathcal{O}_\Lambda(1) \cong V \xrightarrow{B} V^* \cong K_\Lambda(-1)$$

is an isomorphism of sheaves of $\mathcal{O}_\Lambda$-modules.

This generalizes the classical notion: If Λ corresponds to an ordinary basis $\{p_i\}_{\{i=1,\ldots,r+1\}}$ of V, then $\mathcal{O}_\Lambda \cong k \times k \times \ldots \times k$ as a ring, and $\mathcal{O}_\Lambda$ has idempotents e_i corresponding to the basis elements p_i. Any sheaf $\mathcal{F}$ over $\mathcal{O}_\Lambda$ decomposes as $\oplus e_i\mathcal{F}$, and any morphism of sheaves preserves this decomposition. Thus B satisfies condition *b)* above iff the matrix of B with respect to the basis $\{p_i\}$ is diagonal, and $B(p_i)(p_j) = 0$ for all $i \neq j$.

In the classical case the existence of an orthogonal basis implies that the bilinear form corresponding to B is symmetric. This remains also true in our generality: For if B is a sheaf homomorphism, then for any generator $f \in \mathcal{O}_\Lambda(1)$ and section $g \in K_\Lambda(-1)$ the quotient is a section $g/f \in \mathcal{O}_\Lambda$, and we have

$$B(f)(g) = B(f)((g/f)f) = B((g/f)f)(f) = B(g)(f).$$

Since any section $f' \in \mathcal{O}_\Lambda(1)$ may be written as $f' = rf$ for some $r \in \mathcal{O}_\Lambda$, we get

$$B(f')(g) = rB(f)(g) = rB(g)(f) = B(g)(rf) = B(g)(f'),$$

as required.

We will generalize Theorem 8.1 as follows:

Theorem 8.3 *Suppose that a finite Gorenstein scheme $\Gamma \subset \mathbf{P}^r_k = \mathbf{P}(V)$ of degree $2r+2$ decomposes as the disjoint union of two subschemes Γ_1 and Γ_2 that are bases. Then the scheme Γ is self-associated iff Γ_1 and Γ_2 are both orthogonal bases for the same nonsingular bilinear form on V. The bilinear form is unique iff Γ is arithmetically Gorenstein.*

Proof. Suppose first that Γ is self-associated, so there is a sheaf isomorphism $\lambda : \mathcal{O}_\Gamma(1) \longrightarrow K_\Gamma(-1)$ carrying V to $V^\perp$. As λ is a sheaf homomorphism, it decomposes as a direct sum of isomorphisms

$$\lambda_i : \mathcal{O}_{\Gamma_i}(1) \longrightarrow K_{\Gamma_i}(-1), \quad i = 1, 2.$$

Write $a_i : V \longrightarrow \mathcal{O}_{\Gamma_i}(1)$ for the inclusion corresponding to the embedding of Γ_i in $\mathbf{P}(V)$. The bilinear forms $B_i = a_i^* \circ \lambda_i \circ a_i : V \longrightarrow V^*$, $i = 1, 2$, are nonsingular since each of the three maps in the composition is an isomorphism. Thus each Γ_i is an orthogonal basis for B_i. As $\lambda = \lambda_1 \oplus \lambda_2$ maps V to $V^\perp$, the bilinear form

$$(a_1^* \quad a_2^*) \circ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \circ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

is zero. Thus $B_1 = -B_2$, and each Γ_i is an orthogonal basis for B_1.

The bilinear symmetric forms B making V isotropic correspond to elements of the dual of the cokernel of $\mathrm{Sym}_2(V) \longrightarrow \mathcal{O}_\Gamma(2)$. If Γ is self-associated then, by Theorem 7.3, Γ is arithmetically Gorenstein iff it fails by exactly one to impose independent conditions on quadrics. Thus the bilinear form is uniquely determined up to a scalar factor exactly in this case.

Conversely, if both Γ_i are orthogonal bases for a nonsingular form B, then B induces an isomorphism of $\mathcal{O}_\Gamma$-modules

$$\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} : \mathcal{O}_\Gamma(1) = \oplus_{i=1,2} \mathcal{O}_{\Gamma_i}(1) \longrightarrow \oplus_{i=1,2} K_{\Gamma_i}(-1) = K_\Gamma(-1),$$

whose associated bilinear form λ satisfies the conditions of Theorem 7.1. ∎

As a consequence of Theorem 8.3 we can give a new proof of the following result:

Corollary 8.4 *(Coble, Dolgachev-Ortland) The variety of ordered arithmetically Gorenstein sets of $2r+2$ distinct k-rational points in $\mathbf{P}^r_k$ whose first $r+1$ elements span, up to projective equivalence, is isomorphic to an open subset in the variety of complete flags in $\mathbf{P}^r_k$. In particular it is irreducible and rational of dimension $\binom{r+1}{2}$, and thus the variety of unordered self-associated sets of distinct points is also irreducible and unirational of the same dimension.*

Remark 8.5 In the case of 6 points in the plane, the unordered self-associated sets of 6 points form a rational variety, isomorphic to the moduli space of genus 2 curves (Igusa [1964]): such a set of points lies on a conic, whose double cover branched over the 6 points is a curve of genus 2. Is the variety of unordered self-associated sets always rational?

Proof of Corollary 8.4. If $\Gamma = \Gamma_1 \cup \Gamma_2$ is an arithmetically Gorenstein set in $\mathbf{P}^r_k$ decomposed into its subsets of the first $r+1$ and last $r+1$ points, and Γ_1 spans $\mathbf{P}^r_k$, then, by Proposition 2.2, the set Γ_2 also spans. By Theorem 8.3 there is a unique symmetric bilinear form B for which both Γ_1 and Γ_2 are orthogonal bases. There is a unique projective equivalence taking Γ_1 to the standard simplex of $\mathbf{P}_k^r$ and taking B to the form whose matrix with respect to this basis is the identity matrix. To the set Γ we may associate the flag consisting of the spaces spanned by the first i elements of Γ_2, for all $i = 1, \ldots, r+1$.

Conversely, let Γ_1 be a basis in V, and let B be the bilinear form whose matrix with respect to this basis is the identity. Suppose that $V_1 \subset \ldots \subset V_{r+1}$ is a flag with $\dim_k(V_i) = i$ transverse to the flag of coordinate subspaces defined by Γ_1. Assume that the restriction of B to each subspace V_i is nonsingular. By the Gram-Schmidt process, we may choose an orthogonal basis $v_1, \ldots, v_{r+1}$ for B such that $V_i =< v_1, \ldots, v_i >$, $i \in \{1, \ldots, r+1\}$. For an open set of flags, the points in $\mathbf{P}^r$ corresponding to the vectors v_i will be distinct from the points of the standard simplex. Let Γ_2 be the set of these points, and set $\Gamma = \Gamma_1 \cup \Gamma_2$. By Theorem 8.3, the set Γ is arithmetically Gorenstein. ∎

By using Theorem 8.3 we may partially decide when is it possible to extend a finite set (or more generally a locally Gorenstein finite scheme) to an arithmetically Gorenstein one.

Theorem 8.6 *A general set $\Gamma \subset \mathbf{P}_k^r$ of $\gamma = r + 1 + d \leq 2r + 2$ points can be extended to an arithmetically Gorenstein set $\Gamma \cup \Gamma' \subset \mathbf{P}_k^r$ of $2r+2$ points iff $\binom{d}{2} \leq r$. Moreover in case $\binom{d}{2} = r$, there is a unique linear subspace $L \subset \mathbf{P}^r$ of dimension $(r - d)$ such that if $\Gamma \cup \Gamma'$ is arithmetically Gorenstein then Γ' spans L.*

Proof. Changing coordinates if necessary, we may assume without loss of generality that Γ contains as a subset the standard basis $\Gamma_1 = \{e_0, e_1, \ldots, e_r\}$ of the ambient vector space V. We write $\Gamma = \Gamma_1 \cup \Sigma$, where Σ consists of the remaining d points.

To find an arithmetically Gorenstein scheme $\Gamma_1 \cup \Sigma \cup \Sigma'$ we search for bilinear forms B such that both Γ_1 and $\Sigma \cup \Sigma'$ are orthogonal bases. First, Γ_1 is an orthogonal basis iff the matrix of B is diagonal. The mutual orthogonality of the elements of Σ imposes $\binom{d}{2}$ homogeneous linear equations on the $r + 1$ diagonal elements of B. In particular, the system has a non-trivial solution whenever $\binom{d}{2} \leq r$. Since Γ is general and self-associated sets do exist, there is a solution B which is nonsingular. Choosing Γ_2 to be any orthogonal basis containing Σ, and using Theorem 8.3, we prove the first statement of the theorem.

To complete the proof we show that for a general set of points Γ the $\binom{d}{2}$ linear equations on the coefficients of B above are of maximal rank. It follows that in case $r + 1 = \binom{d}{2}$ the form B is unique, and Σ' spans the orthogonal complement of Σ with respect to B. On the other hand, if $r + 1 < \binom{d}{2}$ then the equations for B have only the trivial solution.

Consider the incidence variety $\mathcal{I}$ whose points are pairs consisting of a nonsingular bilinear form B on V such that Γ_1 is an orthogonal basis, and a d-tuple of distinct points in $\mathbf{P}^r$ which are non-isotropic and mutually orthogonal with respect to the bilinear form B. To show that the set of linear equations above is linearly independent, we must show that the fiber of $\mathcal{I}$ over a general set Σ has the expected dimension, which is $r - \binom{d}{2}$. We know from the argument above that the general fiber has dimension at least the expected dimension. Further, the set of d-tuples of points Σ has dimension rd. Thus it suffices to show that $\dim \mathcal{I} = r(d+1) - \binom{d}{2}$.

Projecting a pair $(B, \Sigma) \in \mathcal{I}$ onto the first factor we obtain a surjection $\mathcal{I} \longrightarrow (k^*)^r$. The fiber over a point $B \in (k^*)^r$ may be identified with the set of flags $V_1 \subset \ldots \subset V_d \subset V$, where $\dim_k(V_i) = i$ and B restricted to each V_i is nonsingular: given any $\Sigma = \{v_1, \ldots, v_d\}$ we let $V_i = \langle v_1, \ldots, v_i \rangle$, and conversely given the flag we use the Gram-Schmidt process to produce an orthogonal basis. Thus the fiber is an open set in a flag variety of dimension $rd - \binom{d}{2}$, and so $\dim \mathcal{I} = r(d+1) - \binom{d}{2}$ as required. (One can show further that the incidence variety is irreducible and nonsingular, but we do not need this.) ∎

Example 8.7 Five general points in $\mathbf{P}^2$ lie on a unique conic. Any sixth point on the conic gives an arithmetically Gorenstein set.

Example 8.8 Let Γ be a set of seven points in linearly general position in $\mathbf{P}^3$. They lie on just three independent quadrics. If these form a complete intersection, then there exists a unique extension of the seven points to a self-associated scheme of degree 8. If not, the three quadrics cut out a twisted cubic curve and Γ lies on it. In this case there are many possible extensions: we can add any further point on the rational normal curve, and these are the only possibilities.

Example 8.9 Consider now a set $\Gamma \subset \mathbf{P}^6$ of 11 general points. By Theorem 8.6, the set Γ may be completed to a self-associated set of 14 points in $\mathbf{P}^6$. For all possible completions the linear span of the extra three points is a distinguished plane $\Pi = \mathbf{P}^2 \subset \mathbf{P}^6$. The Koszul complex built on the equations defining this 2-dimensional linear subspace is the complex $E_\bullet(\mu)^*(-8)$ embedded at the back end of the minimal free resolution of the homogeneous ideal I_Γ (see the end of the introduction of Eisenbud-Popescu [1996] for notation and details). A similar remark holds for a general set Γ of $\binom{s+2}{2} + 1$ points in $\mathbf{P}^{\binom{s+2}{2} - s - 1}$, yielding a distinguished linear subspace of dimension $\binom{s+2}{2} - 2s - 2$ whose equations contribute to the resolution of Γ in an interesting way.

Since we have been dealing with the condition of forming two orthogonal bases, we comment on the condition that a set of $2r+2$ points in $\mathbf{P}^r_k$ correspond to the union of 2 bases. First recall the criteria of stability and semistability:

Proposition 8.10 *(Dolgachev-Ortland). Let $\Gamma \subset \mathbf{P}^r_k$ be a set of γ points. Then Γ is semistable if and only if for all m with $1 \le m \le \gamma - 1$ the projective linear span of any subset of m points of Γ has dimension at least $m(r+1)/\gamma - 1$. Similarly, Γ is a stable set of points if and only if all previous inequalities are strict.* ∎

The special case when $\gamma = 2r + 2$ has a nice linear algebra interpretation:

Lemma 8.11 *(Edmonds [1965]; see also Eisenbud-Koh [1987]) A set Γ of $2r+2$ points in $\mathbf{P}^r$ is semistable iff the points of Γ form two bases for the underlying vector space of the ambient projective space.* ∎

By the remarks at the beginning of this section, any self-associated set is semistable.

9 Classification of Self-associated Schemes in Small Projective Spaces

In this section we give a complete classification of self-associated schemes in $\mathbf{P}^r$ for $r \leq 3$, and we review classification results of Coble, Bath, and Babbage for $\mathbf{P}^4$ and $\mathbf{P}^5$. We begin with some examples valid in all dimensions, coming from Corollary 3.2 and Corollary 7.5.

Proposition 9.1 *The following are families of arithmetically Gorenstein nondegenerate schemes of degree $2r+2$ in $\mathbf{P}^r$:*

a) *A Cartier divisor in the class $2H - K_C$ on a rational normal curve $C \subset \mathbf{P}^r$ (defined by the minors of a $2 \times r$ matrix with linear entries).*

b) *A quadric section of a nondegenerate reduced irreducible curve of degree $r+1$, and arithmetic genus 1 in $\mathbf{P}^r$, $r \geq 2$.*

c) *A hyperplane section of a canonical curve of genus $r+2$, $r \geq 1$.*

d) *A Cartier divisor Γ in the class $K_C + D$ on a curve C of genus $g = r+1$, $r \geq 2$, where D is effective of degree 2, and Γ doesn't contain D.* ∎

The families listed in Proposition 9.1 account in fact for the general self-associated sets of points in small projective spaces, as we will see bellow. An easy count of parameters shows that in Proposition 9.1, the families described in *a*) and *b*) have dimensions $2r-1$, $2r+2$ when $r \geq 4$, and 6 when $r = 3$, respectively. For the last claim we use:

Proposition 9.2 *Let $\Gamma \subset \mathbf{P}^r$ be a quadric section of an elliptic normal curve in $\mathbf{P}^r$. If $r > 3$, then there is no other elliptic normal curve containing Γ.*

By contrast, if $r = 3$, there are many elliptic normal curves containing such a Γ; indeed, the set is parametrized by an open subset of $\mathbf{P}^2$.

Proof Sketch. Suppose that $\Gamma \subset E \cap E'$, where E and E' are elliptic normal curves in $\mathbf{P}^r$, and assume Γ is equivalent to twice the hyperplane section of E. It follows that the quadrics vanishing on $E \cup E'$ form a codimension 1 subspace of those vanishing on E.

Suppose $r > 3$. The threefold that is the union of the secant lines of E has E as its singular locus, so if $E \neq E'$ there is a secant line to E that is not secant to E'. It follows that we can find distinct rational normal scrolls X and X', of codimension 2, containing E and E' respectively.

But the intersection of any two distinct quadrics containing a codimension 2 scroll is the union of the scroll and a codimension 2 linear subspace; there is no room for another scroll. Therefore $E = E'$ is the unique elliptic normal curve in $\mathbf{P}^r$ containing Γ. ∎

We now turn to the classification results. In $\mathbf{P}^1$ the matter is trivial: every degree 4 scheme is self-associated, and of course all are arithmetically Gorenstein. The problem is already more challenging in $\mathbf{P}^2$ and $\mathbf{P}^3$, and we begin with some general remarks. Since the classification of Gorenstein schemes in these codimensions is well-known, the difficult point here is to decide what examples exist that are not arithmetically Gorenstein.

Let $\Gamma \subset \mathbf{P}^r$ be a finite self-associated Gorenstein scheme. By Proposition 2.2 every codegree 2 subscheme of Γ spans $\mathbf{P}^r$, and in particular Γ is nondegenerate. By Theorem 7.2 and Proposition 5.2, Γ is Gorenstein unless Γ is contained in the scheme defined by the ideal of minors of a matrix of the form

$$(*) \qquad \begin{pmatrix} x_0 & \dots & x_t & x_{t+1} & \dots & x_r \\ 0 & \dots & 0 & l_{t+1} & \dots & l_r \end{pmatrix},$$

where $0 \leq t < r$ and the l_i are linearly independent linear forms. In particular, Γ_{red} lies in the union of the planes $L_1 = V(x_0, \dots, x_t)$ and $L_2 = V(l_{t+1}, \dots, l_r)$. If $L_1 \cap L_2 = \emptyset$, then Γ must be decomposable, and by Proposition 7.6 $\Gamma \cap L_i$ is self-associated in L_i for each i. It seems plausible that something of this sort happens more generally:

Problem 9.3 Suppose that Γ is self-associated and the ideal of Γ contains the 2×2 minors of the matrix $(*)$ above. Under what circumstances is $\Gamma \cap V(x_0, \dots, x_t)$ self-associated in its span?

From the classification below we see that for $\mathbf{P}^3$, the first projective space in which a non-trivial example arises, the answer is "always!" The following includes a weak result of this type, which still suffices to eliminate many possibilities:

Lemma 9.4 *Suppose that $\Gamma \subset \mathbf{P}^r$ is a self-associated scheme.*

a) If $\Gamma' \subset \Gamma$ has degree $r + d$, then Γ' spans at least a subspace of dimension d.

b) If the homogeneous ideal of Γ contains a product of ideals $(l_1, \dots, l_s) \cdot (m_1, \dots, m_u)$ where the l_i are linearly independent linear forms, and similarly for the m_j, and $s + u > r$, then $2 \leq u \leq r - 1$ and $2 \leq s \leq r - 1$.

Proof. *a)* If Γ is self-associated then, as the embedding series is very ample, Corollary 2.4 shows that no subscheme of Γ of degree $2r$ can lie in a hyperplane. Thus no subscheme of Γ of degree $r + d$ can lie in a subspace of dimension $d - 1$.

b) Let $\Gamma' = \Gamma \cap V(l_1, \dots, l_s)$. Since the homogeneous coordinate ring S_Γ is Cohen-Macaulay, any linear form l vanishing on Γ' annihilates the ideal $(m_1, \dots, m_u)$, so we may harmlessly assume that the span of Γ' is the $(r - s)$-plane $V(l_1, \dots, l_s)$. The residual to Γ' in Γ lies inside $V(m_1, \dots, m_u)$, so by Proposition 2.2, Γ' fails by at least u to impose independent conditions on hyperplanes. If follows that $\deg(\Gamma') \geq u + (r + 1 - s) = r + (u - s + 1)$. By the result of part *a)*, we have $r - s \geq u - s + 1$, or equivalently $u \leq r - 1$, one of the desired inequalities. By symmetry $s \leq r - 1$, and since $s + u > r$, we derive $2 \leq u$ as well. ∎

We can now complete the classification in $\mathbf{P}^2$ and $\mathbf{P}^3$:

Theorem 9.5 *A finite Gorenstein scheme in $\mathbf{P}^2$ is self-associated iff it is a complete intersection of a conic and cubic.*

Proof. If Γ is arithmetically Gorenstein then since it has codimension 2 it must be a complete intersection. It cannot lie on a line, and it has degree 6, so it is the complete intersection of a conic and a cubic.

If Γ is not arithmetically Gorenstein, then the ideal of Γ contains the ideal of the minors of a matrix of the form $(*)$ above. In particular it contains $(x_0, \ldots, x_t)(l_{t+1}, \ldots, l_2)$, so by Lemma 9.4 part *b*), we get $2 \leq t+1 \leq 1$, a contradiction. ∎

Proposition 9.6 *A finite Gorenstein scheme $\Gamma \subset \mathbf{P}^3$ self-associated if and only if either*

a) *Γ is a complete intersection of type $(2,2,2)$ (thus the general such Γ is a quadric section of an "elliptic normal quartic" curve in $\mathbf{P}^3$), or*

b) *Γ is cut out by the Pfaffians of a 5×5 skew symmetric matrix with entries of degrees*

$$\begin{pmatrix} - & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 \\ 1 & 1 & - & 2 & 2 \\ 1 & 1 & 2 & - & 2 \\ 1 & 1 & 2 & 2 & - \end{pmatrix},$$

where the dashes denote zero entries, or

c) *There is a smooth quadric Q and a divisor C of type $(2,0)$ on Q such that Γ is a Cartier divisor on C in the class $4H_C$, where H_C denotes the hyperplane class (that is, Γ consists of a degree 4 subscheme on each of two disjoint lines, or a subscheme of degree 8 on a double line meeting the reduced line in a degree 4 subscheme).*

Proof. The schemes Γ described in *a*) and *b*) are arithmetically Gorenstein of degree 8, and thus self-associated. For part *c*) we may apply Corollary 3.2 to the linearly normal curve C. The restriction map $\mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(C_{\mathrm{red}})$ is an isomorphism and writing H_C for the hyperplane class on C it follows that $\Gamma + K_C - H_C = H_C$, so Γ is indeed self-associated.

For the converse, suppose first that Γ is arithmetically Gorenstein. By the structure theorem (Buchsbaum-Eisenbud [1977]) for codimension three arithmetically Gorenstein schemes, Γ has ideal I_Γ generated by the $2n \times 2n$-Pfaffians of a $(2n+1) \times (2n+1)$ skew symmetric matrix. From the Hilbert function we know that I_Γ contains three quadrics and is moreover 3-regular. If $n = 1$, the ideal is generated by these three quadrics, and is thus a complete intersection (case *a*)). If $n = 2$ there must be 2 cubic generators in addition to the 3 quadrics, and the given degree pattern is easy to deduce (case *b*)). Finally, if $n > 2$, then the Pfaffians would all have degree > 2, which is impossible.

On the other hand, suppose that Γ is not arithmetically Gorenstein. By Proposition 5.2, Γ lies on the scheme defined by the 2×2 minors of a matrix of the form $(*)$. By Lemma 9.4 we have $t = 1$. If $V(l_2, l_3)$ is disjoint from $V(x_0, x_1)$, then Γ lies on the disjoint union of two lines, and is of course a degree 8 Cartier divisor there. Any two disjoint lines lie on a smooth quadric, so we are done in this case.

If on the contrary $V(l_2, l_3)$ meets or coincides with $V(x_0, x_1)$ then the matrix $(*)$ can be reduced by a linear change of variables and columns to the form

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_0 & l_3 \end{pmatrix},$$

with l_3 equal to x_1, x_2, or x_3.

If $l_3 = x_2$ or $l_3 = x_3$, then we see that x_0 corresponds to an element of the socle of the local ring of Γ at the point $V(x_0, x_1, l_3)$, and vanishes on any component of Γ supported away from this point. Since Γ is nondegenerate, $x_0 \neq 0$ in this local ring. Since $\mathcal{O}_\Gamma$ is Gorenstein, x generates the socle of the local ring. It follows that the line $x = 0$ contains a codegree 1 subscheme of Γ, contradicting part a) of Lemma 9.4.

Thus we may suppose $l_3 = x_1$. In this case Γ lies on a double line on the smooth quadric $V(x_0x_3 - x_1x_2)$, and it remains to see that Γ is a Cartier divisor there. By Lemma 9.4 a) the reduced line can intersect Γ in a subscheme of degree at most 4. Passing to the affine case, we may take a polynomial f in the ideal of Γ in the double line which restricts to the reduced line to define the same scheme of degree 4. Since f is a nonzero-divisor in the ideal of the double line, it defines a subscheme of degree 8, and thus f generates the ideal of Γ in the double line. It follows that Γ is Cartier, which concludes the proof of the Proposition. ∎

Remark 9.7 The classification in Proposition 9.6 is also the classification by numerical type of the free resolution, or, as it turns out, by the length of the 2-linear part of the resolution, the "resolution Clifford index" in an obvious sense (see Eisenbud [1992]). The analogue here of Green's conjecture might be to show that the resolution Clifford index is always determined by the "geometric Clifford index," that is, the types of matrices of the form $(*)$ that arise. Be this as it may in general, the possible free resolutions of S_Γ over $k[x_0, \ldots, x_3]$ in cases a), b), and c) respectively, are

degree				
0	1	–	–	–
1	–	3	–	–
2	–	–	3	–
3	–	–	–	1

degree				
0	1	–	–	–
1	–	3	2	–
2	–	2	3	–
3	–	–	–	1

degree				
0	1	–	–	–
1	–	4	4	1
2	–	–	–	–
3	–	2	4	2

Remark 9.8 Case *b*) in Proposition 9.6 above corresponds in fact to schemes of degree 8 on a (possibly degenerate) twisted cubic curve. Indeed, the determinantal ideal in the first two rows of the 5×5 skew symmetric matrix must actually have codimension 2 (Proof: Its minors appear among the 5 minimal generators and are thus linearly independent. We may reduce modulo a general linear form and reduce to a problem in 3 variables. If the three minors had a common divisor x, then x would be in the socle module of the (reduced) ideal of the points, which is impossible, as the socle is entirely in degree 3.) Therefore, the Pfaffians define a scheme of degree 8 on a determinantal curve of degree 3 in $\mathbf{P}^3$. Note also, that two general quadrics in the ideal of the curve define in general an arithmetic genus 1 quartic curve containing the eight points; but they are not a quadric section of this quartic.

Here is a geometric description of a special case of case *c*):

Example 9.9 Suppose $\Gamma \subset \mathbf{P}^3$ is a scheme of degree 8 consisting of four double points. Suppose further that the degree 4 scheme Γ_{red} is contained in a line R. Then Γ is self-associated iff the components of Γ are tangent to four rulings on a smooth quadric surface iff the four points of $\mathbf{P}^1$ corresponding to the tangent vectors to Γ in the normal bundle of R have the same cross-ratio as the corresponding points of Γ_{red} in R. ∎

In $\mathbf{P}^4$ we have a less complete result. The extra hypothesis of linear general position excludes in particular the non arithmetically Gorenstein cases such as the union of 4 points on a line and 6 points on a conic spanning a disjoint plane. The result was enunciated by Bath in the reduced "sufficiently general" case.

Theorem 9.10 *Let $\Gamma \subset \mathbf{P}^4$ be a finite, local complete intersection scheme, which is in linearly general position. Then Γ is self-associated (and in fact arithmetically Gorenstein) if and only if either*

a) Γ is a quadric section of an elliptic normal quintic curve (equivalently a hyperplane section of a non-trigonal canonical curve of genus 6 in $\mathbf{P}^5$), or

b) Γ is a scheme of degree ten on a rational normal quartic curve.

Proof. From the general position hypothesis, Corollary 5.3 and Theorem 7.2 it follows that Γ is self-associated iff Γ is arithmetically Gorenstein. In particular, Γ fails exactly by one to impose independent condition on quadrics, and thus $\mathrm{h}^0(\mathcal{I}_\Gamma(2)) = 6$.

A structure theorem of Kustin-Miller [1985], Herzog-Miller [1986], and Vasconcelos-Villareal [1986] asserts that a generic local complete intersection Gorenstein ideal I, of grade 4 and deviation 2, is of the type $I = \langle J, f\rangle$, where J is a (Gorenstein) codimension 3 ideal defined by the 4×4 Pfaffians of a skew symmetric matrix, and f is a non-zero divisor modulo J. Thus, in case the six quadrics in I_Γ generate the homogeneous ideal, that is I_Γ is Gorenstein of grade 4 and deviation 2, then Γ is a quadric section of an arithmetically Gorenstein scheme $\Lambda \subset \mathbf{P}^4$ defined by the Pfaffians of a 5×5-skew symmetric matrix with linear entries, which is case a) in the statement of the theorem.

Assume now that the quadrics in $\mathrm{H}^0(\mathcal{I}_\Gamma(2))$ do not generate the Gorenstein ideal I_Γ. In this case, there are also cubic generators in the ideal, and by symmetry their number matches the dimension of $\mathrm{Tor}_3^S(I_\Gamma, S)_2$, which is thus nonzero. By the "Strong Castelnuovo Lemma" of Green [1984], Yanagawa [1994] and Cavaliere-Rossi-Valla [1994]; see also Eisenbud-Popescu [1997], it follows that Γ is divisor of degree 10 on a smooth rational normal quartic curve, which is case b) in the statement of the proposition. ∎

Remark 9.11 Bath [1938] claims that the general self-associated set in $\mathbf{P}^4$ is a quadric section of a quintic elliptic normal curve, (case 1) in Theorem 9.10. (See also Babbage [1948].) Here is an outline of his argument:

A general self-associated ordered set $\Gamma = \{p_1, \ldots, p_{10}\} \subset \mathbf{P}^4$ fails by one to impose independent conditions on quadrics, so $\mathrm{h}^0(\mathcal{I}_\Gamma(2)) = 6$. Either Γ is contained in a rational normal quartic curve, or three general quadrics in $\mathrm{H}^0(\mathcal{I}_\Gamma(2))$ meet along a genus 5 canonical curve $C \subset \mathbf{P}^4$, passing through the set Γ. In the latter case, the quadrics in $\mathrm{H}^0(\mathcal{I}_\Gamma(2))$ cut out a g^2_6 residual to Γ on the curve C. However, a g^2_6 on C is special, and this means that any divisor in this linear system spans only a $\mathbf{P}^3$. Let Σ be a (general) divisor in the g^2_6, so that Σ is reduced, disjoint from Γ, and in linearly general position in its span (since C is cut out by quadrics, and is not hyperelliptic). By Castelnuovo's lemma (see for instance Theorem 4.1) there is a unique twisted cubic curve $D \subset \mathbf{P}^3$ (the linear span of Σ) which passes through Σ. Now $\Gamma \cup \Sigma$ is the complete intersection of 4 quadrics from $\mathrm{H}^0(\mathcal{I}_\Gamma(2))$, and since it is only necessary to make a quadric contain one more point of D for the whole twisted cubic D to lie on the quadric, it follows that D lies on three independent quadrics in $\mathrm{H}^0(\mathcal{I}_\Gamma(2))$. They define a complete intersection curve in $\mathbf{P}^4$, which has as components D and another (arithmetically Gorenstein) curve E of degree 5, passing through the ten points $\Gamma = \{p_1, \ldots, p_{10}\}$. The curve E is an elliptic quintic curve, and Γ is a quadric section of it.

In a general self-associated, ordered set $\Gamma = \{p_1, \ldots, p_{10}\} \subset \mathbf{P}^4$, one can always arbitrarily prescribe the first eight points. As in the proof of Theorem 8.6, one sees that among the non-singular bilinear diagonal forms, for which $\Sigma = \{p_1, \ldots, p_5\}$

forms an orthogonal basis, there is a pencil $B_{(s:t)}$ of bilinear forms for which the points $\Sigma' = \{p_6, p_7, p_8\}$ are also mutually orthogonal. The conditions that p_9 is orthogonal on Σ' are expressed by a system of three bilinear equations in $(s : t)$ and the coordinates of the ambient $\mathbf{P}^4$. The system has a solution B iff the 3×2 matrix of the linear system drops rank iff the point p_9 lies on $X \subset \mathbf{P}^4$, the variety defined by the maximal minors of the 3×2 matrix (which has linear entries in the coordinates of the ambient $\mathbf{P}^4$). For general choices, the bilinear form B is unique and nonsingular, and $X \subset \mathbf{P}^4$ is a smooth rational cubic scroll. Analogously, the point p_{10} is orthogonal on Σ', with respect to B, iff p_{10} lies also on the scroll X. Interpreting $X \subset \mathbf{P}^4$ as the image of $\mathbf{P}^2$ via conics through a point, one sees readily that there is a pencil of elliptic quintic normal curves through $\{p_1, \ldots, p_8\}$, which are all bisections for the ruling of the scroll X, and for any given choice of such an elliptic quintic normal curve E one has to pick $\{p_9, p_{10}\}$ as the intersection points of E with a ruling of X.

Remark 9.12 Mukai [1995] proved that every canonical curve of genus 7 and Clifford index 3 (i.e., the general canonical curve of genus 7) is a linear section of the the spinor variety $S_{10} \subset \mathbf{P}^{15}$ of isotropic $\mathbf{P}^4$'s in the 8-dimensional quadric $Q \subset \mathbf{P}^9$. In the same spirit, Ranestad and Schreyer [1997] showed that the "general empty arithmetically Gorenstein scheme of degree 12 in $\mathbf{P}^4$" (i.e., the general graded Artinian Gorenstein with Hilbert function $(1, 5, 5, 1)$) is always a linear section of the same spinor variety. It seems natural to expect a similar result in our case:

Conjecture 9.13 *The general arithmetically Gorenstein, nondegenerate zero--dimensional scheme of degree 12 in $\mathbf{P}^5$ is a linear section of the spinor variety $S_{10} \subset \mathbf{P}^{15}$.*

Remark 9.14 We refer to Babbage [1948] for a description in the spirit of Remark 9.11 of the general set of twelve self-associated points in $\mathbf{P}^5$.

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