SOLUTIONS TO PRACTICE MIDTERM II, FALL 2014, MAT 319/320, COURTESY OF CHENGJIAN YAO

Problem 1

The "only if" part is relatively easier. If $a \in \mathbb{R}$ has a square root, which means that $a = b^2$ for some $b \in \mathbb{R}$, and we know from the theorems in the book about the order structure of $\mathbb{R}$ that $b^2 \geq 0$.

We now prove the "if" part. For any real number a, define a subset of $\mathbb{R}$:

$$F_a = \{x \in \mathbb{R} | x^2 \leq a\}$$

Since $a \geq 0$, $0 \in F_a$, thus F_a is an nonempty subset. Since $(a+1)^2 = a^2 + 2a + 1 > a$, we claim that $a+1$ is an upper bound of F_a. To see this, if for some $x \in F_a$ we have $x > a+1$, then $x^2 > x(a+1) > (a+1)^2 > a$, contradicts to the fact that $x \in F_a$. Now, we get a nonempty subset of $\mathbb{R}$ which is bounded from above, by the *Completeness Axiom for Real Numbers*, there exists a supremum, denote $\sup F_a = b$.

We would like to prove $b^2 = a$. The proof is done by *Contradiction*. If $b^2 > a$, by the *Archimedean Property* of $\mathbb{R}$, there exists for some positive integer n, such that $n(b^2 - a) > 2b$, therefore $b^2 - \frac{2b}{n} > a$, which implies that $(b - \frac{1}{n})^2 = b^2 - \frac{2b}{n} + \frac{1}{n^2} > b^2 - \frac{2b}{n} > a$, therefore $b - \frac{1}{n}$ is also an upper bound for F_a, contradicting the fact that b is the *least upper bound* of F_a. If $b^2 < a$, also by the *Archimedean Property* of $\mathbb{R}$, for some positive integer n_1, $n_1 . \frac{1}{2}(a - b^2) > 2b$, and for some positive integer n_2, $n_2 . \frac{1}{2}(a - b^2) > 1$. Let $n = \max(n_1, n_2)$, then $\frac{2b}{n} < \frac{1}{2}(a - b^2)$ and $\frac{1}{n^2} < \frac{1}{2}(a - b^2)$, therefore $(b + \frac{1}{n})^2 = b^2 + \frac{2b}{n} + \frac{1}{n^2} < b^2 + \frac{1}{2}(a - b^2) + \frac{1}{2}(a - b^2) = a$, therefore $b + \frac{1}{n} \in F_a$, which contradicts the fact that b is an upper bound for F_a since $b < b + \frac{1}{n}$. This concludes that any nonnegative number has a square root in $\mathbb{R}$.

Problem 2

a). "$c = \sup A$" means that $\forall a \in A$, $c \geq a$ and $\forall c' < c$, $\exists a \in A$ such that $a > c'$.

b). By the definition from part a), we know that $\forall \epsilon > 0$, $\exists a \in A$ such that $a > \sup A - \epsilon$, and also we have $a \leq \sup A$, therefore $\sup A - \epsilon < a \leq \sup A$.

c). By the *Complete Axiom for Real Numbers*, the bounded set $\{a_n | n \in \mathbb{N}\}$ has a supremum, denoted by $a = \sup\{a_n | n \in \mathbb{N}\}$. By part b), $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $a - \epsilon < a_N \leq a$, since the sequence is increasing, for all $n > N$, $a_n \geq a_N$, therefore $a - \epsilon < a_n \leq a$, which implies that

$$|a_n - a| < \epsilon$$

for all $n > N$, therefore $\lim_{n\to\infty} a_n = a = \sup\{a_n | n \in \mathbb{N}\}$.

Problem 3

a) Since (a_n) and (b_n) are both bounded, there exist $M_1, M_2 > 0$ such that

$$|a_n| < M_1, |b_n| < M_2$$

for all n, then $|a_n b_n| < M_1 M_2$ for all n, which means the product sequence $(a_n b_n)$ is bounded.

b) "$(a_n b_n)$ is bounded" does not imply (a_n) and (b_n) are bounded. Take for n even $a_n = n$ and $b_n = 0$, and for n odd take $a_n = 0$ and $b_n = 0$. Then for any n we have $a_n b_n = 0$, while neither (a_n) nor (b_n) is bounded.

Problem 4

There are two ways to prove the convergence of the sequence and compute the limit. The more directly computational one is by applying various limit laws for sequences, while alternatively one could verify the sequence is Cauchy and then determine the limit from the definition. The first is easier, but pedagogically we encourage you to understand both approaches.

Applying Limit Theorems:

First simplify the sequence to be

$$a_n = \frac{1 + \frac{1}{n2^n}}{2 + \frac{1}{\sqrt{n}}}.$$

Then note that

$$\frac{1}{n2^n} \leq \frac{1}{n},$$

and since $\lim \frac{1}{n} = 0$, we must also have $\lim \frac{1}{n2^n} = 0$ by the squeeze theorem between $\frac{1}{n}$ and 0. We also know that $\lim \frac{1}{\sqrt{n}} = 0$. Thus by applying the limit laws for quotients, and then the limit laws for sums we get

$$\lim_{n\to\infty} \frac{n + 2^{-n}}{2n + \sqrt{n}} = \frac{\lim\limits_{n\to\infty} \left(1 + \frac{1}{n2^n}\right)}{\lim\limits_{n\to\infty} \left(2 + \frac{1}{\sqrt{n}}\right)} = \frac{1+0}{2+0} = \frac{1}{2}.$$

Cauchy Sequence Proof:

For $n, k \in \mathbb{N}$,

$$
\begin{aligned}
|a_n - a_{n+k}| =& |\frac{n+2^{-n}}{2n+\sqrt{n}} - \frac{(n+k)+2^{-(n+k)}}{2(n+k)+\sqrt{n+k}}| \\
=& \frac{|(2(n+k)+\sqrt{n+k})(n+2^{-n}) - (2n+\sqrt{n})((n+k)+2^{-(n+k)})|}{(2n+\sqrt{n})(2(n+k)+\sqrt{n+k})} \\
\leq& \frac{2^{1-n}(n+k)}{4n(n+k)} + \frac{2^{1-(n+k)}n}{4n(n+k)} + \frac{2^{-n}\sqrt{n+k}}{4n(n+k)} + \frac{2^{-(n+k)}\sqrt{n}}{4n(n+k)} + \frac{|\sqrt{n+k}-\sqrt{n}|}{4\sqrt{n(n+k)}} \\
=& \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4n} + \frac{1}{4\sqrt{n}} \\
\leq& \frac{2}{\sqrt{n}}
\end{aligned}
$$

Thus, $\forall \epsilon > 0$, pick $N = \lceil (\frac{2}{\epsilon})^2 \rceil$, then $\forall n > N$, $k \in \mathbb{N}$,

$$|a_n - a_{n+k}| < \epsilon$$

which implies that (a_n) is a *Cauchy sequence*.

$\forall \epsilon > 0$, pick $N = \lceil (\frac{1}{4\epsilon})^2 \rceil$, then $\forall n > N$,

$$|\frac{n+2^{-n}}{2n+\sqrt{n}} - \frac{1}{2}| = \frac{\sqrt{n}-2^{1-n}}{2(2n+\sqrt{n})} \leq \frac{\sqrt{n}}{4n} = \frac{1}{4\sqrt{n}} < \epsilon$$

which means

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$