The Schottky Problem

Samuel Grushevsky

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Schottky problem is the following question:

Which principally polarized abelian varieties are Jacobians of curves?

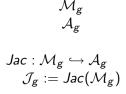
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\mathcal{A}_{g}^{-}	moduli space of g-dimensional abelian varieties
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$\mathit{Jac}:\mathcal{M}_{g}\hookrightarrow\mathcal{A}_{g}$	Torelli map
$\mathcal{J}_{g} := Jac(\mathcal{M}_{g})$	locus of Jacobians

(Recall that A is a projective variety with a group structure; Θ is an ample divisor on A with $h^0(A, \Theta) = 1$; $Jac(C) = Pic^{g-1}(C) \simeq Pic^0(C)$)

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Schottky problem.

Describe/characterize $\mathcal{J}_g \subset \mathcal{A}_g$.

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There do not exist any complex geodesics for the natural metric on \mathcal{A}_g that are contained in $\overline{\mathcal{J}_g}$ (and intersect \mathcal{J}_g). [Work on this by Möller-Viehweg-Zuo; Hain, Toledo...]

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• (Super)string scattering amplitudes [D'Hoker-Phong], ...

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1 2 3	1 3 6	$ \begin{array}{ccc} = & 1 \\ = & 3 \\ = & 6 \end{array} \mathcal{M}_g = \mathcal{A}_g^{indecomposable} $		

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g	3g - 3	$+\frac{(g-3)(g-2)}{2} =$	$\frac{g(g+1)}{2}$	"weak" solutions (up to extra components)
	0	2	2	(up to extra components)
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 $\begin{aligned} \mathcal{H}_g &:= \text{Siegel upper half-space of dimension } g \\ &= \{ \tau \in \textit{Mat}_{g \times g}(\mathbb{C}) \mid \tau^t = \tau, \ \text{Im}\tau > 0 \}. \end{aligned}$

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For
$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}(2g, \mathbb{Z})$$
 let $\gamma \circ \tau := (C\tau + D)^{-1}(A\tau + B)$.

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Definition

A modular form of weight k with respect to $\Gamma \subset \text{Sp}(2g, \mathbb{Z})$ is a function $F : \mathcal{H}_g \to \mathbb{C}$ such that

$$F(\gamma \circ \tau) = \det(C\tau + D)^k F(\tau) \qquad \forall \gamma \in \Gamma, \forall \tau \in \mathcal{H}_g$$

For $\varepsilon, \delta \in \frac{1}{n} \mathbb{Z}^g / \mathbb{Z}^g$ the theta function with characteristic ε, δ is $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z) := \sum_{N \in \mathbb{Z}^g} \exp \left[\pi i (N + \varepsilon, \tau (N + \varepsilon)) + 2\pi i (N + \varepsilon, z + \delta) \right]$

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For $\varepsilon, \delta \in \frac{1}{n}\mathbb{Z}^g/\mathbb{Z}^g$ (or $m = \tau \varepsilon + \delta \in A_\tau[n]$) the theta function with characteristic ε, δ or m is

$$\theta_m(\tau, z) := \sum_{N \in \mathbb{Z}^g} \exp\left[\pi i(N + \varepsilon, \tau(N + \varepsilon)) + 2\pi i(N + \varepsilon, z + \delta)\right]$$

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For $\varepsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ the theta function of the second order is $\Theta[\varepsilon](\tau, z) := \theta \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} (2\tau, 2z).$

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• Theta functions of the second order generate $H^0(A_{\tau}, 2\Theta)$.

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Theorem (Igusa, Mumford, Salvati Manni) For any $n \ge 2$ theta constants embed $\mathcal{A}_g(2n, 4n) := \mathcal{H}_g/\Gamma(2n, 4n) \hookrightarrow \mathbb{P}^{n^{2g}-1}$ $\tau \mapsto \left\{ \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \right\}_{\text{all } \varepsilon, \delta \in \frac{1}{n} \mathbb{Z}^g/\mathbb{Z}^g}$

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 Theta constants of the second order define a generically injective Th : A_g(2,4) → P^{2g-1} (conjecturally an embedding).

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Classical Riemann-Schottky problem

Write the defining equations for

$$\overline{\mathit{Th}(\mathcal{J}_g(2,4))}\subset\overline{\mathit{Th}(\mathcal{A}_g(2,4))}\subset\mathbb{P}^{2^g-1}.$$

g	$\deg Th(\mathcal{J}_g(2,4))$	deg $Th(\mathcal{A}_g(2,4))$
1	1 =	1
2	1 =	1
3	16 =	16

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g	deg $Th(\mathcal{J}_g(2,4))$		deg $Th(\mathcal{A}_g(2,4))$
1	1	=	1
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4	208896	$= 16 \cdot$	13056

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Theorem (Schottky, Igusa)

The defining equation for $\mathcal{J}_4 \subset \mathcal{A}_4$ is

$$F_{4}(\tau) := 2^{4} \sum_{\varepsilon, \delta \in \frac{1}{2} \mathbb{Z}^{g} / \mathbb{Z}^{g}} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau)^{16} - \left(\sum_{\varepsilon, \delta \in \frac{1}{2} \mathbb{Z}^{g} / \mathbb{Z}^{g}} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau)^{8} \right)^{2}.$$

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Exercise. Check if there exist \mathcal{A}_4 -geodesics lying in \mathcal{M}_4 .

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Open Problem

Construct all geodesics for the metric on \mathcal{A}_4 contained in $\overline{\mathcal{M}_4}$.

$$\sum \theta_m^{16}(\tau) = \theta_{D_{16}^+}(\tau), \qquad \sum \theta_m^8(\tau) = \theta_{E_8}(\tau).$$

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Physics conjecture (Belavin, Knizhnik, D'Hoker-Phong, ...)

The ... SO(32) ... type ... superstring theory ... and ... $E_8 \times E_8$ theory ...

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$$F_{g} := 2^{g} \sum \theta_{m}^{16} - \left(\sum \theta_{m}^{8}\right)^{2}$$

vanishes on \mathcal{J}_g for any g (this is true for $g \leq 4$).

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Theorem (G.-Salvati Manni) This conjecture is false for any $g \ge 5$.

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Theorem (G.-Salvati Manni)

This conjecture is false for any $g \ge 5$. In fact the zero locus of F_5 on \mathcal{M}_5 is the divisor of trigonal curves.

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Corollary

 $Th(\mathcal{J}_g(2,4)) \subset Th(\mathcal{A}_g(2,4))$ is not a complete intersection for g = 5, 6, 7. (previously proven by Faber)

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Challenge

Write at least one modular form vanishing on \mathcal{J}_5 .

g	deg $Th(\mathcal{J}_g(2,4))$	deg $Th(\mathcal{A}_g(2,4))$
1	1	1
2	1	1
3	16	16
4	208896	13056
5	282654670848	1234714624
6	23303354757572198400	25653961176383488
7	87534047502300588892024209408	197972857997555419746140160

Corollary

 $Th(\mathcal{J}_g(2,4)) \subset Th(\mathcal{A}_g(2,4))$ is not a complete intersection for g = 5, 6, 7. (previously proven by Faber)

Challenge

Write at least one (nice/invariant) modular form vanishing on \mathcal{J}_5 .

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The hyperelliptic Schottky problem

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The hyperelliptic Schottky problem

Theorem (Mumford, Poor)

For any g there exist sets of characteristics $S_1, \ldots, S_N \subset \frac{1}{2} \mathbb{Z}^{2g} / \mathbb{Z}^{2g}$ such that $\tau \in \mathcal{A}_g$ is the period matrix of a hyperelliptic Jacobian ($\tau \in Hyp_g$) if and only if for some $1 \leq i \leq N$

$$\forall m \qquad \{\theta_m(\tau) = 0 \Longleftrightarrow m \in S_i\}$$

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Definition

The Prym variety for an étale double cover $\tilde{C} \to C$ of $C \in \mathcal{M}_g$ (given by a point $\eta \in Jac(C)[2] \setminus \{0\}$) is

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Theorem (Schottky-Jung, Farkas-Rauch proportionality) Let τ be the period matrix of C and let π be the period matrix of the Prym (for the simplest choice of η). Then

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\pi)^2 = \operatorname{const} \theta \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix} (\tau) \cdot \theta \begin{bmatrix} 0 \\ 1 \\ \delta \end{bmatrix} (\tau) \quad \forall \varepsilon, \delta \in \frac{1}{2} \mathbb{Z}^{g-1} / \mathbb{Z}^{g-1}$$

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Using this allows us to get some equations for $Th(\mathcal{J}_g(2,4))$ from equations for $Th(\mathcal{A}_{g-1}(2,4))$.

Theorem (van Geemen)

The locus \mathcal{J}_g is an irreducible component of the Schottky-Jung locus — the locus obtained by taking the ideal of equations defining $Th(\mathcal{A}_{g-1}(2,4))$ and applying the proportionality for all double covers η .

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Conjecture

 \mathcal{J}_5 is equal to the Schottky-Jung locus in genus 5.

Theorem (van Geemen / Donagi)

The locus \mathcal{J}_g is an irreducible component of the small / big Schottky-Jung locus — the locus obtained by taking the ideal of equations defining $Th(\mathcal{A}_{g-1}(2,4))$ and applying the proportionality for all / for just one double cover(s) η .

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• The locus of intermediate Jacobians of cubic threefolds is contained in the "big" (if we take just one η) Schottky-Jung locus in genus 5.

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- The locus of intermediate Jacobians of cubic threefolds is contained in the "big" (if we take just one η) Schottky-Jung locus in genus 5.
- For g ≥ 7, P_{g-1} ⊊ A_{g-1}, so may have more equations ⇒ need to solve the Prym Schottky problem if the above is not enough.

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Equations for theta constants: recap

- + We get explicit algebraic equations for theta constants. We do get the one defining equation for \mathcal{J}_4 .
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Boundary degeneration of Pryms is hard
[Alexeev-Birkenhake-Hulek]
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 $\begin{array}{ll} \mbox{For } C \in Hyp_g & \mbox{have } \dim(\mbox{Sing } \Theta_{Jac(C)}) = g - 3. \\ \mbox{For } C \in \mathcal{M}_g \setminus Hyp_g & \mbox{have } \dim(\mbox{Sing } \Theta_{Jac(C)}) = g - 4. \\ \mbox{(By Riemann's theta singularity theorem)} \end{array}$

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Definition (Andreotti-Mayer loci)

$$N_k := \left\{ (A, \Theta) \in \mathcal{A}_g \mid \dim \operatorname{Sing} \Theta \geq k
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Theorem (Debarre)

$$\mathcal{P}_{g}$$
 is an irreducible component of N_{g-6} .

•
$$N_0 \subsetneq \mathcal{A}_g$$

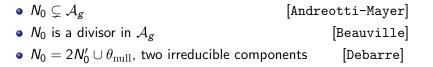
[Andreotti-Mayer]

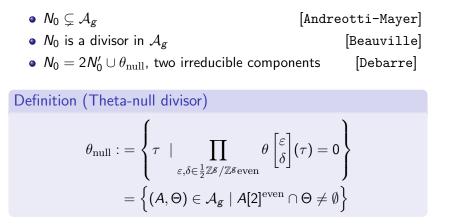
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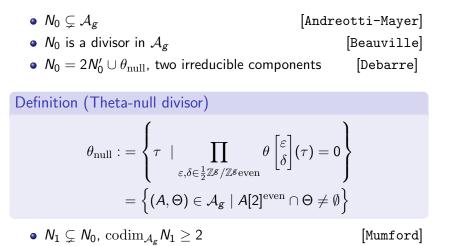
- $N_0 \subsetneq \mathcal{A}_g$
- N_0 is a divisor in \mathcal{A}_g

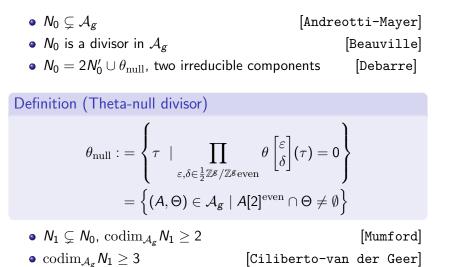
[Andreotti-Mayer] [Beauville]

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Thus interested in $\mathcal{J}_g \cap \theta_{\text{null}}$. For genus 4 have $N_0 = \mathcal{J}_4 \cup \theta_{\text{null}}$, so $\mathcal{J}_4 \setminus \theta_{\text{null}} = N_0 \setminus \theta_{\text{null}}$.

Conjecture (Beauville, Debarre, ...)

$$N_{g-3} = Hyp_g; \quad N_{g-4} \setminus \mathcal{J}_g \subset heta_{\mathrm{null}} \quad ext{ within } \mathcal{A}_g^{indec}$$

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Conjecture (H. Farkas) Theorem (G.-Salvati Manni)

$$\mathcal{J}_4 \cap \theta_{\mathrm{null}} = \left\{ \exists m \in A[2]^{\mathrm{even}} \ \theta(\tau, m) = \det_{i,j} \partial_{z_i} \partial_{z_j} \theta(\tau, m) = 0 \right\}$$

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Theorem (G.-Salvati Manni, Smith-Varley + Debarre) $(\mathcal{J}_g \cap \theta_{\text{null}}) \subset \theta_{\text{null}}^3 \subset \theta_{\text{null}}^{g-1} \subset (\theta_{\text{null}} \cap N'_0) \subset \text{Sing } N_0$

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 $\begin{array}{ll} \text{Note that} & \Theta_{A_1 \times A_2} = (\Theta_{A_1} \times A_2) \cup (A_1 \times \Theta_{A_2}).\\ \text{Thus} & \text{Sing}(\Theta_{A_1 \times A_2}) \supset \Theta_{A_1} \times \Theta_{A_2}. \end{array}$

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$$N_{g-2} = \mathcal{A}_g^{decomposable}$$

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Question

Is it possible that $N_k = N_{k+1}$ for some k?

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Theorem (Kollár)

For any $(A, \Theta) \in \mathcal{A}_g$, any $z \in A$ we have $\operatorname{mult}_z \Theta \leq g$.

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• Thus the conjecture is true for $g \leq 5$ $(\overline{\mathcal{P}_5} = \mathcal{A}_5)$

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+ Geometric conditions for an abelian variety to be a Jacobian.
 Geometric solution in genus 4.

- dim Sing Θ_{τ} hard to compute for an explicitly given $\tau \in \mathcal{H}_g$. Only a weak solution (at least so far) in higher genera.

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Here we start with a curve and solve an easier version of Schottky: Given $C \subset A$, is A = Jac(C)?

Definition The Kummer variety is the image of $Kum := |2\Theta| : A_{\tau} / \pm 1 \hookrightarrow \mathbb{P}^{2^g - 1}$ $z \to \left\{ \Theta[\varepsilon](\tau, z) \right\}_{\mathrm{all} \, \varepsilon \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g},$

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Trisecant formula (Fay, Gunning) $\forall p, p_1, p_2, p_3 \in C \subset Jac(C) = Pic^0(C)$ the following are collinear: $Kum(p+p_1-p_2-p_3), Kum(p+p_2-p_1-p_3), Kum(p+p_3-p_1-p_2)$ (*)

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Theorem (Gunning)

If for some $A \in \mathcal{A}_g^{indec}$ there exist infinitely many p such that (*) (p_i fixed, in general position), then $A \in \mathcal{J}_g$.

Definition The Kummer variety is the image of $Kum := |2\Theta| : A_{\tau} / \pm 1 \hookrightarrow \mathbb{P}^{2^g - 1}$ $z \to \left\{ \Theta[\varepsilon](\tau, z) \right\}_{\mathrm{all} \, \varepsilon \in \frac{1}{2} \mathbb{Z}^g / \mathbb{Z}^g},$

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This is a solution to the Schottky problem, already given a curve.

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 $\begin{array}{l} \text{Conjecture (Buchstaber-Krichever)}\\ \text{Theorem (G., Pareschi-Popa)}\\ \\ \text{Given } A \in \mathcal{A}_g^{indec} \text{ and } p_1, \ldots, p_{g+2} \in A \text{ in general position, if}\\ \\ \forall z \in A \qquad \{Kum(2p_i+z)\}_{i=1\ldots g+2} \subset \mathbb{P}^{2^g-1} \end{array}$

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implies that A is a Jacobian.

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So what would be the Prym analog of the trisecant conjecture?

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For $A \in \mathcal{A}_{g}^{indec}$ and $p, p_{1}, p_{2}, p_{3} \in A$, if (**), and (**) also holds for $-p, p_{1}, p_{2}, p_{3}$, then $A \in \overline{\mathcal{P}_{g}}$. (Characterization by a symmetric pair of quadrisecants)

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Challenge

Use these characterizations to approach Coleman's conjecture, or solve the Torelli problem for Pryms (period map generically injective — conjecturally the non-injectivity is due only to the tetragonal construction), or ...

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+ Gives a way to tell that some abelian varieties are *not* Jacobians. - Does not possibly give a way to show that a given abelian variety *is* a Jacobian, or does it?

Can characterize Hyp_g by the value of their Seshadri constant, if the Γ_{00} conjecture holds [Debarre, Lazarsfeld]

Γ_{00} conjecture

Definition

$$\Gamma_{00} = \{ f \in H^0(A, 2\Theta) \mid mult_0 f \ge 4 \}$$

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- Holds for a generic Prym for $g \ge 8$. [Izadi]
- Holds for a generic abelian variety for g = 5 or $g \ge 14$. [Beauville-Debarre-Donagi-van der Geer]

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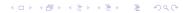
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Idea (Muñoz-Porras)

If Γ_{00} conjecture holds, then \mathcal{J}_g =small Schottky-Jung locus (methods to prove this by degenerating to the boundary).

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Dubrovin, Krichever, Novikov, Arbarello, De Concini, Shiota, Mulase, Marini, Muñoz Porras, Plaza Martin, ...

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- For Pryms, G.-Krichever needed a new hierarchy, etc.