MAT 142
Take home final

due Monday, May 16, 2005

This is part of your final exam, and should be treated as such. You may refer to your textbook and any notes that you have taken in class. However, you may not consult any outside sources. You may not look up answers in any other book or online. You may not discuss the content of the exam with anyone, including your classmates.

1. A sequence, a_k, of real numbers is said to be *Cauchy* if for every $\epsilon > 0$ there exists an N so that $|a_m - a_n| < \epsilon$ for all $m, n > N$. Intuitively, a sequence is Cauchy if, as you go farther out in the sequence, the numbers get closer and closer to one another.
   (a) Prove that if a sequence converges, then it is Cauchy.
   (b) Prove that if a sequence is Cauchy, then it converges.
2. Let $f(x)$ be a differentiable function such that $\lim_{x\to\infty} \frac{f(x)}{e^x} = 0$. The *Laplace Transform* of f is the function, $\mathcal{L}(f)$, defined by

$$\mathcal{L}(f)(s) = \int_0^\infty f(x)e^{-xs}\,dx$$

   (a) Compute the Laplace transforms of the functions $f(x) = x$ and $g(x) = \sin x$.
   (b) Prove that $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$. What is $\mathcal{L}(f'')(s)$?
   (c) The utility of the Laplace transform is that it converts differential equations into algebraic equations. Let $f(x)$ be the solution to the differential equation $y'' + 2y' + 5y = x$ with the initial conditions $f(0) = 4$ and $f'(0) = 2$. Show that $\mathcal{L}(f)(s) = \frac{4s^3+10s^2+1}{s^2(s^2+2s+5)}$. (Hint: You do *not* need to solve the differential equation.)
3. A *Fourier series* is an infinte series of trigonometric functions. In particular, we will consider the Fourier sine series

$$\sum_{n=0}^{\infty} a_n \sin nx$$

   where the a_n's are constants.
   (a) Prove that for any integers, m and n,

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

   (b) Prove that for any integers, m and n,

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

   (c) Prove that if $\sum_{n=0}^{\infty} a_n \sin nx$ converges uniformly to $f(x)$, then the coefficients must be $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

(d) Although there does not exist a Fourier series which converges to $f(x) = x$ for all values of x, there does exist one which converges for $x \in (-\pi, \pi)$. Find it.

4. Compute the following

(a) $\int \sqrt{1+x^2}\, dx$

(b) $\int \ln \sqrt{1+x^2}\, dx$

(c) $\int \frac{dx}{2+\tan x}$

(d) $\int e^{\sqrt{x}}\, dx$