The Morse-Witten complex via dynamical systems

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Abstract: Given a smooth closed manifold M, the Morse-Witten complex associated to a Morse function f and a Riemannian metric g on M consists of chain groups generated by the critical points of f and a boundary operator counting isolated flow lines of the negative gradient flow. Its homology reproduces singular homology of M. The geometric approach presented here was developed in [We93] and is based on tools from hyperbolic dynamical systems. For instance, we apply the Grobman-Hartman theorem and the λ-lemma (Inclination Lemma) to analyze compactness and define gluing for the moduli space of flow lines.
Keywords: Morse homology, Morse theory, hyperbolic dynamical systems.

Contents

1 Introduction

Throughout let M be a smooth closed[1] manifold of finite dimension n and f a smooth function on M. Assume that all critical points of f are nondegenerate, denote by $\mathrm{Crit}_k f$ those of Morse index k and let c_k be their total number (for definitions see Section 2.1). Denoting the k^{th} *Betti number*[2] *of* M by $b_k = b_k(M;\mathbb{Z})$, the *strong Morse inequalities* are given by

$$\begin{aligned} &c_k - c_{k-1} + \cdots \pm c_0 \geq b_k - b_{k-1} + \cdots \pm b_0, \quad k = 0, \ldots, n-1, \\ &c_n - c_{n-1} + \cdots \pm c_0 = b_n - b_{n-1} + \cdots \pm b_0. \end{aligned} \tag{1}$$

Consider the free abelian groups $\mathrm{CM}_k := \mathbb{Z}^{\mathrm{Crit}_k f}$, for $k = 0, \ldots, n$. It is well known that the strong Morse inequalities are equivalent to the existence of boundary homomorphisms $\partial_k : \mathrm{CM}_k \to \mathrm{CM}_{k-1}$ whose homology groups are of rank b_k; see e.g. [Th49, Sm60, Mi65, Fr79].

In 1982 Witten [Wi82] brought to light a geometric realization of such a boundary operator ∂_k. Choosing as auxiliary data a (generic) Riemannian metric g on M, he looked at the negative gradient flow associated to (f,g). Given $x \in \mathrm{Crit}_k f$ and $y \in \mathrm{Crit}_{k-1} f$, there are only finitely many so called *isolated* flow lines running from x to y. Choosing orientations of all unstable manifolds one can associate a characteristic sign $n_u \in \{\pm 1\}$ to every isolated flow line u. Witten defined the boundary operator ∂_k on x by counting all isolated flow lines with signs emanating from x.

To simplify matters one can ignore the signs by taking $\mathbb{Z}_2$-coefficients and counting modulo two. Here is a first example.

[1] Compact and without boundary.
[2] Rank of $\mathrm{H}_k(M;\mathbb{Z})$ = cardinality of a basis of its free part.

Example 1.1. Consider the manifold shown in Figure 1 (for now ignore the gray arrows indicating orientations). The manifold is supposed to be

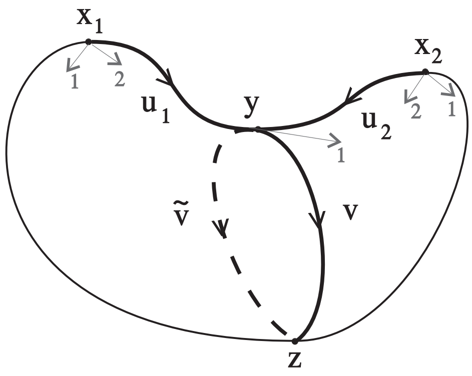

Figure 1: Deformed S^2 embedded in $\mathbb{R}^3$

embedded in $\mathbb{R}^3$ and the function f is given by measuring height with respect to the horizontal coordinate plane. The metric is induced by the euclidean metric on the ambient space $\mathbb{R}^3$. There are four critical points x_1, x_2, y, z with Morse indices $2, 2, 1, 0$, respectively. The four isolated flow lines satisfy

$$\begin{aligned} \partial_2(x_1 + x_2) &= 0 \pmod 2, \\ \partial_1 y &= 0 \pmod 2, \qquad\qquad y = \partial_2 x_1 = \partial_2 x_2, \\ \partial_0 z &= 0, \end{aligned}$$

and the resulting homology groups

$$\mathrm{HM}_2 = \langle x_1 + x_2 \rangle = \mathbb{Z}_2, \qquad \mathrm{HM}_1 = 0, \qquad \mathrm{HM}_0 = \langle z \rangle = \mathbb{Z}_2,$$

are equal to singular homology with $\mathbb{Z}_2$-coefficients of the 2-sphere.

In the early 90's several approaches towards rigorously setting up Witten's complex[3] and the resulting *Morse homology theory* emerged. The approach by Floer [Fl89] and Salamon [Sa90] is via Conley index theory. The one taken by Schwarz [Sch93] is to consider the negative gradient equation in the spirit of Floer theory as a section in an appropriate Banach bundle over the set of paths in M (see also [Sa90] for partial results). A third approach from

[3] In [Wi82] there is also another definition via a deformed deRham complex. In 1985 Helffer and Sjöstrand [HS85] gave a rigorous treatment using semiclassical analysis.

a dynamical systems point of view, namely via intersections of unstable and stable manifolds, was taken by the present author. In [We93] we applied the Grobman-Hartman theorem and the λ-lemma to set up the Morse-Witten complex. Poźniak's work on the more general Novikov complex carries elements of the second and third approach and we shall present his definition of the continuation maps in Section 4.2.

Writing the present paper was motivated by recent developments. *Although* unpublished, the dynamical systems methods developed in [We93] proved useful in the work of Ludwig [Lu03] on stratified Morse theory. *Because* unpublished, they were rediscovered independently by Jost [Jo02] and – in the far more general context of Hilbert manifolds – by Abbondandolo and Majer (see [AM04] and references therein).

All results in this paper are part of mathematical folklore, unless indicated differently. In fact, the proofs in Sections 3.2–3.3 are due to the author [We93] and the ones in Section 4.2 to Poźniak [Po91].

This paper is organized as follows: Section 2 recalls relevant elements of Morse theory and negative gradient flows. Section 3 is at the heart of the matter where we introduce and analyze the moduli spaces of connecting flow lines. Morse-Smale transversality assures that they are manifolds. Then we show how to compactify, glue and orient them consistently using tools from dynamical systems (for which our main reference will be the excellent textbook of Palis and de Melo [PM82]). In Section 4 we define the Morse-Witten complex, investigate its dependence on (f,g) and arrive at the theorem equating its homology to singular homology. Finally we provide some remarks concerning Morse-inequalities, relative Morse homology, Morse cohomology, and Poincaré duality (for details we refer the reader to the original source [Sch93]). We conclude by computing Morse homology and cohomology of real projective space $\mathbb{R}\mathrm{P}^2$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$.

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2 Morse theory

2.1 Critical points

Given a smooth function $f : M \to \mathbb{R}$ consider the set of its critical points

$$\mathrm{Crit} f := \{x \in M \,|\, df(x) = 0\}.$$

Near $x \in \mathrm{Crit} f$ choose local coordinates $\varphi = (u^1, \dots, u^n) : U \subset M \to \mathbb{R}^n$ and define a symmetric bilinear form on T_xM, the *Hessian of f at x*, by

$$H_x^f(\xi, \eta) := \sum_{i,j=1}^{n} S_{ij}\, \xi^i \eta^j, \qquad S_{ij} = S_{ij}(f, x; \varphi) := \frac{\partial^2 f}{\partial u^i \partial u^j}(x).$$

The symmetric matrix $S(f, x; \varphi) := (S_{ij})_{i,j=1}^n$ is the *Hessian matrix* and the number of its negative eigenvalues $\mathrm{ind}_f(x)$ is called the *Morse index of x*. If $S(f, x; \varphi)$ is nonsingular, we say that x is a *nondegenerate* critical point. It is an exercise to check that the notions of Hessian, Morse index and nondegeneracy do not depend on the choice of local coordinates as long as $x \in \mathrm{Crit} f$. (Hint: Use $df(x) = 0$ to show that for another choice of coordinates $(\tilde{u}^1, \dots, \tilde{u}^n)$ the matrix S transforms according to $\tilde{S} = T^t S T$, where T denotes the derivative of the coordinate transition map. To see that the Morse index is well defined apply Sylvester's law; see e.g. [La95, Ch. XV Thm. 4.1]).

Lemma 2.1. *Every nondegenerate critical point x of f is isolated.*

Proof. Choose local coordinates (φ, U) near x as above. Consider the map

$$F := \left(\frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^n}\right) : U \to \mathbb{R}^n$$

whose zeroes correspond precisely to the critical points of f in U. In particular $F(x) = 0$. It remains to show that there are no other zeroes nearby. Since the derivative of F at x equals $S(f, x; \varphi)$, it is an isomorphism. Therefore F is locally near x a diffeomorphism by the inverse function theorem. $\square$

Definition 2.2. A smooth function $f : M \to \mathbb{R}$ is called *Morse* if all its critical points are nondegenerate.

Corollary 2.3. *If M is closed and f is a Morse function, then $\#\mathrm{Crit} f < \infty$.*

Proof. Assume not and let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence of pairwise distinct critical points of f. By compactness of M, there is a convergent subsequence with limit, say x. By continuity of df, the limit x is again a critical point. This contradicts Lemma 2.1. $\square$

Viewing df as a section of the cotangent bundle T^*M, the nondegeneracy of a critical point x is equivalent to the transversality of the intersection of the two closed submanifolds M and graph df at x. The intersection is compact and, in the Morse case, also discrete (complementary dimensions). This reproves Corollary 2.3. Transversality is a generic property (also open in the case of closed submanifolds) and so this point of view is appropriate to prove the following theorem (see e.g. [Hi76]).

Theorem 2.4. *If M is closed, then the set of Morse functions is open and dense in $C^\infty(M,\mathbb{R})$.*

2.2 Gradient flows and (un)stable manifolds

Let X be a smooth vector field on M. For $q \in M$ consider the initial value problem for smooth curves $\gamma : \mathbb{R} \to M$ given by

$$\dot\gamma(t) = X(\gamma(t)), \qquad \gamma(0) = q. \tag{2}$$

Because M is closed, the solution $\gamma = \gamma_q$ exists for all $t \in \mathbb{R}$. It is called the *trajectory or flow line through q*. The *flow generated by X* is the smooth map $\phi : \mathbb{R} \times M \to M$, $(t,q) \mapsto \gamma_q(t)$. For every $t \in \mathbb{R}$, it gives rise to the diffeomorphism $\phi_t : M \to M, q \mapsto \phi(t,q)$, the so called *time-$t$-map*. The family of time-t-maps satisfies $\phi_{t+s} = \phi_t\phi_s$ and $\phi_0 = id$, i.e. it is a one-parameter group of diffeomorphisms of M.

The *orbit $\mathcal{O}(q)$ through $q \in M$* is defined by $\phi_{\mathbb{R}}q := \{\phi_t q \mid t \in \mathbb{R}\}$. There are three types of orbits, namely singular, closed and regular ones. A *singular* orbit is one which consists of a single point q (which is necessarily a singularity of X). An orbit is called *closed* if there exists $T \neq 0$ such that $\phi_T q = q$ and $\phi_T q \neq q$ whenever $t \in (0,T)$. In this case T is called the *period* of the orbit. Nonsingular and nonclosed orbits are called *regular*. They are injective immersions of $\mathbb{R}$ into M. Hence it is natural to ask if they admit limit points at their ends. For $q \in M$, define its α- and *ω-limit* by

$$\begin{aligned}\alpha(q) &:= \{p \in M \mid \ \phi_{t_k} q \to p \text{ for some sequence } t_k \to -\infty\},\\ \omega(q) &:= \{p \in M \mid \ \phi_{t_k} q \to p \text{ for some sequence } t_k \to \infty\}.\end{aligned}$$

The α-limit of q is the ω-limit of q for the vector field $-X$. Hence the properties of α translate into those of ω and vice versa. Because $\omega(q) = \omega(\tilde{q})$, whenever $\tilde{q}$ belongs to the orbit through q, it makes sense to define $\omega(\mathcal{O}(q)) := \omega(q)$. It is a consequence of closedness of M that $\alpha(q)$ and $\omega(q)$ are nonempty, closed, connected and invariant by the flow (i.e. a union of orbits); see e.g. [PM82, Ch. 1 Prop. 1.4].

Let us now restrict to the case of *gradient flows*, which exhibit a number of key features. Let g be a Riemannian metric and f a smooth function on M. The identity $g(\nabla f, \cdot) = df(\cdot)$ uniquely determines the *gradient vector field* ∇f. The flow associated to $X = -\nabla f$ is called *negative gradient flow*. If γ is a trajectory of the negative gradient flow, then

$$\frac{d}{dt} f \circ \gamma(t) = g(\nabla f(\gamma(t)), \dot{\gamma}(t)) = -\left|\nabla f(\gamma(t))\right|^2 \leq 0, \qquad \forall t \in \mathbb{R}.$$

This shows that f is strictly decreasing along nonsingular orbits. Therefore closed orbits cannot exist and any regular orbit $\mathcal{O}(q)$ intersects a level set $f^{-1}(f(q))$ at most once. Moreover, such an intersection is orthogonal with respect to g. Using these properties one can show that $\alpha(q) \cup \omega(q) \subset \mathrm{Crit} f$; see e.g. [PM82, Ch. 1 Expl. 3]. The idea of proof is to assume by contradiction that there exists $p \in \omega(q)$ with $X(p) \neq 0$. Hence there exists a sequence $q_k \in \mathcal{O}(q)$ converging to p and $f^{-1}(f(p))$ is locally near p a codimension one submanifold orthogonal to X. Then, by continuity of the flow, the orbit through q intersects the level set in infinitely many points, which cannot be true. Example 3 in [PM82, Ch. 1] shows that $\omega(q)$ may indeed contain more than one critical point. However, in this case it must contain infinitely many by connectedness of $\omega(q)$. Hence Corollary 2.3 implies Lemma 2.6 below.
The composition of the linearization of $-\nabla f$ at a singularity x with the projection onto the second factor defines the linear operator

$$-D\nabla f(x) : T_x M \xrightarrow{-d\nabla f(x)} T_{\nabla f(x)} TM \simeq T_x M \oplus T_x M \xrightarrow{pr_2} T_x M.$$

With respect to geodesic normal coordinates (φ, U) near x the operator $D\nabla f(x)$ is represented by the Hessian matrix $S(f, x; \varphi)$. Moreover, these coordinates are convenient to prove the identity

$$H^f_x(\xi, \eta) = g(D\nabla f(x)\, \xi, \eta), \qquad \forall \xi, \eta \in T_x M.$$

Hence $D\nabla f(x)$ is a symmetric operator and the number of its negative eigenvalues coincides with $k := \mathrm{ind}_f(x)$. Let E^u denote the sum of eigenspaces corresponding to negative eigenvalues and similarly define E^s with respect to positive eigenvalues. The superscripts abbreviate *unstable* and *stable* and this terminology arises as follows. The time-t-map associated to the linear vector field $-D\nabla f(x)$ on T_xM is given by the symmetric linear operator $A_t := exp(-tD\nabla f(x))$ on T_xM. Moreover, if λ is an eigenvalue of $D\nabla f(x)$, then $e^{-t\lambda}$ is eigenvalue of A_t and the eigenspaces are the same. This shows that A_t leaves the subspaces E^u and E^s invariant and acts on them strictly expanding and contracting, respectively.

Lemma 2.5. *For $f \in C^\infty(M,\mathbb{R})$, let ϕ_t be the time-t-map generated by $X = -\nabla f$. If $x \in \mathrm{Crit} f$, then*

$$d\phi_t(x) = exp(-tD\nabla f(x)).$$

Proof. The map $d\phi_t(x)$ coincides with A_t, because it satisfies the two characterizing identities for the time-t-map associated to $-D\nabla f(x)$: pick $\xi \in T_xM$ and let c be a smooth curve in M satisfying $c(0) = x$ and $c'(0) = \xi$. Then use $\phi_0 = id$, $\partial_t\phi_t = -\nabla f(\phi_t)$, and $\phi_t(x) = x$ to obtain

$$\begin{aligned} d\phi_0(x)\xi &= \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \phi_0(c(\tau)) = \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} c(\tau) = \xi \\ \frac{\partial}{\partial t} d\phi_t(x)\xi &= \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \frac{\partial}{\partial t}\phi_t(c(\tau)) = -D\nabla f(x) \circ d\phi_t(x)\xi. \end{aligned}$$

□

From now on we shall assume in addition that the negative gradient flow is generated by a *Morse function.* Hence, as observed above, the following lemma is a consequence of Corollary 2.3.

Lemma 2.6. *Let M be closed and $X = -\nabla f$, where f is Morse. Then $\alpha(q)$ and $\omega(q)$ consist each of a single critical point of f, for every $q \in M$.*

The *stable* and the *unstable manifold of* $x \in \mathrm{Crit} f$ are defined by

$$W^s(x) := \{q \in M \mid \omega(q) = x\}, \qquad W^u(x) := \{q \in M \mid \alpha(q) = x\}.$$

Lemma 2.6 shows $\omega(q) = \lim_{t\to\infty} \phi_t q$. The map $H : [0,1] \times W^s(x) \to W^s(x)$, $(\tau, q) \mapsto \phi_{t/(1-t)} q$, provides a homotopy between the identity map on $W^s(x)$ and the constant map $q \mapsto x$. Hence the (un)stable manifolds are contractible

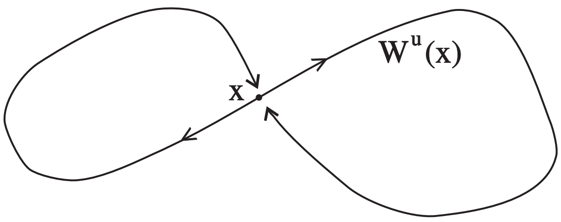

Figure 2: Unstable manifold injectively immersed, but not embedded

sets. Whereas for general vector fields X with hyperbolic singularity x these sets are only injectively immersed (Figure 2), they are embedded in the Morse case. The reason is, roughly speaking, that the stable manifold cannot return to itself, since f is strictly decreasing along regular orbits.

Theorem 2.7 (Stable manifold theorem). *Let f be a Morse function and $x \in \mathrm{Crit} f$. Then $W^s(x)$ is a submanifold of M without boundary and its tangent space at x is given by the stable subspace $E^s \subset T_xM$.*

The theorem holds for $W^u(x)$ with tangent space E^u (replace f by $-f$).

Proof. 1) We sketch the proof that $W^s(x)$ is locally near x the graph of a smooth map and E^s is its tangent space at x (see e.g. [PM82, Ch. 2 Thm. 6.2] for a general hyperbolic flow and [Jo02, Thm. 6.3.1] for full details in our case). Then, using the flow, it follows that $W^s(x)$ is injectively immersed. In local coordinates (φ, U) around x the initial value problem (2) is given by

$$\dot\gamma = A\gamma + h(\gamma), \qquad \gamma(0) = \gamma_0. \tag{3}$$

Here $\varphi(x) = 0$, the linear map A represents $-D\nabla f(0)$ in these local coordinates, and h satisfies $h(0) = 0$ and $dh(0) = 0$. Consider the splitting $\mathbb{R}^n = E^u \oplus E^s$ induced by $A = (A^u, A^s)$, fix a metric on $\mathbb{R}^n$ *compatible with the splitting*[4] (exists by [PM82, Ch. 2, Prop. 2.10]), and let $P^u : \mathbb{R}^n \to E^u$ and $P^s : \mathbb{R}^n \to E^s$ denote the orthogonal projections. For $\delta, \mu > 0$ define

$$C^0_{\delta,\mu} := \left\{ \gamma \in C^0([0,\infty), \mathbb{R}^n) \,\middle|\, \sup_{t\ge 0} e^{\delta t} \left|\gamma(t)\right| \le \mu \right\}.$$

If $\delta, \mu > 0$ are sufficiently small, then the map $\mathcal{F} : E^s \times C^0_{\delta,\mu} \to C^0_{\delta,\mu}$ given by

$$\left(\mathcal{F}(\gamma_0^s, \gamma)\right)(t) := e^{tA}\gamma_0^s + \int_0^t e^{(t-\tau)A} P^s h(\gamma(\tau))\, d\tau - \int_t^\infty e^{(t-\tau)A} P^u h(\gamma(\tau))\, d\tau$$

[4]This means $\|A^s\| < 1$ and $\|A^u\| > 1$.

is a strict contraction in γ, whenever $|\gamma_0^s|$ is sufficiently small. The key fact is that the unique fixed point $\hat{\gamma}$ of $\mathcal{F}(\gamma_0^s,\cdot)$ is precisely the unique solution of (3) such that $P^s\hat{\gamma}(0)=\gamma_0^s$. Consequently this solution converges exponentially fast to zero as $t\to\infty$. Define the desired graph map $E^s\to E^u$ locally near zero by

$$\gamma_0^s \mapsto -\int_0^\infty e^{-\tau A}P^u h(\hat{\gamma}(\tau))\,d\tau.$$

2) It remains to prove that $W^s(x)$ is an embedding. Assume that its codimension is at least one, otherwise we are done. An immersion is locally an embedding: there exists an open neighbourhood W of x in $W^s(x)$ which is a submanifold of M of dimension $\ell := \dim E^s$. Let $\mu := \min_{q\in\partial W} f(q) - f(x)$, then $\mu>0$ and $f|_{W^s(x)\setminus W}\geq f(x)+\mu$. Denote by B_ε the open ε-neighbourhood of x with respect to the Riemannian distance on M. For $\varepsilon>0$ sufficiently small, it holds $f|_{B_\varepsilon}<f(x)+\mu/2$ and therefore

$$B_\varepsilon\cap(W^s(x)\setminus W)=\emptyset. \tag{4}$$

The goal is to construct smooth submanifold coordinate charts for every $p\in W^s(x)$. Assume $p\in W^s(x)\setminus W$, otherwise we are done. Define the open neighbourhood $W_\varepsilon := W\cap B_\varepsilon$ of x in W. There exists $T>0$ such that $\phi_T p\in W_\varepsilon\subset W$. Choose a submanifold chart (φ,U) for W around $\phi_T p$. In particular, the set U is open in M, it contains $\phi_T p$ and $\varphi(U\cap W)=0\times V$. Here $V\subset\mathbb{R}^\ell$ is an open neighbourhood of 0. Shrinking U, if necessary, we may assume without loss of generality that a) $U\subset B_\varepsilon$ and b) $U\cap W=U\cap W_\varepsilon$. Condition a) and (4) imply $U\cap(W^s(x)\setminus W)=\emptyset$ and condition b) shows

$$U\cap(W\setminus W_\varepsilon)=U\cap W\cap(M\setminus W_\varepsilon)=U\cap W_\varepsilon\cap(M\setminus W_\varepsilon)=\emptyset.$$

Use these two facts and represent $W^s(x)$ in the form

$$W^s(x)=W_\varepsilon\cup(W\setminus W_\varepsilon)\cup(W^s(x)\setminus W)$$

to conclude ('no-return')

$$U\cap W^s(x)=U\cap W_\varepsilon. \tag{5}$$

Define the submanifold chart for $W^s(x)$ at p by $(\psi,U_p):=(\varphi\circ\phi_T,\phi_{-T}U)$. The set U_p is indeed an open neighbourhood of p in M and ψ satisfies

$$\begin{aligned}\psi^{-1}(0\times V)&=\phi_{-T}\circ\varphi^{-1}(0\times V)=\phi_{-T}(U\cap W)\\&=\phi_{-T}(U\cap W_\varepsilon)\qquad=\phi_{-T}(U\cap W^s(x))=U_p\cap W^s(x).\end{aligned}$$

The third equality follows by condition b) and equality four by (5). □

3 Spaces of connecting orbits

Given $x, y \in \mathrm{Crit} f$, define the *connecting manifold of x and y* by

$$\mathcal{M}_{xy} = \mathcal{M}_{xy}(f, g) := W^u(x) \cap W^s(y).$$

Let $a \in (f(y), f(x))$ be a regular value. The *space of connecting orbits from x to y* is defined by

$$\widehat{\mathcal{M}}_{xy} = \widehat{\mathcal{M}}_{xy}(f, g, a) := \mathcal{M}_{xy} \cap f^{-1}(a). \tag{6}$$

This set represents precisely the orbits of the negative gradient flow running from x to y, because every orbit intersects the level hypersurface exactly once. For two different choices of a there is a natural identification between the corresponding sets $\widehat{\mathcal{M}}_{xy}(f, g, a)$ which is provided by the flow.

The structure of this section is the following. In Subsection 3.1 we observe that it is possible to achieve transversality[5] of all intersections of stable and unstable manifolds simultaneously by an arbitrarily small C^1-perturbation of the gradient vector field within the set of gradient vector fields. Then the connecting manifolds and the spaces of connecting orbits are submanifolds of M without boundary and their dimensions are given by

$$\dim \mathcal{M}_{xy} = \mathrm{ind}_f(x) - \mathrm{ind}_f(y), \quad \dim \widehat{\mathcal{M}}_{xy} = \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1. \tag{7}$$

In Subsection 3.2 we investigate the structure of the topological boundary of the connecting manifolds and show how this leads to a natural compactification of the spaces of connecting orbits. In case of index difference $+1$ they are already compact, hence finite. The other important case is index difference $+2$. Here the connected components of $\widehat{\mathcal{M}}_{xz}$ are either diffeomorphic to S^1 or to $(0, 1)$. This dichotomy follows from the fact that these are the only two types of 1-dimensional manifolds without boundary. We shall see that to each end of an open component there corresponds a unique pair of connecting orbits $(u, v) \in \widehat{\mathcal{M}}_{xy} \times \widehat{\mathcal{M}}_{yz}$, where y is a critical point of intermediate index.

The main implication of Subsection 3.3 is that every such pair (u, v) corresponds to precisely one of the ends of all open components. The main

[5]Two submanifolds A and B of M are said to *intersect transversally* if

$$T_qA + T_qB = T_qM, \qquad \forall q \in A \cap B.$$

In this case $A \cap B$ is a submanifold of M whose codimension equals the sum of the codimensions of A and B; see e.g. [Hi76, Ch.1 Thm. 3.3].

ingredient is the so called gluing map. More precisely, we shall define a C^1-map which assigns to (u, v) and to a positive real parameter ρ a unique element of $\widehat{\mathcal{M}}_{xz}$. Moreover, the limit $\rho \to 0$ in the sense of Section 3.2 corresponds to the original pair (u, v).

In Subsection 3.4 we prove that a choice of orientations of all unstable manifolds induces orientations of the spaces of connecting orbits and that they are compatible with the gluing maps of Section 3.3.

3.1 Transversality

Definition 3.1. We say that a gradient vector field $\nabla^g f$ satisfies the *Morse-Smale condition* if $W^u(x)$ and $W^s(y)$ intersect transversally, for all $x, y \in \mathrm{Crit} f$. In this case (g, f) is called a *Morse-Smale pair.*

Here is an example which shows how the Morse-Smale condition can be achieved by an arbitrarily small perturbation of the Morse function.

Example 3.2. Consider a 2-torus T embedded upright in $\mathbb{R}^3$ as indicated in Figure 3 and let $f : T \to \mathbb{R}$ be given by measuring height with respect to the horizontal coordinate plane. This function admits four critical points M, s_1, s_2 and m of Morse indices $2, 1, 1, 0$, respectively. Let the metric on T be induced from the ambient euclidean space. The negative gradient flow is not Morse-Smale, because $W^u(s_1)$ and $W^s(s_2)$ do intersect and therefore the intersection cannot be transversal. However, Morse-Smale transversality can be achieved by slightly tilting the torus as indicated in Figure 4, in other words by perturbing f and thereby destroying the annoying flow lines between s_1 and s_2.

Theorem 3.3 (Morse-Smale transversality). *Let f be a smooth Morse function and g a smooth Riemannian metric on a closed manifold M. Then $\nabla^g f$ can be C^1 approximated by a smooth gradient vector field $X = \nabla^{\tilde{g}} \tilde{f}$ satisfying the Morse-Smale condition.*

Theorem 3.3 is due to Smale [Sm61]. Actually $\tilde{f}$ can be chosen such that its value at any critical point equals the Morse index. Note also that the metric $\tilde{g}$ is generally not close to g anymore. It is an exercise to check that if (g, f) is a Morse-Smale pair, then $\tilde{f}$ is necessarily Morse. The *type of a Morse-Smale vector field* $\nabla^g f$, or equivalently of the Morse function f, is by definition the number of critical points together with their Morse indices.

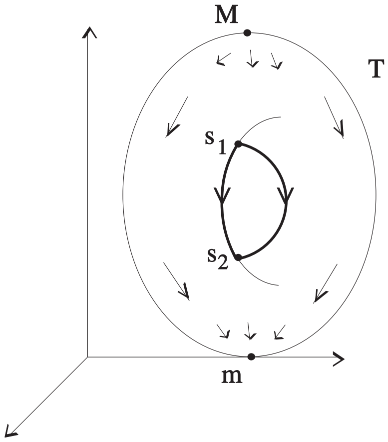

Figure 3: Not Morse-Smale

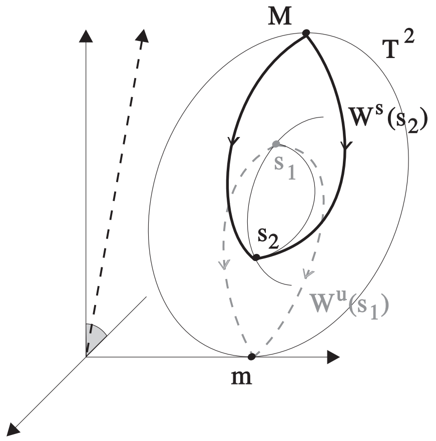

Figure 4: Morse-Smale

Morse-Smale vector fields are in particular hyperbolic vector fields and those have the property that their type is locally constant with respect to the C^1-topology. This can be seen by combining Proposition 3.1 and the corollary to Proposition 2.18 in Chapter 2 of [PM82], which in addition shows that the critical points of f and $\tilde{f}$ in Theorem 3.3 are C^0-close to each other. In fact, one can even keep the Morse function f fixed and achieve Morse-Smale transversality by perturbing only the metric (see [Sch93]).

Theorem 3.4. *If $-\nabla f$ is Morse-Smale, then all spaces $\mathcal{M}_{xy}$ and $\widehat{\mathcal{M}}_{xy}$ are submanifolds of M without boundary and their dimensions are given by (7).*

Lemma 3.5. *If $W^u(x)$ and $W^s(y)$ intersect transversally, then the following are true.*

1. *If $\operatorname{ind}_f(x) < \operatorname{ind}_f(y)$, then $\mathcal{M}_{xy} = \emptyset$.*
2. *$\mathcal{M}_{xx} = \{x\}$.*
3. *If $\operatorname{ind}_f(x) = \operatorname{ind}_f(y)$ and $x \neq y$, then $\mathcal{M}_{xy} = \emptyset$.*
4. *If $\mathcal{M}_{xy} \neq \emptyset$ and $x \neq y$, then $\operatorname{ind}_f(x) > \operatorname{ind}_f(y)$.*

Proof. 1. Transversality. 2. Any additional element must be noncritical and therefore gives rise to a closed orbit, which is impossible for gradient flows. 3. Assume the contrary, then $\mathcal{M}_{xy}$ contains a 1-dimensional submanifold of the form $\phi_{\mathbb{R}} q$, but $\dim \mathcal{M}_{xy} = 0$. 4. Assume the contrary and apply statements one and three of the lemma to obtain a contradiction. $\square$

3.2 Compactness

Assume throughout this section that $-\nabla f$ is Morse-Smale and $x, y \in \mathrm{Crit} f$. In case that a connecting manifold $\mathcal{M}_{xy}$ is noncompact we shall investigate the structure of its topological boundary as a subset of M. This gives rise to a canonical compactification of the associated orbit space $\widehat{\mathcal{M}}_{xy}$. In case of index difference $+1$, the manifold $\widehat{\mathcal{M}}_{xy}$ itself is already compact, hence a finite set. For self-indexing f this is easy to prove.

Proposition 3.6. *If* $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) = 1$, *then* $\#\widehat{\mathcal{M}}_{xy} < \infty$.

Proof. Assume that there is no critical value between $f(y)$ and $f(x)$. If $\mathcal{M}_{xy} \neq \emptyset$, fix $a \in (f(y), f(x))$ and let $\widehat{\mathcal{M}}_{xy} := \mathcal{M}_{xy} \cap f^{-1}(a)$. For $\varepsilon > 0$ sufficiently small define two closed sets

$$S^u := f^{-1}(f(x) - \varepsilon) \cap W^u(x), \qquad S^s := f^{-1}(f(y) + \varepsilon) \cap W^s(x).$$

Let them flow sufficiently long time, such that $f|_{\phi_T S^u} < a$ and $f|_{\phi_{-T} S^s} > a$ (here we use our assumption). Being the intersection of three closed sets, it follows that the set

$$\phi_{[0,T]} S^u \cap f^{-1}(a) \cap \phi_{[-T,0]} S^s$$

is closed. On the other hand, it coincides with the 0-dimensional submanifold $\widehat{\mathcal{M}}_{xy}$. A discrete closed subset of a compact set is finite. The general case follows from Theorem 3.8 below. $\square$

Definition 3.7. A subset $K \subset \widehat{\mathcal{M}}_{xy}$ is called *compact up to broken orbits*, if

$$\begin{aligned}
&\forall \text{ sequence } \{p_k\}_{k\in\mathbb{N}} \subset K,\\
&\exists \text{ critical points } x = x_0, x_1, \dots, x_\ell = y,\\
&\exists \text{ connecting orbits } u^j \in \widehat{\mathcal{M}}_{x_{j-1}x_j},\ j = 1, \dots, \ell,\\
&\text{such that } p_k \longrightarrow (u^1, \dots, u^\ell) \text{ as } k \to \infty.
\end{aligned} \tag{8}$$

Here convergence means, by definition, geometric convergence with respect to the Riemannian distance d on M of the orbits through p_k to the union of orbits through the u^j's. More precisely,

$$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N}, \forall k \geq k_0 : \ \mathcal{O}(p_k) \subset U_\varepsilon \left(\mathcal{O}(u^1) \cup \dots \cup \mathcal{O}(u^\ell)\right).$$

Here $U_\varepsilon(A)$ denotes the open ε-neighbourhood of a subset $A \subset M$. We say that the sequence p_k *converges to the broken orbit* $(u^1, \dots, u^\ell)$ *of order* ℓ.

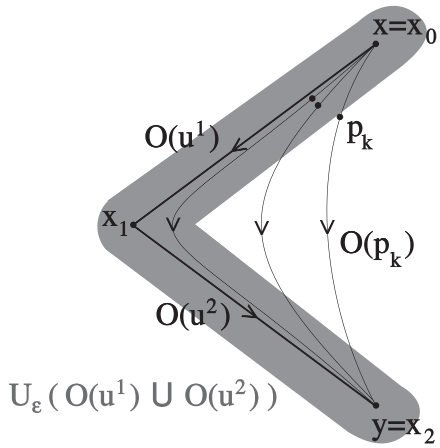

Figure 5: Convergence of sequence p_k to broken orbit (u^1, u^2) of order 2

Theorem 3.8 (Compactness). *If the Morse-Smale condition is satisfied, then the spaces of connecting orbits $\widehat{\mathcal{M}}_{xy}$ are compact up to broken orbits of order at most* $\mathrm{ind}_f(x) - \mathrm{ind}_f(y)$.

Proof. Fix a regular value a of f and define $\widehat{\mathcal{M}}_{xy} \subset f^{-1}(a)$ by (6). Assume $\mathrm{ind}_f(x) > \mathrm{ind}_f(y)$, otherwise $\widehat{\mathcal{M}}_{xy} = \emptyset$ by Lemma 3.5 and we are done. Given a sequence $\{p_k\}_{k\in\mathbb{N}} \subset \widehat{\mathcal{M}}_{xy} \subset f^{-1}(a)$, there exists a subsequence converging to some element u of the compact set $f^{-1}(a)$. We use the same notation for the subsequence. By Lemma 2.6 we have $u \in \mathcal{M}_{z'z}$, for some $z', z \in \mathrm{Crit} f$. By continuity of ϕ_t, it follows that $\phi_t u$ lies in the closure $cl(\mathcal{M}_{xy})$ of $\mathcal{M}_{xy}$ for every $t \in \mathbb{R}$, and therefore $z \in cl(\mathcal{M}_{xy})$. The proof proceeds in two steps.

STEP 1. *If $z \neq y$, then there exists $v \in W^u(z) \cap cl(\mathcal{M}_{xy})$ with $v \neq z$.*

The key tool is the Grobman-Hartman theorem for flows which states that the flows associated to $-\nabla f$ and $-D\nabla f(z)$, respectively, are locally conjugate. (See e.g. [PM82, Ch. 2 Thm. 4.10] where only locally equivalent is stated, but in fact locally conjugate is proved; see also [Rb95, Thm.5.3]). This means that there exist neighborhoods U_z of z in M and V_0 of 0 in T_zM, as well as a homeomorphism $h : U_z \to V_0$, such that

$$h(\phi_t q) = (D\phi_t(z) \circ h)(q), \tag{9}$$

for all (q, t) such that $\phi_t q \in U_z$ and $D\phi_t(z) \circ h(q) \in V_0$. Observe that h identifies a neighbourhood of z in $W^s(z)$ with one of zero in E^s; similarly for the unstable spaces. (If the eigenvalues of $-D\nabla f(z)$ satisfy certain nonresonance conditions, then h can be chosen to be a diffeomorphism; see [St58]). We may assume without loss of generality that u and the p_k are elements of U_z, otherwise apply ϕ_T with $T > 0$ sufficiently large and choose a subsequence. Now apply the Grobman-Hartman homeomorphism h and consider the image of u and of the p_k in $V_0 \subset T_zM$. We continue using the same notation (see Figure 6). To prove Step 1 assume the contrary. Since h con-

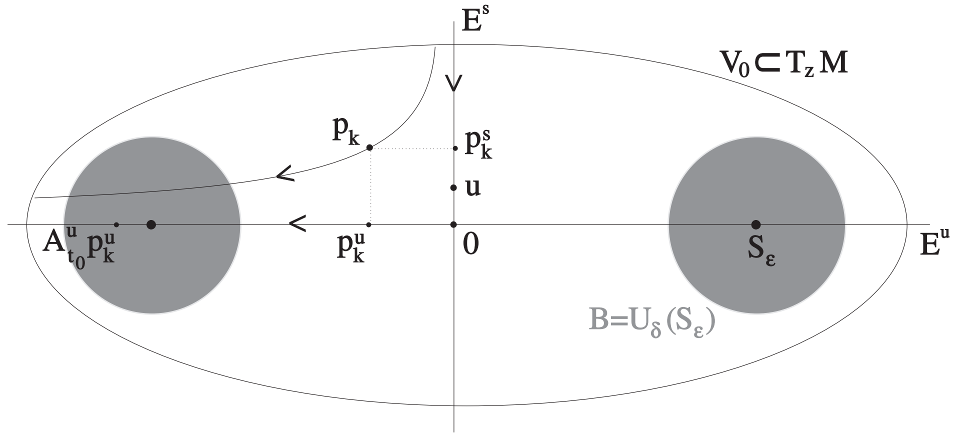

Figure 6: The flow in a Grobman-Hartman (C^0-) chart

jugates ϕ_t and the linearized flow, the contrary means that every sphere S_ε of radius ε in E^u admits a δ-neighbourhood B in T_zM which is disjoint from $\mathcal{M}_{xy}$. Fix $\varepsilon > 0$ and $0 < \delta < \varepsilon$ sufficiently small, such that S_ε and B are contained in V_0. We may also assume $|u| < \delta/2$, otherwise apply ϕ_T again. By linearity of the flow on $T_zM = E^s \oplus E^u$ we can write $D\phi_t(z)p_k$ in the form $(A_t^s p_k^s, A_t^u p_k^u)$, where $p_k = (p_k^s, p_k^u)$. The linear operators $A_t^s \in \mathcal{L}(E^s)$ and $A_t^u \in \mathcal{L}(E^u)$ introduced in Section 2.2 are, for $t > 0$, a strict contraction and a strict dilatation, respectively. For every sufficiently large k we have $|p_k^s| < \delta$, hence $|A_t^s p_k^s| < \delta$ for positive t. Because A_t^u is expanding and 0 is the only fixed point of A_t, it follows that $\left|A_{t_0}^u p_k^u\right| > \varepsilon$ for some $t_0 > 0$. Hence the orbit $\mathcal{O}(p_k)$ runs through B and this contradicts our assumption and therefore proves Step 1. Furthermore, the argument shows that the orbit through p_k converges locally near z to (u, v) in the sense of (8).

STEP 2. *We prove the theorem.*

Assume $z \neq y$, then by Lemma 2.6 we conclude that $v \in \mathcal{M}_{z\tilde{z}}$, for some $\tilde{z} \in \mathrm{Crit} f$ with $\mathrm{ind}_f(\tilde{z}) < \mathrm{ind}_f(z)$. Repeating the arguments in the proof of

Step 1 leads to an iteration which can only terminate at y. It must terminate, because Crit$f$ is a finite set by Corollary 2.3 and the index in each step of the iteration strictly decreases by Lemma 3.5 (d). Start again with the sequence $\{p_k\}_{k\in\mathbb{N}}$ and repeat the same arguments for the flow in backward time. This proves existence of critical points and connecting orbits as in Definition 3.7.

It remains to prove uniform convergence. Near critical points the argument was given in the proof of Step 1. Outside fixed neighbourhoods of the critical points this is a consequence of the estimate

$$d\left(\phi_t q, \phi_t \tilde{q}\right) \leq e^{\kappa|t|} d\left(q, \tilde{q}\right), \quad \forall q, \tilde{q} \in M, \forall t \in \mathbb{R},$$

where $\kappa = \kappa(M, -\nabla f) > 0$ is a constant; see e.g. [PM82, Ch. 2 Lemma 4.8]. The estimate shows that on compact time intervals the orbits through p_k converge uniformly to the orbit through u. Now set $b := f^{-1}(v)$ and view $\widehat{\mathcal{M}}_{xy}$ as a subset of $f^{-1}(b)$. Every point p_k determines a unique point $\tilde{p}_k$ by intersecting the orbit through p_k with $f^{-1}(b)$. Arguing as above, including choosing further subsequences, shows that the orbits through the points $\tilde{p}_k$ converge uniformly on compact time intervals to the orbit through v. Repeating this argument a finite number of times concludes the proof of Theorem 3.8. □

3.3 Gluing

Theorem 3.9 (Gluing). *Assume the Morse-Smale condition is satisfied and choose $x, y, z \in$ Crit$f$ of Morse indices $k+1, k, k-1$, respectively. Then there exists a positive real number ρ_0 and an embedding*

$$\# : \widehat{\mathcal{M}}_{xy} \times [\rho_0, \infty) \times \widehat{\mathcal{M}}_{yz} \to \widehat{\mathcal{M}}_{xz}, \qquad (u, \rho, v) \mapsto u\#_\rho v,$$

such that

$$u\#_\rho v \longrightarrow (u, v) \quad \text{as } \rho \to \infty.$$

Moreover, no sequence in $\widehat{\mathcal{M}}_{xz} \setminus (u\#_{[\rho_0,\infty)} v)$ converges to (u, v).

Proof. The proof has three steps. It consists of local constructions near y. Therefore we restrict to the case where ϕ_t is defined near $y = 0 \in \mathbb{R}^n$.

Step 1 (Local Model). *We may assume without loss of generality that a sufficiently small neighbourhood of y in the stable manifold is a neighbourhood of 0 in E^s and similarly for the unstable manifold.*

The stable subspace E^s associated to $d\phi_t(y) \in \mathcal{L}(\mathbb{R}^n)$ is independent of the choice of $t > 0$ and similarly for the unstable subspace E^u. By Theorem 2.7

they are the tangent spaces at y to the stable and unstable manifold W^s and W^u of y, respectively. The proof of the theorem shows that locally near y the stable and unstable manifolds are graphs. More precisely, there exist small neighbourhoods $U^s \subset E^s$ and $U^u \subset E^u$ of y and smooth maps $\eta_s : U^s \to E^u$ and $\eta_u : U^u \to E^s$ such that $\eta_s(0) = 0$, $d\eta_s(0) = 0$ and similarly for η_u (see Figure 7). The graphs of η_u and η_s, denoted by W^u_{loc} and W^s_{loc}, are called

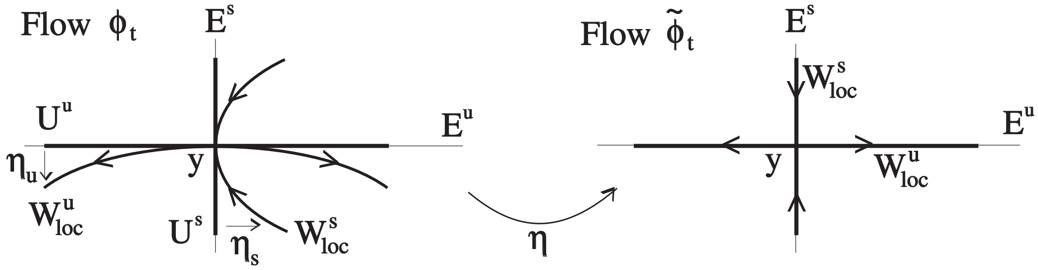

Figure 7: Local stable and unstable manifolds

local unstable and stable manifold, respectively. The smooth map

$$\eta : U^u \times U^s \to E^u \oplus E^s, \qquad (x_u, x_s) \mapsto (x_u - \eta_s(x_s), x_s - \eta_u(x_u))$$

satisfies $\eta(0) = 0$ and $d\eta(0) = \mathbb{1}$ (see [PM82, Section 7]). Hence it is a diffeomorphism when restricted to some neighbourhood of zero. The family of local diffeomorphisms defined by $\tilde{\phi}_t := \eta \circ \phi_t \circ \eta^{-1}$, $t > 0$, satisfies $\tilde{\phi}_t(0) = 0$ and $d\tilde{\phi}_t(0) = d\phi_t(0)$. Moreover, a small neighborhood of zero in the stable manifold of $\tilde{\phi}_t$ is a small neighbourhood of zero in E^s and a similar statement holds for the unstable manifold.

For later reference we shall fix a metric $|\cdot|$ on $\mathbb{R}^n$ compatible with the splitting $\mathbb{R}^n = E^u \oplus E^s$, as in the proof of Theorem 2.7.

Step 2 (Unique intersection point). *Fix closed balls $B^u \subset W^u_{loc} \subset E^u$ and $B^s \subset W^s_{loc} \subset E^s$ around y and let $V := B^u \times B^s$. Choose $u \in \widehat{\mathcal{M}}_{xy}$ and assume without loss of generality that $u \in B^u$ (otherwise replace u by $\phi_T u$ for some $T > 0$ sufficiently large). Choose a k-dimensional disc $D^k \subset W^u(x)$ which transversally intersects the orbit through u precisely at u. For $t \geq 0$ let D^k_t denote the connected component of $\phi_t(D^k) \cap V$ containing $\phi_t(u)$. Choose $v \in \widehat{\mathcal{M}}_{yz}$ and define the $(n-k)$-dimensional disc $D^{n-k}_{-t} \subset W^s(z)$ similarly, but with respect to the backward flow ϕ_{-t} (see Figure 8). Then there exists $t_0 \geq 0$ such that for every $t \geq t_0$ there is a unique point p_t of intersection of D^k_t and D^{n-k}_{-t}.*

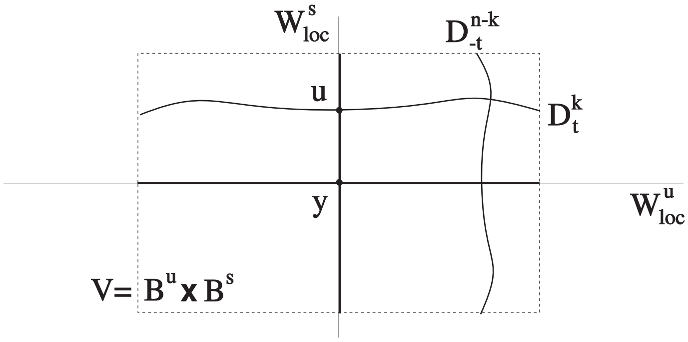

Figure 8: Unique intersection point $p_t \in D^k_t \cap D^{n-k}_{-t} \subset W^u(x) \cap W^s(z)$

The idea of proof is to represent D^k_t and D^{n-k}_{-t}, for $t > 0$ sufficiently large, as graphs of smooth maps $F_t : B^u \to B^s$ and $G_t : B^s \to B^u$, respectively. Since $D^k_t \cap D^{n-k}_{-t}$ corresponds to the fixed point set of $G_t \circ F_t : B^u \to B^u$, it remains to prove that this map is a strict contraction.

Because D^k intersects B^s transversally, we are in position to apply our key tool, namely the λ-Lemma [PM82, Ch. 2 Lemma 7.2]. Given $\varepsilon > 0$, it asserts existence of $t_0 > 0$ such that D^k_t is ε C^1-close to B^u, for every $t \geq t_0$ (and similarly for D^{n-k}_{-t} and B^s). This means that there exist diffeomorphisms

$$\varphi_t : B^u \to D^k_t, \qquad q \mapsto (\varphi^u_t(q), \varphi^s_t(q)),$$
$$\gamma_t : B^s \to D^{n-k}_{-t}, \qquad p \mapsto (\gamma^u_t(p), \gamma^s_t(p)),$$

such that

$$\left| \begin{pmatrix} q \\ 0 \end{pmatrix} - \begin{pmatrix} \varphi^u_t(q) \\ \varphi^s_t(q) \end{pmatrix} \right| < \varepsilon, \qquad \left\| \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix} - \begin{pmatrix} d\varphi^u_t(q) \\ d\varphi^s_t(q) \end{pmatrix} \right\| < \varepsilon, \qquad \forall q \in B^u,$$

$$\left| \begin{pmatrix} 0 \\ p \end{pmatrix} - \begin{pmatrix} \gamma^u_t(p) \\ \gamma^s_t(p) \end{pmatrix} \right| < \varepsilon, \qquad \left\| \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} - \begin{pmatrix} d\gamma^u_t(p) \\ d\gamma^s_t(p) \end{pmatrix} \right\| < \varepsilon, \qquad \forall p \in B^s.$$

Hence, for $\varepsilon > 0$ sufficiently small, the maps φ^u_t and γ^s_t are invertible and the required graph maps are given by (see Figure 9)

$$F_t(q) := \varphi^s_t \circ (\varphi^u_t)^{-1}(q), \qquad G_t(p) := \gamma^u_t \circ (\gamma^s_t)^{-1}(p).$$

The existence of a fixed point of the smooth map $G_t \circ F_t : B^u \to B^u$ follows, for instance, from the Brouwer fixed point theorem or the fact that D^k_t and D^{n-k}_{-t} are homotopic to B^u and B^s, respectively, and the latter have

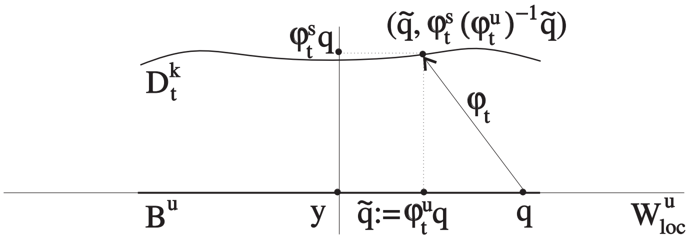

Figure 9: The disc D_t^k as graph of the map $F_t := \varphi_t^s \circ (\varphi_t^u)^{-1}$

intersection number one (see e.g. [Hi76, Ch. 5 Sec. 2] for a definition of the intersection number in case of manifolds with boundary). Using again ε C^1-closedness we obtain

$$\begin{aligned}
\|d\,(G_t \circ F_t)\,|_q\| &= \left\| d\gamma_t^u|_{(\gamma_t^s)^{-1}F_t q} \circ d(\gamma_t^s)^{-1}|_{F_t q} \circ d\varphi_t^s|_{(\varphi_t^u)^{-1}q} \circ d(\varphi_t^u)^{-1}|_q \right\| \\
&\le \varepsilon^2 \left\| d(\gamma_t^s)^{-1}|_{F_t q} \right\| \cdot \left\| d(\varphi_t^u)^{-1}|_q \right\| \\
&\le \varepsilon^2/(1-\varepsilon)^2.
\end{aligned}$$

The last expression is strictly less than one, whenever $0 < \varepsilon < 1/2$. To obtain the final step let $S := d(\varphi_t^u)^{-1}|_q$ and apply the triangle inequality to obtain

$$1 = \|S^{-1} - S^{-1}(\mathbb{1} - S)\| \ge \|S^{-1}\| - \|S^{-1}\| \cdot \|\mathbb{1} - S\|,$$

hence an estimate for $\|S^{-1}\|$. Then the contracting mapping principle (see e.g. [RS80, Thm. V.18]) guarantees a *unique* fixed point and $|p_t| < \sqrt{2}\varepsilon$.

Step 3 (Gluing map). *Using the notation of Step 2 we define $\rho_0 := t_0$ and $u\#_\rho v := p_\rho$, $\forall \rho \in [\rho_0, \infty)$. This map satisfies the assertions of the theorem.*

The negative gradient vector field is transverse to the discs D_t^k and D_{-t}^{n-k} (otherwise choose D^k and D^{n-k} in Step 2 smaller). This implies that they are displaced from themselves by the flow, so their intersection p_t cannot remain constant: $\frac{d}{dt}p_t \ne 0$. This shows that $u\#.v$ is an immersion into $\mathcal{M}_{xz}$. In order to show that it is an immersion into $\widehat{\mathcal{M}}_{xz}$ we need to make sure that p_t does *not* vary along flow lines, in other words $\frac{d}{dt}p_t$ and $-\nabla f(p_t)$ need to be linearly independent. This is true, since otherwise the discs must be either *both* moved in direction $-\nabla f$ or *both* opposite to it. However, in our case D_t^k moves in direction $-\nabla f$ and D_{-t}^{n-k} opposite to it (see Figure 10). Hence $\dim \widehat{\mathcal{M}}_{xz} = 1$ implies that the map $u\#.v : [t_0, \infty) \to \widehat{\mathcal{M}}_{xz}$ is also a homeomorphism onto its image (no self-intersections or returns) and therefore an embedding.

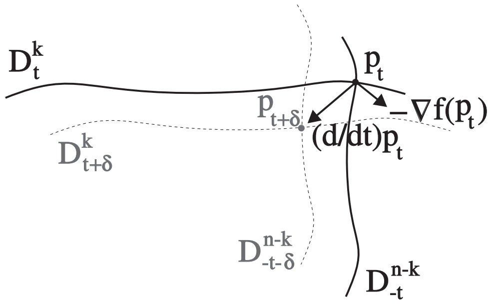

Figure 10: Variation of the point of intersection p_t

Choose a sequence of positive reals $\varepsilon_\ell \to 0$. The λ-Lemma yields a sequence $\{t_{0,\ell}\}_{\ell\in\mathbb{N}}$ such that D_t^k is ε_ℓ C^1-close to B^u, whenever $t \geq t_{0,\ell}$, and similarly for D_{-t}^{n-k}. Given any sequence $t_i \to \infty$ of sufficiently large reals, we can choose a subsequence $\{t_{0,\ell_i}\}_{i\in\mathbb{N}}$ such that $t_i \geq t_{0,\ell_i}$. It follows that $D_{t_i}^k$ and B^u are ε_{ℓ_i} C^1-close and similarly for $D_{-t_i}^{n-k}$ and B^s. Hence $|p_{t_i}| < \sqrt{2}\varepsilon_{\ell_i} \to 0$, as $i \to \infty$. This proves

$$|p_t| \to 0, \quad \text{as } t \to 0.$$

Convergence of $u\#.v$ to the broken orbit (u, v) then follows by the same arguments as in the proof of Theorem 3.8. Uniqueness of the intersection point p_t proves the final claim of Theorem 3.9. $\square$

3.4 Orientation

Assume the Morse-Smale condition is satisfied. Recall from Section 2.2 that the stable and unstable manifolds are contractible and therefore orientable.

Proposition 3.10 (Induced orientation). *Fix an orientation of $W^u(x)$ for every $x \in \mathrm{Crit} f$ of Morse index larger than zero. Then, for all $x, y \in \mathrm{Crit} f$, the connecting manifolds $\mathcal{M}_{xy}$ and orbits spaces $\widehat{\mathcal{M}}_{xy}$* inherit *induced orientations $[\mathcal{M}_{xy}]_{ind}$ and $[\widehat{\mathcal{M}}_{xy}]_{ind}$.*

Proof. The main idea is that the transversal intersection of an oriented and a cooriented submanifold is orientable[6]. An orientation of $W^u(x)$ is by defini-

[6]In contrast to the case of two orientable submanifolds: consider $\mathbb{R}\mathrm{P}^4 = \{[x_0, \ldots, x_4]\}$ with submanifolds $\{[0, x_1, x_2, x_3, x_4]\} \simeq \mathbb{R}\mathrm{P}^3$ and $\{[x_0, x_1, x_2, x_3, 0]\} \simeq \mathbb{R}\mathrm{P}^3$ intersecting in $\{[0, x_1, x_2, x_3, 0]\} \simeq \mathbb{R}\mathrm{P}^2$. Here, by definition, an equivalence class consists of all vectors which become equal after scalar multiplication with some nonzero real number.

tion an orientation of its tangent bundle[7]. Transversality of the intersection implies that this tangent bundle splits along $\mathcal{M}_{xy}$. This means

$$T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy} \oplus \mathcal{V}_{\mathcal{M}_{xy}} W^s(y). \tag{10}$$

Here the last term denotes the normal bundle of $W^s(y)$ restricted to $\mathcal{M}_{xy}$. It remains to show that two of the vector bundles are oriented and therefore determine an orientation of $T\mathcal{M}_{xy}$ denoted by $[\mathcal{M}_{xy}]_{ind}$. Firstly, the restriction of the oriented vector bundle $TW^u(x)$ to any submanifold, for instance to $\mathcal{M}_{xy}$, is an oriented vector bundle. Secondly, contractibility of the base manifold $W^s(y)$ implies that $\mathcal{V}W^s(y)$ is orientable. Hence an orientation is determined by an orientation of a single fiber. The natural choice is the one over y, because it is isomorphic to the oriented vector space $T_y W^u(y)$ via

$$T_y W^u(y) \oplus T_y W^s(y) \simeq T_y M \simeq \mathcal{V}_y W^s(y) \oplus T_y W^s(y).$$

Now restrict the oriented vector bundle $\mathcal{V}W^s(y) \to W^s(y)$ to the submanifold $\mathcal{M}_{xy} \hookrightarrow W^s(y)$ to obtain the required oriented vector bundle $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$. The orientation $[\widehat{\mathcal{M}}_{xy}]_{ind}$ is determined by the splitting

$$T_{\widehat{\mathcal{M}}_{xy}} \mathcal{M}_{xy} \simeq \mathbb{R} \oplus T\widehat{\mathcal{M}}_{xy},$$

as the orientation $[T_{\widehat{\mathcal{M}}_{xy}} \mathcal{M}_{xy}]_{ind}$ is given by restriction of the oriented bundle $T\mathcal{M}_{xy}$, and the orientation of the line bundle is provided by $-\nabla f$. □

Given $x, y, z \in \mathrm{Crit} f$ of Morse indices $k+1, k, k-1$, respectively, as well as $u \in \widehat{\mathcal{M}}_{xy}$ and $v \in \widehat{\mathcal{M}}_{yz}$, the gluing map of orbits $u \#_\rho v = p_\rho$ induces a gluing map of orientations

$$\sigma^\# : \mathrm{Or}(\mathcal{M}^u_{xy}) \times \mathrm{Or}(\mathcal{M}^v_{yz}) \to \mathrm{Or}(\mathcal{M}^{u\#_\rho v}_{xz}), \quad \rho \in [\rho_0, \infty).$$

Here $\mathcal{M}^u_{xy}$ denotes the connected component of $\mathcal{M}_{xy}$ containing u. Let $[\dot{u}]$ denote the orientation of $\mathcal{M}^u_{xy}$ provided by the flow. The orientation of a k-dimensional fibre determined by an ordered k-tuple of vectors is denoted by $\langle v_1, \cdots, v_k \rangle$. Let $[\langle v_1, \ldots, v_k \rangle]$ be the resulting orientation of the whole orientable vector bundle. The map $\sigma^\#$ is defined in case of flow orientations by (see Figure 10)

$$\sigma^\# ([\dot{u}], [\dot{v}]) := \left[\langle -\nabla f(p_\rho), -\tfrac{d}{d\rho} p_\rho \rangle \right],$$

[7] A good reference concerning orientations of vector bundles is [Hi76].

and in the general case by

$$\sigma^{\#}\left([\mathcal{M}_{xy}^{u}],[\mathcal{M}_{yz}^{v}]\right) := ab\,\sigma^{\#}\left([\dot{u}],[\dot{v}]\right), \tag{11}$$

where $a, b \in \{\pm 1\}$ are determined by $[\mathcal{M}_{xy}^{u}] = a[\dot{u}]$ and $[\mathcal{M}_{yz}^{v}] = b[\dot{v}]$.

Theorem 3.11 (Coherence). *The gluing map (11) and the orientations provided by Proposition 3.10 are compatible in the sense that*

$$\sigma^{\#}\left([\mathcal{M}_{xy}^{u}]_{ind},[\mathcal{M}_{yz}^{v}]_{ind}\right) = [\mathcal{M}_{xz}^{u\#_\rho v}]_{ind}.$$

Proof. Define $n_u \in \{\pm 1\}$ by the identity $[\mathcal{M}_{xy}^{u}]_{ind} = n_u[\dot{u}]$, then

$$\begin{aligned}\sigma^{\#}\left([\mathcal{M}_{xy}^{u}]_{ind},[\mathcal{M}_{yz}^{v}]_{ind}\right) &= n_u n_v \sigma^{\#}\left([\dot{u}],[\dot{v}]\right)\\ &= n_u n_v \left[\langle -\nabla f(p_\rho), -\tfrac{d}{d\rho}p_\rho\rangle\right].\end{aligned} \tag{12}$$

The second equality holds by definition of $\sigma^{\#}$. To compare the right hand side with $[\mathcal{M}_{xz}^{u\#_\rho v}]_{ind}$ we need to relate the induced orientations of the bundles $T\mathcal{M}_{xy}^{u}$, $T\mathcal{M}_{yz}^{v}$ and $T\mathcal{M}_{xz}^{u\#_\rho v}$. Unfortunately, the base manifolds do not have a common point. On the other hand, the point y lies in the closure of all three base manifolds and all three tangent bundles can be extended to y. This is due to the existence of the limits[8] (see [Sch93, Lemma B.5] and [KMP])

$$\lim_{t\to\infty}\frac{\frac{d}{dt}\phi_t u}{\|\frac{d}{dt}\phi_t u\|} =: \dot{u}(+\infty), \qquad \lim_{t\to-\infty}\frac{\frac{d}{dt}\phi_t v}{\|\frac{d}{dt}\phi_t v\|} =: \dot{v}(-\infty).$$

Repeatedly using (10) shows that the orientations of the fibres over y are related by (we use the same notation for the bundles extended to y)

$$\begin{aligned}[T_yW^u(x)] &= [T_y\mathcal{M}_{xy}^{u}]_{ind} \oplus [\mathcal{V}_yW^s(y)]\\ &= [T_y\mathcal{M}_{xy}^{u}]_{ind} \oplus [T_yW^u(y)]\\ &= [T_y\mathcal{M}_{xy}^{u}]_{ind} \oplus [T_y\mathcal{M}_{yz}^{v}]_{ind} \oplus [\mathcal{V}_yW^s(z)]\end{aligned}$$

and (since $y = \lim_{\rho\to\infty} p_\rho$)

$$[T_yW^u(x)] = [T_y\mathcal{M}_{xz}^{u\#_\rho v}]_{ind} \oplus [\mathcal{V}_yW^s(z)].$$

Hence

$$[T_y\mathcal{M}_{xz}^{u\#_\rho v}]_{ind} = [T_y\mathcal{M}_{xy}^{u}]_{ind} \oplus [T_y\mathcal{M}_{yz}^{v}]_{ind} = n_u n_v[\dot{u}(+\infty)] \oplus [\dot{v}(-\infty)].$$

The ordered pairs $\langle \dot{u}(+\infty), \dot{v}(-\infty)\rangle$ and $\langle -\nabla f(p_\rho), -\frac{d}{d\rho}p_\rho\rangle$ represent the same orientation of $\mathcal{M}_{xz}^{u\#_\rho v}$. By (12) this proves the theorem. □

[8] Here nondegeneracy of y is crucial and $\dot{u}(+\infty)$ and $\dot{v}(-\infty)$ are eigenvectors of the Hessian of f at y corresponding to a positive and a negative eigenvalue, respectively.

4 Morse homology

4.1 Morse-Witten complex

Definition 4.1. The *Morse chain groups associated to a Morse function f, with integer coefficients and graded by the Morse index*, are the free abelian groups generated by the critical points of f of Morse index k

$$\mathrm{CM}_k(M,f) := \bigoplus_{x\in \mathrm{Crit}_k f} \mathbb{Z}x, \qquad k \in \mathbb{Z}.$$

A sum over the empty set is understood to be zero.

The chain groups are finitely generated by Corollary 2.3. Let us choose a Riemannian metric g on M. If (g,f) is not a Morse-Smale pair, replace it by a C^1-close Morse-Smale pair according to Theorem 3.3. Since the type of a Morse function is locally constant, both chain groups are canonically isomorphic. From now on we assume that (g,f) is Morse-Smale. Choose an orientation for every unstable manifold and denote this set of choices by Or.

Definition 4.2. Assume $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) = 1$ and let $u \in \widehat{\mathcal{M}}_{xy}$. The orbit $\mathcal{O}(u)$ is a connected component of $\mathcal{M}_{xy}$ and hence carries the induced orientation $[\mathcal{O}(u)]_{ind}$ provided by Proposition 3.10. Denoting the flow orientation by $[\dot{u}]$, the *characteristic sign* $n_u = n_u(Or)$ is defined by

$$[\mathcal{O}(u)]_{ind} = n_u[\dot{u}].$$

Definition 4.3. The *Morse-Witten boundary operator*

$$\partial_k = \partial_k(M,f,g,Or) : \mathrm{CM}_k(M,f) \to \mathrm{CM}_{k-1}(M,f)$$

is given on a generator x by

$$\partial_k x := \sum_{y\in \mathrm{Crit}_{k-1} f} n(x,y)y, \qquad n(x,y) := \sum_{u\in \widehat{\mathcal{M}}_{xy}} n_u,$$

and extended to general chains by linearity.

Both sums in the definition of ∂_k are finite by Corollary 2.3 and Proposition 3.6, respectively. To prove that ∂ satisfies $\partial^2 = 0$ we need to investigate the 1-dimensional components of the space of connecting orbits. Fix $x \in \mathrm{Crit}_k f$ and $z \in \mathrm{Crit}_{k-2} f$. By Theorem 3.4 the orbit space $\widehat{\mathcal{M}}_{xz}$ is a manifold (without boundary) of dimension 1 and therefore its connected components $\mathcal{M}^i_{xz}$ are diffeomorphic either to $(0,1)$ or to S^1 (see Figs. 11–12).

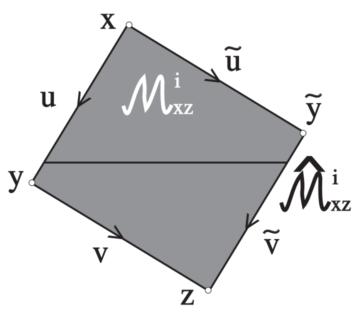

Figure 11: $\widehat{\mathcal{M}}^i_{xz} \simeq (0,1)$

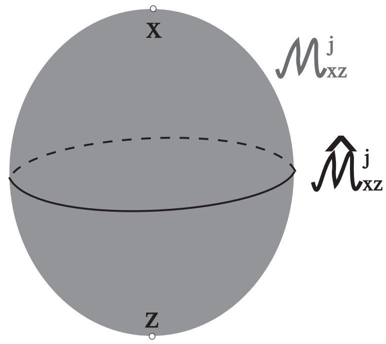

Figure 12: $\widehat{\mathcal{M}}^j_{xz} \simeq S^1$

Proposition 4.4. *Let $x \in \mathrm{Crit}_k f$ and $z \in \mathrm{Crit}_{k-2} f$, then the following are true. (i) The set*

$$\mathcal{B}^1_{xz} := \{(u,v) \mid u \in \widehat{\mathcal{M}}_{xy},\ v \in \widehat{\mathcal{M}}_{yz},\ \textit{for some } y \in \mathrm{Crit}_{k-1} f\}$$

of broken orbits of order two between x and z corresponds precisely to the ends of the noncompact connected components of $\widehat{\mathcal{M}}_{xz}$.
(ii) Two broken orbits (u,v) and $(\tilde{u},\tilde{v})$ corresponding to the same connected component $\widehat{\mathcal{M}}^i_{xz}$ are called cobordant. *Their characteristic signs satisfy*

$$n_u n_v + n_{\tilde{u}} n_{\tilde{v}} = 0.$$

Proof. By Theorem 3.8 a connected component $\widehat{\mathcal{M}}^i_{xz} \simeq (0,1)$ is compact up to broken orbits of order two. Hence we obtain two broken orbits (u,v) and $(\tilde{u},\tilde{v})$ – one corresponding to each end. The last statement in Theorem 3.9 implies $(u,v) \neq (\tilde{u},\tilde{v})$. (However $u = \tilde{u}$ and $v \neq \tilde{v}$, or vice versa, is possible as Example 1.1 shows). The Gluing Theorem 3.9 also tells that each broken orbit (u,v) corresponds to a noncompact end of $\widehat{\mathcal{M}}_{xz}$. This concludes the proof of part (i). To prove part (ii) use the definition (11) of $\sigma^\#$, Theorem 3.11, and the fact that $u\#_\rho v, \tilde{u}\#_\rho \tilde{v} \in \widehat{\mathcal{M}}^i_{xz}$ to obtain the following identities for the orientation of $\mathcal{M}^i_{xz}$, namely

$$\begin{aligned} n_u n_v[\langle \nabla f(p_\rho), \tfrac{d}{d\rho} p_\rho \rangle] &= n_u n_v \sigma^\# \left([\dot{u}], [\dot{v}]\right) && = \sigma^\# \left([u]_{ind}, [v]_{ind}\right) \\ &= [\widehat{\mathcal{M}}^{u\#_\rho v}_{xz}]_{ind} && = [\widehat{\mathcal{M}}^{\tilde{u}\#_\rho \tilde{v}}_{xz}]_{ind} \\ &= \sigma^\# \left([\tilde{u}]_{ind}, [\tilde{v}]_{ind}\right) && = n_{\tilde{u}} n_{\tilde{v}} \sigma^\# \left([\dot{\tilde{u}}], [\dot{\tilde{v}}]\right) \\ &= n_{\tilde{u}} n_{\tilde{v}} [\langle \nabla f(\tilde{p}_\rho), \tfrac{d}{d\rho} \tilde{p}_\rho \rangle] && = -n_{\tilde{u}} n_{\tilde{v}} [\langle \nabla f(p_\rho), \tfrac{d}{d\rho} p_\rho \rangle]. \end{aligned}$$

The last step follows, because $\frac{d}{d\rho}\tilde{p}_\rho$ and $\frac{d}{d\rho}p_\rho$ both point outward along the boundary of $\widehat{\mathcal{M}}^i_{xz}$. □

Theorem 4.5 (Boundary operator). $\partial_{k-1}\partial_k = 0, \quad \forall k \in \mathbb{Z}$.

Proof. By linearity, definition of ∂ and $\mathcal{B}^1_{xz}$, and Proposition 4.4 it follows

$$\begin{aligned}
\partial_{k-1}\partial_k x &= \sum_{z\in \mathrm{Crit}_{k-2} f} \left(\sum_{y\in \mathrm{Crit}_{k-1} f} \sum_{u\in \widehat{\mathcal{M}}_{xy}} \sum_{v\in \widehat{\mathcal{M}}_{yz}} n_u n_v \right) z \\
&= \sum_{z\in \mathrm{Crit}_{k-2} f} \left(\sum_{(u,v)\in \mathcal{B}^1_{xz}} n_u n_v \right) z \\
&= \sum_{z\in \mathrm{Crit}_{k-2} f} \left(\sum_{\substack{\text{connected components } \widehat{\mathcal{M}}^i_{xz} \\ \text{of } \widehat{\mathcal{M}}_{xz} \text{ diffeomorphic to } (0,1)}} (n_{u_i} n_{v_i} + n_{\tilde{u}_i} n_{\tilde{v}_i}) \right) z \\
&= 0.
\end{aligned}$$

□

Definition 4.6. Given a closed smooth finite dimensional manifold M, a Morse function f and a Riemannian metric g on M such that the Morse-Smale condition holds, denote by Or a choice of orientations of all unstable manifolds associated to the vector field $-\nabla f$. Then the *Morse homology groups with integer coefficients* are defined by

$$\mathrm{HM}_k(M; f, g, Or; \mathbb{Z}) := \frac{\ker \partial_k}{\mathrm{im}\, \partial_{k+1}}, \qquad k \in \mathbb{Z}.$$

In view of the Smale Transversality Theorem 3.3 and the Continuation Theorem 4.8, we can in fact define these homology groups for every pair (f, g), Morse-Smale or not, and every choice of orientations Or. Since they are all naturally isomorphic, we shall denote them by $\mathrm{HM}_*(M; \mathbb{Z})$.

Example 4.7. Going back to Example 1.1 let us calculate the characteristic signs using the orientations indicated in Figure 1. We obtain

$$n_{u_1} = n_{u_2} = n_{\tilde{v}} = -1, \qquad n_v = +1.$$

Hence $\mathrm{HM}_2 = \langle x_1 - x_2 \rangle \simeq \mathbb{Z}$, $\mathrm{HM}_1 = 0$ and $\mathrm{HM}_0 = \langle z \rangle \simeq \mathbb{Z}$.

4.2 Continuation

In this section we present Poźniak's [Po91] construction of continuation maps, i.e. of natural grading preserving isomorphisms

$$\Psi_*^{\beta\alpha} : \mathrm{HM}_*(M; f^\alpha, g^\alpha, Or^\alpha) \to \mathrm{HM}_*(M; f^\beta, g^\beta, Or^\beta)$$

associated to any choice of Morse-Smale pairs (f^α, g^α) and (f^β, g^β) and orientations Or^α and Or^β of all unstable manifolds.

Theorem 4.8 (Continuation). $\Psi_*^{\beta\alpha} = (\Psi_*^{\alpha\beta})^{-1}, \quad \Psi_*^{\gamma\beta}\Psi_*^{\beta\alpha} = \Psi_*^{\gamma\alpha}$.

The remaining part of this section prepares the proof of the theorem.

The chain map associated to a homotopy

Let $(f^\alpha, g^\alpha, Or^\alpha)$ and $(f^\beta, g^\beta, Or^\beta)$ be as above. Fix homotopies $\{f_s\}_{s\in[0,1]}$ and $\{g_s\}_{s\in[0,1]}$ from f^α to f^β and g^α to g^β, respectively. By rescaling and smoothing the homotopies near the endpoints, if necessary, we may assume that they are constant near the end points. More precisely, for some fixed $\delta \in [0, 1/4]$ we have $f_s \equiv f^\alpha$, for $s \in [0, \delta]$, and $f_s \equiv f^\beta$, for $s \in [1-\delta, 1]$, and similarly for g_s. Such homotopies $h^{\alpha\beta} := (f_s, g_s)$ are called *admissible.* Let us parametrize S^1 by the interval $[-1, 1]$ with endpoints being identified.

Lemma 4.9. *The set of critical points of the function $F = F_\kappa : M \times S^1 \to \mathbb{R}$,*

$$F(q, s) := \frac{\kappa}{2}(1 + \cos \pi s) + f_{|s|}(q),$$

coincides with $(\mathrm{Crit} f^\alpha \times \{0\}) \cup \big(\mathrm{Crit} f^\beta \times \{1\}\big)$, *for every positive real*

$$\kappa > \frac{2 \max_{M\times S^1} \partial_s f}{\pi \sin \pi(1-\delta)}. \tag{13}$$

Then all critical points are nondegenerate and their Morse indices satisfy

$$\mathrm{ind}_F(x, 0) = \mathrm{ind}_{f^\alpha}(x) + 1, \qquad \mathrm{ind}_F(y, 1) = \mathrm{ind}_{f^\beta}(y).$$

The proof of the lemma is illustrated by Figure 13 and left as an exercise. Define a product metric on $M \times S^1$ by

$$G_{q,s} := g_{q,|s|} \oplus 1$$

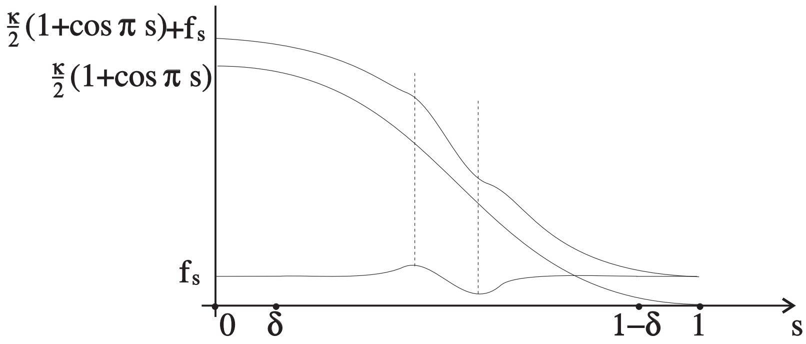

Figure 13: If $\kappa >> 1$, then $\partial_s F$ vanishes only at $s = 0, 1$

and consider the negative gradient flow of F with respect to G. We orient all unstable manifolds by taking the choice

$$Or^{\alpha\beta} := (Or^\alpha \oplus \partial_s) \cup Or^\beta.$$

The Morse-Witten complex associated to $(M \times S^1, F, G, Or^{\alpha\beta})$ has the following properties. Its chain groups split by Lemma 4.9

$$\mathrm{CM}_k(M \times S^1, F) \simeq \mathrm{CM}_{k-1}(M, f^\alpha) \oplus \mathrm{CM}_k(M, f^\beta). \tag{14}$$

Setting $-\nabla F(q, s) =: (\xi_{q,s}, a_{q,s}) \in T_q M \times \mathbb{R}$ calculation shows

$$a_{q,s} = -\frac{\partial}{\partial s} F(q, s) = \frac{\kappa\pi}{2} \sin \pi s - \frac{\partial}{\partial s} f_{|s|}(q)$$

and $\xi_{q,s}$ is determined by the identity

$$df_{|s|}(q)\cdot = g_{q,|s|}(-\xi_{q,s}, \cdot).$$

By (13) the zeroes of a are precisely given by the union of $M \times \{0\}$ and $M \times \{1\}$. In particular, both sets are flow invariant and the restricted flow coincides with the one generated by $-\nabla f^\alpha$ and $-\nabla f^\beta$, respectively. The characteristic sign n_u of an isolated flow line $u \in \widehat{\mathcal{M}}_{xy}$ is related to the characteristic signs associated to the corresponding flow lines $(u, 0)$ and $(u, 1)$ in $M \times \{0\}$ and $M \times \{1\}$, respectively, by

$$n_{(u,1)} = n_u = -n_{(u,0)}.$$

Replacing κ by any number larger than $(\max f^\beta - \min f^\alpha)$ guarantees that there are no flow lines from $M \times \{1\}$ to $M \times \{0\}$, because the negative gradient flow decreases along trajectories.

Concerning the Morse-Smale condition we observe that unstable and stable manifolds of critical points (x, i) and (y, i), respectively, intersect transversally for $i = 0$ and also for $i = 1$. However, this is not necessarily the case for $W^u(x, 0)$ and $W^s(y, 1)$. If necessary, replace (F, G) by a sufficiently C^1-close Morse-Smale pair (without changing notation). The number and Morse indices of critical points as well as the structure of connecting manifolds, as long as they arise from transversal intersections are preserved. In particular, the two chain subcomplexes sitting at $M \times \{0\}$ and at $M \times \{1\}$ remain unchanged.

Flow lines from $M \times \{0\}$ to $M \times \{1\}$ converge at the ends to $(x, 0)$ and $(y, 1)$, where x, y are critical points of f^α and f^β, respectively. In case $\mathrm{ind}_F(x, 0) - \mathrm{ind}_F(y, 1) = 1$ there are finitely many such flow lines. The algebraic count of those which pass through $M \times \{1/2\}$ defines a map

$$\psi_*^{\beta\alpha} = \psi_*^{\beta\alpha}(h^{\alpha\beta}, Or^{\alpha\beta}) : \mathrm{CM}_*(M, f^\alpha) \to \mathrm{CM}_*(M, f^\beta)$$

given on a generator x by

$$x \mapsto \sum_{\substack{(u,c)\in\widehat{\mathcal{M}}_{(x,0)(y,1)} \\ \mathcal{O}((u,c))\cap(M\times\{1/2\})\neq\emptyset}} n_{(u,c)} y. \tag{15}$$

(It is an exercise to show that counting all flow lines produces the zero map). Hence the boundary operator $\Delta_k = \Delta_k(M \times S^1, F, G, Or^{\alpha\beta})$ is of the form

$$\Delta_k \simeq \begin{pmatrix} -\partial_{k-1}^\alpha & 0 \\ \psi_{k-1}^{\beta\alpha} & \partial_k^\beta \end{pmatrix}$$

with respect to the splitting (14). Theorem 4.5 states $\Delta_{k-1}\Delta_k = 0$ and this implies $\psi_{k-2}^{\beta\alpha}\partial_{k-1}^\alpha = \partial_{k-2}^\beta\psi_{k-1}^{\beta\alpha}$. Hence $\psi_*^{\beta\alpha}(h^{\alpha\beta}, Or^{\alpha\beta})$ is a chain map and we denote the induced map on homology by $[\psi_*^{\beta\alpha}(h^{\alpha\beta}, Or^{\alpha\beta})]$. Observe that by $[\cdot]$ we also denote orientations. However, the meaning should be clear from the context.

Remark 4.10 (Constant homotopies). Let h^α denote the pair of constant homotopies (f^α, g^α) and set $Or^{\alpha\alpha} = (Or^\alpha \oplus \partial_s) \cup Or^\alpha$. Then the isolated flow lines of the negative gradient flow on $M \times S^1$ come in quadruples and with characteristic signs as shown in Figure 14. This implies

$$\psi_*^{\alpha\alpha}(h^\alpha, Or^{\alpha\alpha}) = 1\!\!1. \tag{16}$$

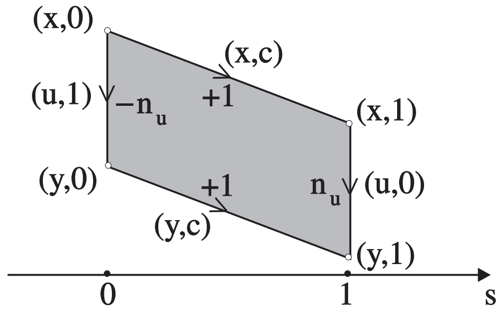

Figure 14: Isolated flow lines in case of constant homotopies

Remark 4.11 (Variation of metric). Morse-Witten chain complexes corresponding to Morse-Smale pairs (f, g^α) and (f, g^β) are *chain* isomorphic, as observed by Cornea and Ranicki [CR03, Rmk. 1.23 (c)]. To see this reorder the generators of $\mathrm{CM}_k(M, f)$ according to descending value of f, choose a homotopy g_s and define F, G and $Or^{\alpha\alpha}$ as above. Now the key observation is that the chain homomorphism $\psi_k^{\alpha\alpha}((f, g_s), Or^{\alpha\alpha})$ has diagonal entries $+1$ and only zeroes above the diagonal: relevant isolated flow lines of $-\nabla F$ from $M \times 0$ to $M \times 1$ are given by pairs $(u, c) : \mathbb{R} \to M \times [0, 1]$ satisfying

$$\begin{pmatrix} \dot{u} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \xi_{u,c} \\ \frac{\kappa\pi}{2} \sin \pi c \end{pmatrix}, \quad df|_u(\cdot) = g_{u,|c|}(-\xi_{u,c}, \cdot), \quad \lim_{t \to \pm\infty} (u, c) \in \mathrm{Crit}_k f \times \{0\}.$$

For every $x \in \mathrm{Crit}_k f$ there is a unique isolated flow line connecting $(x, 0)$ to $(x, 1)$. It is of the form (x, c) and transversality is automatically satisfied (therefore it survives small perturbations possibly required to achieve Morse-Smale transversality for (F, G) and actually defining $\psi_k^{\alpha\alpha}$). Hence every diagonal entry of $\psi_k^{\alpha\alpha}$ equals $+1$. The identity for flow lines (u, c)

$$\frac{d}{dt} f(u(t)) = df|_{u(t)} \dot{u}(t) = g_{u(t),|c(t)|}(-\xi_{u(t),c(t)}, \dot{u}(t)) = -\left|\dot{u}(t)\right|^2_{g_{u(t),|c(t)|}}$$

shows that f is strictly decreasing along nonconstant u-components, hence all elements of $\psi_k^{\alpha\alpha}$ strictly above the diagonal are zero.

Homotopies of homotopies

Given four Morse functions, metrics and choices of orientations (f^i, g^i, Or^i), $i = \alpha, \beta, \gamma, \delta$, and homotopies of homotopies $\{f_{s,r}\}_{s,r\in[0,1]}$ and $\{g_{s,r}\}_{s,r\in[0,1]}$ satisfying, for some fixed $\varepsilon \in [0, 1/4]$,

$$f_{s,r} = \begin{cases} f^\alpha & ,(s,r) \in [0,\varepsilon] \times [0,\varepsilon] \\ f^\beta & ,(s,r) \in [1-\varepsilon,1] \times [0,\varepsilon] \\ f^\gamma & ,(s,r) \in [0,\varepsilon] \times [1-\varepsilon,1] \\ f^\delta & ,(s,r) \in [1-\varepsilon,1] \times [1-\varepsilon,1] \end{cases}$$

and similarly for $g_{s,r}$, we define a function and metric on $M \times S^1 \times S^1$ by

$$\begin{aligned} F(q,s,r) &:= \frac{\kappa}{2}(1+\cos\pi s) + \frac{\rho}{2}(1+\cos\pi r) + f_{|s|,|r|}(q) \\ G_{q,s,r} &:= g_{q,|s|,|r|} \oplus 1 \oplus 1. \end{aligned}$$

Choosing $\kappa, \rho > 0$ sufficiently large it follows as in Lemma 4.9 that F is Morse and there is a splitting

$$\begin{aligned} &\mathrm{CM}_k(M \times S^1 \times S^1, F) \\ &\simeq \mathrm{CM}_{k-2}(M, f^\alpha) \oplus \mathrm{CM}_{k-1}(M, f^\beta) \oplus \mathrm{CM}_{k-1}(M, f^\gamma) \oplus \mathrm{CM}_k(M, f^\delta). \end{aligned} \tag{17}$$

Orient all unstable manifolds of $-\nabla F$ on $M \times S^1 \times S^1$ by

$$Or^{\alpha\beta\gamma\delta} := (Or^\alpha \oplus \partial_s \oplus \partial_r) \cup (Or^\beta \oplus \partial_r) \cup (Or^\gamma \oplus \partial_s) \cup Or^\delta.$$

Arguing similarly as in the former subsection we conclude that the boundary operator is, with respect to the splitting (17), represented by

$$\Delta_k(M \times S^1 \times S^1, F, G, Or^{\alpha\beta\gamma\delta}) \simeq \begin{pmatrix} \partial^\alpha_{k-2} & 0 & 0 & 0 \\ \psi^{\beta\alpha}_{k-2} & -\partial^\beta_{k-1} & 0 & 0 \\ -\psi^{\gamma\alpha}_{k-2} & 0 & -\partial^\gamma_{k-1} & 0 \\ \Lambda^{\delta\alpha}_{k-2} & \psi^{\delta\beta}_{k-1} & \psi^{\delta\gamma}_{k-1} & \partial^\delta_k \end{pmatrix},$$

where Λ and the ψ's are defined similar to (15) by counting isolated flow lines. The signs arise as follows. The characteristic sign n_u of an isolated flow line u in one of the subcomplexes, e.g. the first summand in (17), and the sign $n_{(u,0,0)}$ of the corresponding element of the full complex are related by

$$n_{(u,0,0)} = n^\alpha_u, \quad n_{(u,1,0)} = -n^\beta_u, \quad n_{(u,0,1)} = -n^\gamma_u, \quad n_{(u,1,1)} = n^\delta_u.$$

Similarly the signs $n_{(u,c)}$ in the definition (15) of ψ_* and the corresponding ones in the full complex are related by

$$n_{(u,c,0)} = n^{\beta\alpha}_{(u,c)}, \quad n_{(u,c,1)} = n^{\delta\gamma}_{(u,c)}, \quad n_{(u,0,c)} = -n^{\gamma\alpha}_{(u,c)}, \quad n_{(u,1,c)} = n^{\delta\beta}_{(u,c)}.$$

The identity $\Lambda^{\delta\alpha}_{k-3}\partial^{\alpha}_{k-2} + \partial^{\delta}_{k-1}\Lambda^{\delta\alpha}_{k-2} = \psi^{\delta\gamma}_{k-2}\psi^{\gamma\alpha}_{k-2} - \psi^{\delta\beta}_{k-2}\psi^{\beta\alpha}_{k-2}$ is a consequence of $\Delta_{k-1}\Delta_k = 0$ and shows that Λ is a chain homotopy between $\psi^{\delta\gamma}_{k-2}\psi^{\gamma\alpha}_{k-2}$ and $\psi^{\delta\beta}_{k-2}\psi^{\beta\alpha}_{k-2}$. Therefore

$$\left[\psi^{\delta\gamma}_*\right]\left[\psi^{\gamma\alpha}_*\right] = \left[\psi^{\delta\beta}_*\right]\left[\psi^{\beta\alpha}_*\right]. \tag{18}$$

It is important to recall that $\psi^{\beta\alpha}_*$ actually abbreviates $\psi^{\beta\alpha}_*(h^{\alpha\beta}, Or^{\alpha\beta})$.

Proposition 4.12. *The induced map on homology*

$$\Psi^{\beta\alpha}_* := \left[\psi^{\beta\alpha}_*(h^{\alpha\beta}, Or^{\alpha\beta})\right] : \mathrm{HM}_*(M; f^\alpha, g^\alpha, Or^\alpha) \to \mathrm{HM}_*(M; f^\beta, g^\beta, Or^\beta)$$

is independent of the choice of $h^{\alpha\beta}$ and $Or^{\alpha\beta}$ and satisfies $\Psi^{\alpha\alpha}_* = \mathbb{1}$.

Proof. In (18) choose $\gamma := \alpha$, $\delta := \beta$ and homotopies as indicated in Figure 15, where $h^\alpha = (f^\alpha, g^\alpha)$ denotes constant homotopies. By (18) we have

$$\left[\psi^{\beta\alpha}_*(\tilde{h}^{\alpha\beta}, Or^{\alpha\beta})\right]\left[\psi^{\alpha\alpha}_*(h^{\alpha}, Or^{\alpha\alpha})\right] = \left[\psi^{\beta\beta}_*(h^{\beta}, Or^{\beta\beta})\right]\left[\psi^{\beta\alpha}_*(h^{\alpha\beta}, Or^{\alpha\beta})\right],$$

and (16) concludes the proof. □

Proof of Theorem 4.8. Choosing $\gamma := \alpha$ in (18) and homotopies as indicated in Figure 16 proves $\Psi^{\delta\beta}_*\Psi^{\beta\alpha}_* = \Psi^{\delta\alpha}_*$. Setting $\delta := \alpha$ concludes the proof. □

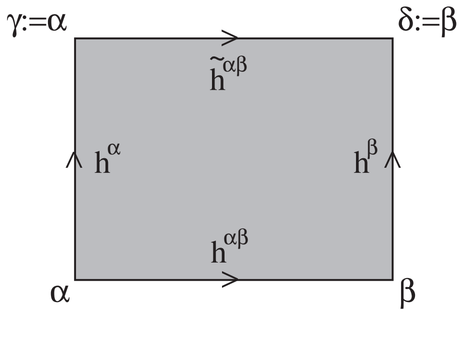

Figure 15: Proof of Proposition 4.12

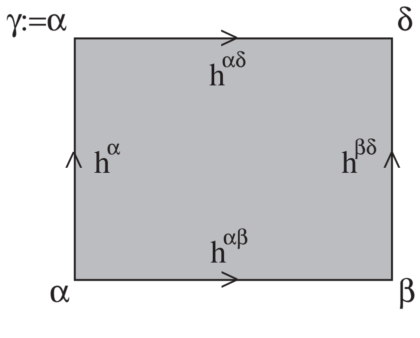

Figure 16: Proof of Theorem 4.8

4.3 Computation

Motivated by our examples in Figures 1, 4 and 12 one might conjecture

Theorem 4.13. $\mathrm{HM}_*(M;\mathbb{Z}) \simeq \mathrm{H}_*^{sing}(M;\mathbb{Z})$.

The examples also suggest that the unstable manifolds corresponding to a Morse cycle are closely related to singular cycles. This observation is the origin of various geometric approaches towards a proof of the theorem.

However, the first proofs given were somewhat less geometrical; see Milnor [Mi65, Lemma 7.2, Theorem 7.4] in case of self-indexing f; for the general case see Floer [Fl89], Salamon [Sa90] and Schwarz [Sch93].

The geometric idea above was made precise by Schwarz [Sch99] constructing pseudo-cycles in an intermediate step.

Another idea for a geometric proof has been around for many years, but to the best of my knowledge never made rigorous: a *smooth triangulation of* M is a pair (K, h) of a simplicial complex K and a homeomorphism h between the union $|K|$ of all simplices of K and M, such that h restricted to any top-dimensional simplex is an embedding. It exists by [Ca35, Wh40]. Given (K, h), the idea is to construct a Morse function f whose critical points are precisely at the barycenters[9] of the simplices and a metric g, such that (f, g) is a Morse-Smale pair and its negative gradient flow has the following property. From each barycenter of a k-face there is precisely one isolated flow line to the barycenter of every $(k-1)$-face in the boundary of the k-face. The orientations of the unstable manifolds have to be chosen, such that the characteristic signs reflect the signs in the definition of the simplicial boundary operator. Then the simplicial and Morse chain complexes are identical and the theorem follows immediately. A difficulty in this approach is the fact that top-dimensional simplices do not in general fit together smoothly.

Yet another idea is to give M the structure of a CW-space viewing the unstable manifolds of a self-indexing Morse function as cells and then relate Morse homology to cellular homology. In case that the metric is euclidean with respect to the local coordinates provided near critical points by the Morse Lemma this has been achieved by Laudenbach [Ld92]. In the general case the difficulty arises how to extend the natural identification of an open disc with an unstable manifold, which is provided by the flow, *continuously* to the boundary: some point might converge to one critical point, but arbitrarily close points to another one.

[9] For simplicity we shall call the images of barycenters under h again barycenters.

4.4 Remarks

Morse inequalities

To prove the Morse-inequalities (1) we may assume without loss of generality that the homology of M is torsion free (otherwise choose rational coefficients and observe that $b_k(M;\mathbb{Z}) = b_k(M;\mathbb{Q})$; see e.g. [SZ88, Satz 10.6.6]). The dimension of the free module $\mathrm{CM}_k(M,f)$ was denoted by c_k. Since the Morse-Witten boundary operator ∂_k is a module homomorphism, there is a splitting of $\mathrm{CM}_k(M,f)$ into the sum of free submodules $\ker\partial_k \oplus \mathrm{im}\,\partial_k$. This implies

$$c_k = \gamma_k + \beta_k, \qquad \gamma_k := \dim\ker\partial_k, \quad \beta_k := \dim\mathrm{im}\,\partial_k, \tag{19}$$

with $\beta_0 = 0 = \beta_{n+1}$. Moreover, by Theorem 4.13

$$b_k = \mathrm{rank}\left(\frac{\ker\partial_k}{\mathrm{im}\,\partial_{k+1}}\right) = \gamma_k - \beta_k. \tag{20}$$

The last identity is due to the assumption that there is no torsion. It follows by induction using (19) and (20) that

$$c_k - c_{k-1} + \cdots \pm c_0 = \beta_{k+1} + b_k - b_{k-1} + \cdots \pm b_0, \qquad k = 0,\ldots,n,$$

and this proves (1).

Relative Morse homology

Given a Morse function f with regular values a and b, one can modify the definition of the Morse chain groups by using as generators only those critical points x with $f(x) \in [a,b]$. The resulting Morse homology groups represent relative singular homology. More precisely, with $M_f^a := \{f \le a\}$ it holds

$$\mathrm{HM}_*^{(a,b)}(M,f,g,Or) \simeq \mathrm{H}_*^{sing}(M_f^b, M_f^a;\mathbb{Z}).$$

Independence of f holds true for such f which are connected by a *monotone* homotopy f_s (meaning that $\partial_s f_s \le 0$ pointwise).

Another point of view is as follows. Given a manifold M with boundary and a negative gradient vector field $-\nabla f$ which is nonzero on ∂M and points outward on some boundary components, denoted by $(\partial M)^{out}$, then

$$\mathrm{HM}_*(M,f,g,Or) \simeq \mathrm{H}_*^{sing}(M,(\partial M)^{out};\mathbb{Z}).$$

As an example consider a 2-sphere in $\mathbb{R}^3$ as in Figure 12, equipped with the height function f and the metric induced by the euclidean metric on $\mathbb{R}^3$. Denote the upper hemisphere by D_u and the lower one by D_ℓ. It is then easy to calculate the absolute and relative homology groups of a disk D, namely

$$\mathrm{H}_0^{sing}(D;\mathbb{Z}) \simeq \mathrm{HM}_0(D_\ell, -\nabla f) = \langle z \rangle = \mathbb{Z},$$
$$\mathrm{H}_2^{sing}(D,\partial D;\mathbb{Z}) \simeq \mathrm{HM}_2(D_u, -\nabla f) = \langle x \rangle = \mathbb{Z}.$$

For all other values of k these homology groups are zero, since there are simply no generators on the chain level.

Morse cohomology

Let (f,g) be a Morse-Smale pair. The Morse chain groups $\mathrm{CM}_k(M,f)$ and cochain groups $\mathrm{CM}^k(M,f) := \mathrm{Hom}(\mathrm{CM}_k(M,f),\mathbb{Z})$ can be identified via the map $\mathrm{Crit}_k f \ni x \mapsto \eta_x$, where $\eta_x(z)$ is one in case $z = x$ and zero otherwise. Define the Morse coboundary operator

$$\delta^k = \delta^k(M,f,g,Or) : \mathrm{CM}^k(M,f) \to \mathrm{CM}^{k+1}(M,f)$$

on a generator y by counting isolated flow lines of the negative gradient flow ending at y (or equivalently those of the *positive* gradient flow emanating from y)

$$\delta^k y := \sum_{x \in \mathrm{Crit}_{k+1} f} n(x,y)x.$$

The homology of this chain complex is called *Morse cohomology* and denoted by $\mathrm{HM}^*(M,f,g,Or)$. It is naturally isomorphic to $\mathrm{H}^*_{sing}(M;\mathbb{Z})$.

Poincaré duality

Consider the natural identifications

$$\mathrm{CM}^k(M,f) \simeq \mathrm{CM}_k(M,f) \simeq \mathrm{CM}_{n-k}(M,-f). \tag{21}$$

Assume that M is orientable and fix an orientation (or restrict to coefficients in $\mathbb{Z}_2$). Taking a choice Or of orientations of all unstable manifolds with respect to $-\nabla f$ determines orientations of the stable manifolds, which equal the unstable manifolds with respect to $-\nabla(-f)$. Let $Or^{-f} = Or^{-f}(Or,[M])$ denote the induced orientations. Under the identification (21) the boundary operators $\delta^k(M,f,g,Or)$ and $\partial_{n-k}(M,-f,g,Or^{-f})$ coincide. Hence

$$\mathrm{HM}^k(M,f,g,Or) \simeq \mathrm{HM}_{n-k}(M,-f,g,Or^{-f}) \simeq \mathrm{HM}_{n-k}(M,f,g,Or).$$

4.5 Real projectice space

Real projective space exhibits enough asymmetry in its (co)homology to be a good testing ground for the (rather informal) remarks above. View $\mathbb{R}\mathrm{P}^2$ as the unit disc in $\mathbb{R}^2$ with opposite boundary points identified and consider a Morse function as in Figure 17 (cf. [Hi76, Ch. 6, § 3, Exc. 6]) having precisely three critical points x, y, z of Morse indices 2,1,0, respectively.

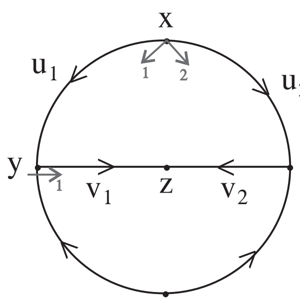

Figure 17: Morse function on $\mathbb{R}\mathrm{P}^2$

Hence

$$\mathrm{CM}_2 = \mathrm{CM}^2 = \langle x \rangle, \quad \mathrm{CM}_1 = \mathrm{CM}^1 = \langle y \rangle, \quad \mathrm{CM}_0 = \mathrm{CM}^0 = \langle z \rangle.$$

Let the orientations of the unstable manifolds be as in Figure 17, then

$$n_{u_1} = n_{u_2} = n_{v_1} = +1, \qquad n_{v_2} = -1,$$

and the (co)boundary operators ∂_k and δ^k act by

$$\begin{aligned} \partial_2 x &= 2y, & \delta^2 x &= 0, \\ \partial_1 y &= 0, & \delta^1 y &= 2x, \\ \partial_0 z &= 0, & \delta^0 z &= 0. \end{aligned}$$

Hence integral Morse (co)homology is given by

$$\begin{aligned} \mathrm{HM}_2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) &= 0, & \mathrm{HM}^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) &= \mathbb{Z}_2, \\ \mathrm{HM}_1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) &= \mathbb{Z}_2, & \mathrm{HM}^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) &= 0, \\ \mathrm{HM}_0(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) &= \mathbb{Z}, & \mathrm{HM}^0(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) &= \mathbb{Z}, \end{aligned} \tag{22}$$

whereas all (co)homology groups with $\mathbb{Z}_2$-coefficients equal $\mathbb{Z}_2$.

It is also interesting to check in this example that, if we replaced in the Morse complex setup the negative by the positive gradient flow (thereby obtaining an ascending boundary operator), then the homology of this new chain complex would reproduce the cohomology groups in (22).

It is an instructive exercise to extend the Morse function in Figure 17 to $\mathbb{R}\mathrm{P}^3$, viewed as the unit three disc in $\mathbb{R}^3$ with opposite boundary points identified, and calculate its Morse (co)homology.

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