# Introduction to Sobolev Spaces

Lecture Notes
MM692 2018-2

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# Contents

# Chapter 1

# Introduction

These are Lecture Notes [1] written for the last third of the course “MM692 Análise Real II” in 2018-2 at UNICAMP. Originally motivated by [MS04, App. B.1] this text is essentially a compilation of the presentations in [AF03, Bre11,Eva98,GT01]. Given a solid background in measure theory, we also highly recommend the book [Zie89] whose geometric measure theory point of view offers deep insights and strong results. An excellent source that analyses Sobolev spaces as part of a wider context is [Ste70, Ch. V].

Differentiation and integration are reverse operations. The notion of weak derivative combines both worlds. It is more general than the usual partial derivative and at the same time it makes available the powerful tool box of integration theory. Weak derivatives and **Sobolev spaces**, the spaces of functions that admit weak derivatives, are typically used in applications as an intermediate step towards solution. For instance, to solve a PDE it is often easier (bigger space) to establish first existence of a weak solution (ask for weak derivatives only) and then show in a second step, regularity, that the weak solution found is actually differentiable in the usual sense.

We made an effort to spell out details in the case of *Lipschitz domains* $D$, because often in the literature one only finds proofs for the $C^1$ case. The relevant theorems are the Approximation Theorem 5.1.7 and the Extension Theorem 5.2.1. The assumptions in these theorems determine the assumptions in Section 6 on the Sobolev inequalities.

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[1] The present manuscript is not complete and won't be in the near future.

## 1.1 Notation and conventions

| partial derivative $\partial_i u = \partial_{x_i}$ or $D^{e_i}$ – weak derivative $u_{x_i}$ or $u_{e_i}$ |
|---|

$\Omega$ open subset of $\mathbb{R}^n$ where $n \geq 1$

$\partial A$ $:= \bar{A} \cap \overline{A^{\mathrm{C}}}$ **boundary of set** $A \subset \mathbb{R}^n$ where $\bar{A}$ is the closure of $A$

$Q \Subset \Omega$ **pre-compact subset** $Q$ of $\Omega$:
An open subset of $\Omega$ whose closure $\bar{Q}$ is compact and contained in $\Omega$

$D \Subset \Omega$ Lipschitz domain ($\partial D$ is locally a Lipschitz graph; see Definition 5.1.5)

$K$ compact set (i.e. bounded and closed)

$\boldsymbol{u} = [u]$ equivalence class; the boldface notation[2] eases $D^\alpha[u]$ to $D^\alpha \boldsymbol{u}$, whereas the bracket notation clarifies the definition $D^\alpha \boldsymbol{u} := [D^\alpha u]$ as opposed to $\boldsymbol{D^\alpha u}$

### $C^k$ spaces

A map taking values in the real line $\mathbb{R}$ is called a **function**.

$C^k(\Omega)$ set of functions on $\Omega$ all of whose partial derivatives up to order $k$ exist and are continuous, equipped with (but incomplete under) the norm

$\|\cdot\|_{C^k}$ maximum of sup-norms of all partial derivatives up to order $k$

$C^k_{\mathrm{b}}(\Omega)$ $:= \{f \in C^k(\Omega) \colon \|f\|_{C^k} < \infty\}$ is the Banach space under $\|\cdot\|_{C^k}$ of $k$ times bounded continuously differentiable functions on $\Omega$ $\quad C^k(K) = C^k_{\mathrm{b}}(K)$

$C^k(\bar{\Omega})$ consists[3] of those functions $f : \Omega \to \mathbb{R}$ in $C^k(\Omega)$ whose partial derivatives up to order $k$ are uniformly continuous on bounded subsets of $\Omega$; cf. [Eva98, §A.3]. So each such derivative continuously extends to $\bar{\Omega}$

**Note.** In $C^k(\bar{\Omega})$ the bar is *notation only*, it does not denote the closure. Otherwise, there is ambiguity for unbounded sets, such as $\Omega = \mathbb{R}^n = \overline{\mathbb{R}^n}$.

$C^k_{\mathrm{b}}(\bar{\Omega})$ $:= \{f \in C^k(\bar{\Omega}) \colon \|f\|_{C^k} < \infty\}$ is a closed (hence Banach) subspace of the Banach space $C^k_{\mathrm{b}}(\Omega)$; cf. [AF03, §1.28] $\quad C^k(\bar{Q}) = C^k_{\mathrm{b}}(\bar{Q})$

$C^\infty_\cdot(\cdot)$ $:= \cap_{k=0}^\infty C^k_\cdot(\cdot)$

[2] concerning notation we distinguish functions $u$ and their equivalence classes $[u] = \boldsymbol{u}$, but for operators on them we use the same symbol, e.g. $D$, since $Du$ and $D\boldsymbol{u}$ indicate context

[3] In the pre-compact case $C^k(\bar{Q})$ is the set of restrictions to $Q$ of smooth functions defined on a neighborhood of $\bar{Q}$: uniformly continuous derivatives up to order $k$ follows from compactness of $\bar{\Omega}$.

### Hölder spaces

Observe: Hölder $\Rightarrow$ uniform continuity $\Rightarrow$ continuous extension to boundary.

$|f|_{C^{0,\mu}(\Omega)}$ $:= \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\mu}$ **Hölder coefficient** of $f$, Hölder exponent $\mu \in (0,1]$. One calls $f$ $\mu$-Hölder continuous on $\Omega$ $:\Leftrightarrow |f|_{C^{0,\mu}(\Omega)} < \infty$. Hölder implies uniform continuity, cf. Figure A.1, hence $f$ extends continuously to $\bar\Omega$

$\|f\|_{C^{k,\mu}(\Omega)}$ $:= \max_{|\beta|\le k} \|D^\beta f\|_{C^0(\Omega)} + \max_{|\alpha|=k} |D^\alpha f|_{C^{0,\mu}(\Omega)}$ Hölder norm[4]

$C^{k,\mu}(\Omega)$ $:= \{f \in C^k(\Omega) \colon |D^\beta f|_{C^{0,\mu}} < \infty \;\forall |\beta| \le k\} = C^{k,\mu}(\bar\Omega)$ vector space of functions which are *uniformly* Hölder continuous on $\Omega$ and so are all derivatives up to order $k$ (hence they all extend continuously to the boundary)

$C^{k,\mu}_{\mathrm{loc}}(\Omega)$ $:= \{f \in C^k(\Omega) \colon f \in C^{k,\mu}(Q) \;\forall Q \Subset \Omega\} \supset C^{k,\mu}(\Omega)$ local Hölder space

$C^{k,\mu}_{\mathrm{b}}(\Omega)$ $:= \{f \in C^{k,\mu}(\Omega) \colon \|f\|_{C^{k,\mu}} < \infty\}$ **Hölder Banach space** with respect to the norm $\|\cdot\|_{C^{k,\mu}}$; idea of proof e.g. here or here $C^{k,\mu}_{\mathrm{b}}(Q) = C^{k,\mu}(Q)$[5]

Integration will always be with respect to Lebesgue measure $m$ on $\mathbb{R}^n$, unless mentioned otherwise. We follow the notation in [Eva98, App. A] for derivatives:

### Derivatives

Suppose $u : \Omega \to \mathbb{R}$ is a function and $x = (x_1, \dots, x_n) \in \Omega$ is a point.

(a) $\frac{\partial u}{\partial x_i}(x) := \lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h}$, provided this limit exists.

(b) We abbreviate $\frac{\partial u}{\partial x_i}$ by $\partial_{x_i} u$ or $\partial_i u$. Also $\partial_{ij} u := \frac{\partial^2 u}{\partial x_i \partial x_j}$ etc.

(c) **Multi-index notation.**

(i) A list $\alpha = (\alpha_1, \dots, \alpha_n)$ of integers $\alpha_i \ge 0$ is a **multi-index** of **order**

$$|\alpha| := \alpha_1 + \dots + \alpha_n.$$

Set $\alpha! := \alpha_1! \dots \alpha_n!$ and for $\alpha \ge \beta$ (all $\alpha_i \ge \beta_i$) set $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

(ii) For a multi-index $\alpha$ the associated **partial derivative** is denoted by

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial {x_1}^{\alpha_1} \dots \partial {x_n}^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x).$$

(iii) The **list of all partial derivatives of order $k$** $\in \mathbb{N}_0$ is denoted by

$$D^k u(x) := \left(D^\alpha u(x)\right)_{|\alpha|=k} \in \mathbb{R}^{n^k}.$$

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[4] Why $|\alpha| = k$ suffices? If $\|f\|_{C^k}$ is finite, so are all Hölder coefficients of $D^\beta f$ with $|\beta| < k$.

[5] '$\supset$' All derivatives of $f \in C^{k,\mu}(Q)$ up to order $k$ are $\mu$-Hölder, thus uniformly continuous, hence they extend to the closure - which is compact - so they are bounded.

(iv) **Special cases.** If $k = 1$, one obtains a list $D^1 u$ denoted by

$$Du = (\partial_1 u, \dots, \partial_n u) = \text{gradient vector}$$

and we set $|Du| := \sqrt{|\partial_1 u|^2 + \dots + |\partial_n u|^2}$. The list $Du$ of first order partial derivatives is also called the **strong gradient**.
If $k = 2$, one obtains a symmetric $n \times n$ matrix

$$D^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \dots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ & \ddots & \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix} = \text{Hessian matrix.}$$

(iv) **Weak derivatives.** We usually denote the weak derivative of $u$ corresponding to $\alpha$ by $u_\alpha$. In the smooth case $u_\alpha = D^\alpha u$ coincides with the ordinary derivative.
**Order one.** Weak derivatives in direction of the $i^{\text{th}}$ coordinate unit vector $\alpha = e_i := (0, \dots, 0, 1, 0, \dots, 0)$ are denoted by $u_{e_i}$.
In abuse of notation we also denote by $Du$ the list $(u_{e_1}, \dots, u_{e_n})$ of weak derivatives. But we call it the **weak gradient** for distinction.

# Chapter 2

# $L^p$-spaces

This chapter is an exposition of notions and results a.e. without proofs.

## 2.1 Borel and Lebesgue measure space on $\mathbb{R}^n$

### Topology

The **standard topology** on euclidean space $\mathbb{R}^n$ is the collection $\mathcal{U}_n$ of subsets of $\mathbb{R}^n$ that arises as follows. Consider the family of all open balls centered at the points of $\mathbb{R}^n$ of all radii. Now add all finite intersections of these balls. Then add all arbitrary unions of members of the enlarged family to obtain $\mathcal{U}_n \subset 2^{\mathbb{R}^n}$. By $2^X$ we denote the **power set** of a set $X$: the collection of all subsets of $X$. The elements $U$ of $\mathcal{U}_n$ are called **open sets** and their complements $U^{\mathrm{C}} := \mathbb{R}^n \setminus U$ are called **closed sets**.

**Lemma 2.1.1.** *Every open subset $\Omega$ of $\mathbb{R}^n$ is of the form $\Omega = \cup_{i=1}^\infty K_i$ for a nested sequence of compact subsets $K_i \subset K_{i+1} \subset \Omega$.*

*Proof.* [Bre11, Cor. 4.23]: $K_i := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Omega^{\mathrm{C}}) \geq 2/i \text{ and } |x| \leq i\}$. □

### Measure compatible with topology – Borel measure

There are many ways to enlarge a given collection $\mathcal{C}$ of subsets of $\mathbb{R}^n$ to obtain a family of subsets that satisfies the axioms of a $\sigma$-algebra; see e.g. [Sal16, Ch. 1]. However, there is a smallest such family, denoted by $\mathcal{A}_{\mathcal{C}} \subset 2^{\mathbb{R}^n}$ and called the smallest $\sigma$-algebra on $\mathbb{R}^n$ that contains the collection $\mathcal{C}$. The smallest $\sigma$-algebra on a topological space that contains all open sets is called the Borel $\sigma$-algebra. In case of $\mathbb{R}^n$ with the standard topology $\mathcal{U}_n$ we use the notation $\mathcal{B}_n := \mathcal{A}_{\mathcal{U}_n}$ for the **Borel $\sigma$-algebra on $\mathbb{R}^n$**.

A **measure** is a function $\mu : \mathcal{A} \to [0, \infty]$ whose domain is a $\sigma$-algebra such that, firstly, at least one family member $A \in \mathcal{A}$ has finite measure $\mu(A) < \infty$

and, secondly, the function $\mu$ is **$\sigma$-additive**.[1] The elements $A$ of a $\sigma$-algebra are called **measurable sets**, those of measure zero, i.e. $\mu(A) = 0$, **null sets**.

On $\mathcal{B}_n$ there is a unique measure $\mu$, called **Borel measure on $\mathbb{R}^n$**, which is translation invariant and assigns measure 1 to the unit cube $[0, 1]^n$.

### Completion of Borel measure – Lebesgue measure

It is reasonable to expect that the measure of a subset should not be larger than the measure of the ambient set. If the ambient set has measure zero, subsets should be of measure zero, too. However, the domain of a measure is a $\sigma$-algebra, a family of sets. But a subset $C$ of a member $A$, even if $A$ is of measure zero, is not necessarily a member itself, hence not in the domain of the measure.

Such annoying kind of incompleteness happens in Borel measure space $(\mathbb{R}^n, \mathcal{B}_n, \mu)$, namely, subsets of null sets are not necessarily measurable. However, there is a completion procedure that results in a larger collection $\mathcal{A}_n \supset \mathcal{B}_n$, the **Lebesgue $\sigma$-algebra**, together with a measure $m : \mathcal{A}_n \to [0, \infty]$, the **Lebesgue measure**. The desired completeness property holds true: subsets of Lebesgue null sets are Lebesgue null sets. Moreover, the restriction of Lebesgue measure to the Borel $\sigma$-algebra $m|_{\mathcal{B}_n} = \mu$ coincides with Borel measure. For details see e.g. [Sal16, Ch. 2].

### Functions – equality almost everywhere and measurability

One says that two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are equal **almost everywhere**, in symbols **$f = g$ a.e.**, if the set $\{f \neq g\} \subset \mathbb{R}^n$ of points on which $f$ and $g$ differ is a **Lebesgue null set**: an element of $\mathcal{A}_n$ of Lebesgue measure zero.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **Lebesgue measurable** if $\mathbb{R}^n$ is equipped with the Lebesgue $\sigma$-algebra $\mathcal{A}_n \subset 2^{\mathbb{R}^n}$ and $\mathbb{R}$ with the Borel $\sigma$-algebra $\mathcal{B}_1 \subset 2^{\mathbb{R}}$ and the pre-images $f^{-1}(B) \in \mathcal{A}_n$ of $\mathcal{B}_1$-measurable sets, i.e. $B \in \mathcal{B}_1$, are $\mathcal{A}_n$-measurable sets; see [Sal16, Def. 2.2]. The function is called **Borel measurable** if the pre-images $f^{-1}(B) \in \mathcal{B}_n \subset \mathcal{A}_n$ even lie in the Borel $\sigma$-algebra.

Recall that a map between topological spaces is **continuous** if pre-images of open sets are open. Every continuous map $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable; see e.g. [Sal16, Thm. 1.20].

### Measurable support

It is common to define the support of a function on a topological space to be the complement of the largest open set on which $f$ vanishes or, equivalently, the closure of the set $\{f \neq 0\}$. The usual symbol is $\operatorname{supp} f$. In contrast, for measurable functions it is useful to replace 'vanishes' by 'vanishes almost everywhere'; cf. [Bre11, §4.4]. The symbol will be $\operatorname{supp}_{\mathrm{m}} f$. This way functions which are equal a.e. will have the same support in the new sense.

[1] If $(A_i)_{i=1}^\infty \subset \mathcal{A}$ is a countable collection of pairwise disjoint sets then $\mu$ assigns to their union the sum of all individual measures, in symbols $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

**Definition 2.1.2** (Support of measurable functions)**.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesque measurable. Consider the family $(\Omega_\lambda)_{\lambda\in\Lambda}$ of all open subsets of $\mathbb{R}^n$ such that $f = 0$ a.e. on each of them. By definition the **(measurable) support of $f$** is the complement of the union $\Omega := \cup_{\lambda\in\Lambda} \Omega_\lambda$, in symbols

$$\operatorname{supp_m} f := \Omega^{\mathrm{C}}.$$

Since $\Omega$ is open $\operatorname{supp_m} f$ is closed.

**Exercise 2.1.3.** Check that $\operatorname{supp} \chi_{\mathbb{Q}} = \mathbb{R}$, while $\operatorname{supp_m} \chi_{\mathbb{Q}} = \emptyset$. The latter seems more reasonable for a function that is zero almost everywhere.

**Exercise 2.1.4** (Ordinary and measurable support)**.**

(i) Show that $f = 0$ a.e. on $\Omega$, equivalently, that $N := \{f \neq 0\}$ is a null set.

(ii) If $f = g$ almost everywhere, then $\operatorname{supp_m} f = \operatorname{supp_m} g$.

(iii) For continuous functions both supports $\operatorname{supp_m} f = \operatorname{supp} f$ coincide.

[Hints: (i) That $f = 0$ a.e. on $\Omega_\lambda$ means $N_\lambda := \{f \neq 0\} \cap \Omega_\lambda$ is a null set. While $N = \cup_{\lambda\in\Lambda} N_\lambda$, the union is not necessarily countable. Recall that the standard topology of $\mathbb{R}^n$ has a countable base: There is a countable family $(O_n)$ of open sets such that any open set, say $\Omega_\lambda$, is the union of some $O_n$'s.]

### Integration of Lebesgue measurable functions

The construction of the Lebesgue integral starts out from characteristic functions $\chi_A : \mathbb{R}^n \to \mathbb{R}$. By intuition, for a constant $c \geq 0$, the value of the integral of $c\chi_A$ should be the area below its graph, that is the measure of $A$ (length if $n = 1$) times the height $c$. But $A$ only has a measure if it is a member of the $\sigma$-algebra. In this case one defines $\int_{\mathbb{R}^n} c\chi_A \, dm := c \cdot m(A)$ and correspondingly in case of finite sums $s = \sum_1^k c_i \chi_{A_i} \geq 0$ (called simple functions).

The fact that the **characteristic function** $\chi_A : \mathbb{R}^n \to \mathbb{R}$ of a subset $A \subset \mathbb{R}^n$ is Lebesgue measurable iff the set $A$ is Lebesgue measurable, brings in measurability of the functions to be integrated. A non-negative Lebesgue measurable function $f : \mathbb{R}^n \to [0, \infty]$ can be approximated from below by simple functions and one defines the **Lebesgue integral of such $f \geq 0$**, denoted by

$$\int_{\mathbb{R}^n} f \, dm \in [0, \infty],$$

as the supremum of the integrals of simple functions $s$ with $0 \leq s \leq f$. Decompose a real valued Lebesgue measurable $f = f^+ - f^- : \mathbb{R}^n \to \mathbb{R}$ into its positive and negative parts $f^\pm(x) := \max\{\pm f(x), 0\} \geq 0$ and define $\int f := \int f^+ - \int f^-$ if at least one of the two terms is finite. Such $f$ is said to **admit Lebesgue integration**. If both terms are finite, that is if $\int |f| = \int f^+ + \int f^- < \infty$, the measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is called (Lebesgue) **integrable**. By $\mathcal{L}^1(\mathbb{R}^n)$ one denotes the set of all integrable functions on $\mathbb{R}^n$. For $A \in \mathcal{A}_n$ define

$$\int_A f\, dm := \int_{\mathbb{R}^n} \chi_A f\, dm.$$

The frequent problem of interchanging integration and limit is settled by the

**Theorem 2.1.5** (Lebesgue dominated convergence theorem)**.** *Let all functions be defined on a measurable set $E$. Let $f_k$ be a sequence of measurable functions that converges pointwise a.e. to a measurable function $f$ and is dominated by an integrable function $g$ in the sense that*

$$\forall k \in \mathbb{N}: \quad |f_k(x)| \le g(x) \quad \text{for a.e. } x \in E.$$

*Then the $f_k$ and also $f$ are integrable and*

$$\int_E f\, dm = \lim_{k\to\infty} \int_E f_k\, dm. \tag{2.1.1}$$

*Proof.* E.g. [Sal16, Thm. 1.45 and text after Cor. 1.56] or [Fol99, Thm. 2.24]. □

**Theorem 2.1.6** (Generalized Lebesgue dominated convergence theorem)**.** *Let all functions be defined on a measurable set $E$. Let $f, \{f_k\}$ be measurable with $f_k \to f$ a.e. and $g, \{g_k\}$ be integrable with $g_k \to g$ a.e. and $|f_k| \le g_k$ pointwise*[2] *for each $k$. Then the $f_k$ and also $f$ are integrable*

$$\lim_{k\to\infty} \int_E g_k = \int_E g, \qquad \Rightarrow \qquad \lim_{k\to\infty} \int_E f_k = \int_E f. \tag{2.1.2}$$

*Proof.* Exercise. □

**Convention 2.1.7.** Throughout we work with Lebesgue measure $m : \mathcal{A}_n \to [0,\infty]$. **Measurable** means Lebesgue measurable, unless mentioned otherwise. Instead of $\int_{\mathbb{R}^n} f\, dm$ we usually write $\int_{\mathbb{R}^n} f$ or even $\int f$. If we wish to emphasize the variable $f = f(x)$ of the function we shall write $\int_{\mathbb{R}^n} f(x)\, dx$ or $\int_{\mathbb{R}^n} f\, dx$.

## 2.2 Definition

### Vector space $\mathcal{L}^p$ of functions

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. For finite $p \in [1,\infty)$ the ***$L^p$*-norm of *$f$*** is the extended real defined by

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f|^p\right)^{1/p} \in [0,\infty]. \tag{2.2.3}$$

[2] here pointwise, as opposed to a.e. pointwise, is crucial so to $g_k - f_k \ge 0$ applies Fatou's Lemma for non-negative measurable functions

On the other hand, consider the (possibly empty) semi-infinite interval

$$I_f := \{c \in [0,\infty) \colon \text{the set } \{|f| > c\} \text{ is of measure zero}\} \subset [0,\infty).$$

The elements $c$ of $I_f$ tell that $|f| > c$ happens only along a null set. The infimum of such $c$ is called the **$L^\infty$-norm of $f$** and denoted by

$$\|f\|_\infty := \inf I_f = \operatorname{ess\,sup} |f|. \tag{2.2.4}$$

By convention $\inf \emptyset = \infty$, hence $\|f\|_\infty \in [0,\infty]$

**Definition 2.2.1** ($\mathcal{L}^p$-spaces)**.** Suppose $p \in [1,\infty]$. A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ of finite $L^p$-norm, i.e. $\|f\|_p < \infty$, is called ***p*-integrable** or an ***L^p*-function**. A 1-integrable function is called **integrable**. The set $\mathcal{L}^p(\mathbb{R}^n)$ of $L^p$-functions is a real vector space. For finite $p$ closedness under addition holds by Minkowski's inequality. Failure of non-degeneracy causes that the function $\|\cdot\|_p$ is actually not a norm, at least not on $\mathcal{L}^p(\mathbb{R}^n)$.

**Definition 2.2.2** (Local $\mathcal{L}^p$-spaces)**.** Suppose $p \in [1,\infty]$. Let $\mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$ be the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\chi_Q f$ is $p$-integrable for every pre-compact $Q \Subset \mathbb{R}^n$ or, equivalently, such that $\chi_K f$ is $p$-integrable for every compact $K \subset \mathbb{R}^n$.

**Definition 2.2.3** (Hölder conjugates)**.** Two reals $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ are called **Hölder conjugates** of one another. It is also common to denote the Hölder conjugate if $p$ by $p'$. Suppose $p, q, r \in [1,\infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $p, q$ are called **Hölder *r*-conjugates** of one another.

**Theorem 2.2.4** (Hölder)**.** *Suppose $p, q \in [1,\infty]$ are Hölder conjugates and $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $g \in \mathcal{L}^q(\mathbb{R}^n)$. Then the product $fg$ is integrable and*

$$\int_{\mathbb{R}^n} |fg| \le \|f\|_p \, \|g\|_q \,. \tag{2.2.5}$$

*Proof.* See e.g. [Sal16, Thm. 4.1]. □

The theorem also applies to functions $f, g : A \to \mathbb{R}$ which are defined on a measurable set $A \in \mathcal{A}_n$ and whose extensions $\tilde{f}, \tilde{g} : \mathbb{R}^n \to \mathbb{R}$ to $\mathbb{R}^n$ by zero are measurable. In this just apply the theorem to the extensions.

**Corollary 2.2.5** (Hölder-$r$)**.** *Suppose $p, q \in [1,\infty]$ are Hölder $r$-conjugates and $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $g \in \mathcal{L}^q(\mathbb{R}^n)$. Then $fg \in \mathcal{L}^r(\mathbb{R}^n)$ and*

$$\|fg\|_r \le \|f\|_p \, \|g\|_q \,. \tag{2.2.6}$$

*Proof.* Apply Hölder to $|u|^r |v|^r \in \mathcal{L}^1$ with Hölder conjugates $p/r$ and $q/r$. □

**Exercise 2.2.6.** Suppose $p_1, \dots p_\ell, r \in [1,\infty]$ satisfy $\frac{1}{p_1} + \dots + \frac{1}{p_\ell} = \frac{1}{r}$ and $f_i \in \mathcal{L}^{p_i}(\mathbb{R}^n)$ for $i = 1, \dots, \ell$. Then the product $f_1 f_2 \dots f_\ell$ lies in $\mathcal{L}^r(\mathbb{R}^n)$ and

$$\|f_1 f_2 \dots f_\ell\|_r \le \|f_1\|_{p_1} \dots \|f_\ell\|_{p_\ell} \,. \tag{2.2.7}$$

**Theorem 2.2.7** (Lebesgue dominated $L^p$ convergence theorem)**.** *Let all functions be defined on a measurable set $E$. Suppose $f_k$ is a sequence of measurable functions that converges pointwise a.e. to a measurable function $f$ and is dominated by a $p$-integrable function $g$ for some $p \in [1, \infty)$ in the sense that*

$$\forall k \in \mathbb{N}: \quad |f_k(x)| \le g(x) \quad \textit{for a.e. } x \in E.$$

*Then the $f_k$ and also $f$ are $p$-integrable and $f_k$ converges to $f$ in $L^p$, in symbols*

$$\lim_{k\to\infty} \|f_k - f\|_p = 0.$$

*Proof.* Theorem 2.1.5 with $F_k := |f_k - f|^p$, $F := 0$, and $G := (2g)^p$. □

**Theorem 2.2.8** (Generalized Lebesgue dominated $L^p$ convergence theorem)**.** *Let $p \in [1, \infty)$. Let all functions be defined on $E$. Let $f, \{f_k\}$ be measurable with $f_k \to f$ a.e. and $g, \{g_k\}$ be $p$-integrable with $g_k \to g$ a.e. and $|f_k| \le g_k$ pointwise for each $k$. Then the $f_k$ and also $f$ are $p$-integrable and*

$$\|g_k - g\|_p \to 0 \qquad \Rightarrow \qquad \|f_k - f\|_p \to 0. \tag{2.2.8}$$

*Proof.* Exercise. □

Looks like a complification of Theorem 2.1.6 only? Well, the theorem can be a real peacemaker, just wait for the action, e.g. in Proposition 4.1.21.

### Banach space $L^p$ of equivalence classes

The function $\mathcal{L}^p(\mathbb{R}^n) \to [0, \infty)$, $f \mapsto \|f\|_p$, satisfies all norm axioms except non-degeneracy. Indeed $\|f\|_p = 0$ only tells that $f = 0$ almost everywhere. Thus it is natural to quotient out by equality almost everywhere to obtain the vector space of equivalence classes

$$L^p(\mathbb{R}^n) := \mathcal{L}^p(\mathbb{R}^n)/\sim, \qquad L^p_{\mathrm{loc}}(\mathbb{R}^n) := \mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}^n)/\sim,$$

where by definition $f \sim g$ if $f = g$ almost everywhere. We use either notation $[f]$ or $\boldsymbol{f}$ for equivalence classes. Because the Lebesgue integral is insensitive to sets of measure zero, the definition

$$\|\boldsymbol{f}\|_p := \|f\|_p$$

makes sense and provides a **norm on $L^p(\mathbb{R}^n)$**.

**Exercise 2.2.9.** Illustrate by examples that the property "*$u$ has a continuous representative*" is not the same as "*$u$ is continuous a.e.*".

## 2.3 Basic properties

**Theorem 2.3.1.** (i) $L^p(\mathbb{R}^n)$ *is a Banach space for* $p \in [1,\infty]$.

(ii) $L^p(\mathbb{R}^n)$ *is* ***separable***[3] *for finite* $p \in [1,\infty)$.

(iii) $L^p(\mathbb{R}^n)$ *is* ***reflexive***[4] *for finite* $p \in (1,\infty)$ *larger* 1.

(iv) $C_0(\mathbb{R}^n)$ *is a dense subset of* $L^p(\mathbb{R}^n)$ *for finite* $p \in [1,\infty)$.[5]

*Proof.* See e.g. [Sal16, (i) Thm. 4.9, (ii) Thm. 4.13, (iv) Thm. 4.15] or [Bre11, (i) Thm. 4.8, (ii) Thm. 4.13, (iii) Thm. 4.10]. □

The properties (i) completeness, (ii) separability, and (iii) reflexivity are **hereditary** to Banach subspaces, that is they are inherited by closed subspaces of a Banach space. Part (iv) will be improved from continuous to smooth as an application of convolution in Theorem 3.2.3 (ii).

**Lemma 2.3.2** (Continuity of shift operator in compact-open topology)**.** *Given* $p \in [1,\infty)$, *the* ***shift operator*** *is for* $\xi \in \mathbb{R}^n$ *defined by*

$$\tau_s : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad \boldsymbol{f} \mapsto \tau_s \boldsymbol{f} := \boldsymbol{f}(\cdot + s\xi), \qquad s \in [0,\infty).$$

*It satisfies* $\tau_t\tau_s = \tau_{t+s}$ *and* $\tau_0 = \mathbb{1}$. *Given* $\boldsymbol{f} \in L^p(\mathbb{R}^n)$, *the path*

$$\gamma : [0,\infty) \to L^p(\mathbb{R}^n), \quad s \mapsto \tau_s \boldsymbol{f},$$

*is continuous.*

*Proof.* It suffices to prove continuity at $s = 0$. Suppose $f \in \boldsymbol{f} \in L^p(\mathbb{R}^n)$ and let $\varepsilon > 0$. It would be nice if $f$ was uniformly continuous. So let's approximate $f$ by $\phi \in C_0(\mathbb{R}^n)$ with $\|f - \phi\|_p < \varepsilon/3$ using Theorem 2.3.1. Since $\phi$ is continuous and of compact support it is uniformly continuous. Hence there is a constant $\delta = \delta(\varepsilon) > 0$ such that $|s| < \delta$ implies $\|\tau_s\phi - \phi\|_p < \varepsilon/3$. We get that

$$\|\tau_s f - f\|_p \le \|\tau_s(f-\phi)\|_p + \|\tau_s\phi - \phi\|_p + \|\phi - f\|_p < \varepsilon/3 + \varepsilon/3 + \varepsilon/3.$$

We used linearity of $\tau_s$ and $\|\tau_s(f-\phi)\|_p = \|f-\phi\|_p$ since $\tau_s$ is an isometry. □

**Exercise 2.3.3** (Shift operator in operator norm topology)**.** Is the shift operator continuous with respect to the norm topology? I.e. is

$$\tau : [0,\infty) \to \mathcal{L}(L^p(\mathbb{R}^n)), \quad s \mapsto \tau_s,$$

continuous? The **operator norm** is given by $\|\tau_s\| := \sup_{f \neq 0} \frac{\|\tau_s f\|_p}{\|f\|_p}$.

[3] there must be a dense sequence

[4] the canonical injection $J : E \rightarrowtail E^{**}$ that assigns to $x \in E$ the continuous linear functional on $E^*$ given by $[E^* \to \mathbb{R}\colon f \mapsto f(x)] \in E^{**}$ must be surjective, hence $E \simeq E^{**}$ canonically

[5] 'subset' actually refers to the image of the natural injection $C_0(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $f \mapsto [f]$

# Chapter 3

# Convolution

Roughly speaking, the convolution product associates to two adequately integrable functions $f$ and $g$ on $\mathbb{R}^n$ an integrable function, denoted by $f * g$, which inherits nice properties from one of the factors, say smoothness of $f$, but still resembles very much the other factor $g$, if $f$ is chosen to have very small support. A major application is to $\|\cdot\|_1$-approximate an integrable $g$ by the smooth function $f * g$ by choosing $f$ appropriately. One obtains the fundamental density result that the smooth compactly supported functions form a dense subset among the integrable ones. This remains valid for the class of functions with domain $\Omega \subset \mathbb{R}^n$. We recommend the presentations in [Sal16, §7.5] and [Bre11, §4.4].

An integrable function $f$ on $\Omega \subset \mathbb{R}^n$ naturally corresponds to an integrable function on $\mathbb{R}^n$ by extending $f$ by zero outside $\Omega$. We denote this **natural zero extension** again by $f$. This does *not* work for locally integrable functions. A way out is to restrict first to a closed subset of $\Omega$ and then extend the restriction by zero. See Section 3.3 on mollification of functions with domain $\Omega$.

## 3.1 Convolution of functions

In this section we deal with functions on $\mathbb{R}^n$. If a function $f$ is given on $\Omega \subset \mathbb{R}^n$ just replace it by the natural zero extension $f : \mathbb{R}^n \to \mathbb{R}$ which assigns constantly the value zero outside $\Omega$.

**Definition 3.1.1.** For Lebesgue measurable $f, g : \mathbb{R}^n \to \mathbb{R}$ define the bad set[1]

$$E(f, g) := \left\{ x \in \mathbb{R}^n \;\middle|\; h(x) := \int_{\mathbb{R}^n} |f(x-y)g(y)| \, dy = \infty \right\}.$$

[1] The function $y \mapsto f(x-y)g(y) = (f \circ \phi_x)(y)g(y)$ is Lebesgue measurable, independent of $x$, since $f, g$ are and $\phi_x(y) := x - y$ is a homeomorphism of $\mathbb{R}^n$. Thus not to be Lebesgue integrable, as required in [Sal16, (7.30)] to define $E(f, g)$, just means 'infinite integral'.

The **convolution of $f$ and $g$** is the function $f * g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(f * g)(x) := \begin{cases} \int_{\mathbb{R}^n} f(x-y)g(y)\,dy & \text{, for } x \in E(f,g)^{\mathrm{C}}, \\ 0 & \text{, for } x \in E(f,g). \end{cases}$$

Of course, to have any chance that the function $f * g : \mathbb{R}^n \to \mathbb{R}$ be measurable, let alone integrable, one needs that $E(f,g)$ is a measurable set to start with. Ideally $E(f,g)$ should be of measure zero, so it becomes invisible when integrating $f * g$ over $\mathbb{R}^n$. Note that $m(E(f,g)) = 0$ means that the function $y \mapsto f(x-y)g(g)$ is integrable for a.e. $x \in \mathbb{R}^n$. If *both* $f$ and $g$ are compactly supported, so is their convolution $f * g$ by Theorem 3.1.6 (i).

**Theorem 3.1.2.** *For Lebesgue measurable $f, g, \tilde{f}, \tilde{g} : \mathbb{R}^n \to \mathbb{R}$ it is true that*

(i) *if $f = \tilde{f}$ a.e. and $g = \tilde{g}$ a.e., then $E(f,g) = E(\tilde{f},\tilde{g})$ and $f * g = \tilde{f} * \tilde{g}$;*

(ii) *the bad set $E(f,g) \in \mathcal{B}_n$ is Borel and $f * g$ is Borel measurable;*

(iii) *convolution is commutative: indeed $E(f,g) = E(g,f)$ and $f * g = g * f$.*

*Proof.* [Sal16, Thm. 7.32]. □

While the theorem asserts that the convolution of measurables is measurable, even Borel measurable, and so is the bad set, the question whether the convolution of integrables is integrable is answered next. The answer is positive, even for generalized ($r \neq \infty$) conjugate exponents.

**Theorem 3.1.3** (Young's inequality)**.** *Given Hölder $r$-conjugates $p, q \in [1, \infty]$, suppose $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then the bad set $E(f,g)$ is of measure zero, empty for $r = \infty$ i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and the convolution satisfies the estimate*

$$\|f * g\|_r \le \|f\|_p \|g\|_q . \tag{3.1.1}$$

*So the Borel measurable function $f * g$ lies in $\mathcal{L}^r(\mathbb{R}^n)$.*

*Proof.* [Sal16, Thm. 7.33]. Emptiness of $E$ for $r = \infty$ holds by Hölder (2.2.5) as $h(x) = \int_{\mathbb{R}^n} |f(x-y)g(y)|\, dm(y) \le \|f(x-\cdot)\|_p \|g\|_q = \|f\|_p \|g\|_q < \infty$. □

**Remark 3.1.4** (Case $r = \infty$)**.** Theorem 3.1.3 asserts that in case of Hölder conjugate exponents, i.e. $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ where $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, one has an everywhere defined convolution $f * g \in \mathcal{L}^\infty(\mathbb{R}^n)$. So $f * g$ is bounded. By Theorem 3.1.6 (ii) below $f * g$ is even uniformly continuous.

**Remark 3.1.5** (Convolution and local integrability)**.** Let $p, q \in [1, \infty]$ be Hölder $r$-conjugates. For locally $p$-integrable $f \in \mathcal{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$ and compactly supported $g \in \mathcal{L}^q(\mathbb{R}^n)$ the bad set $E(f,g)$ is still of measure zero and the convolution $f * g \in \mathcal{L}^r_{\mathrm{loc}}(\mathbb{R}^n)$ is locally $r$-integrable; cf. paragraph after Def. 7.34 in [Sal16].

**Continuity, differentiability, support**

**Theorem 3.1.6.** *Let $p, q \in [1, \infty]$ be conjugate. Then the following is true.*

(i) *If $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $g \in \mathcal{L}^1(\mathbb{R}^n)$, then* $\operatorname{supp}_{\mathrm{m}}(f * g) \subset \overline{\operatorname{supp}_{\mathrm{m}} f + \operatorname{supp}_{\mathrm{m}} g}$.

(ii) *If $f \in \mathcal{L}^p(\mathbb{R}^n)$ and $g \in \mathcal{L}^q(\mathbb{R}^n)$, then $f * g$ is uniformly continuous.*

(iii) *If $f \in \mathcal{L}^p(\mathbb{R}^n)$ has compact support and $g \in \mathcal{L}^q_{\mathrm{loc}}(\mathbb{R}^n)$, then $f * g \in C^0(\mathbb{R}^n)$.*

(iv) *If $\varphi \in C^k_0(\mathbb{R}^n)$, $k \geq 1$, and $g \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $\varphi * g \in C^k(\mathbb{R}^n)$ and*

$$\partial^\alpha(\varphi * g) = (\partial^\alpha \varphi) * g \tag{3.1.2}$$

*for every multi-index with $|\alpha| \leq k$.*

*Proof.* [Bre11, (i) Prop. 4.18, (iv) Prop. 4.20] and [Sal16, (ii-iv) Thm. 7.35]. □

## 3.2 Convolution of equivalence classes

By Theorem 3.1.2 and A.1.1 convolution of functions descends to a map

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n), \quad ([f], [g]) \mapsto [f * g]. \tag{3.2.3}$$

whenever $p, q, r \in [1, \infty]$ are such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.[2]

**Remark 3.2.1.** Convolution provides

- a commutative (and associative) product on $L^1(\mathbb{R}^n)$; $(p = q = r = 1)$
- an action of $L^1(\mathbb{R}^n)$ on $L^p(\mathbb{R}^n)$. $(q = 1,\ r = p)$

**Definition 3.2.2** (Mollifier)**.** A **mollifier on $\mathbb{R}^n$** is a smooth symmetric[3] function $\rho : \mathbb{R}^n \to [0, \infty)$ supported in the open unit ball $B_1$ and with $\int_{\mathbb{R}^n} \rho = 1$. For each $\delta \in (0, 1]$ define the **rescaling** $\rho_\delta(x) := \delta^{-n}\rho(\delta^{-1}x)$ for $x \in \mathbb{R}^n$. While $\rho_\delta$ is even supported in $B_\delta \subset B_1$, it is still of unit integral $1 = \int_{\mathbb{R}^n} \rho_\delta = \|\rho_\delta\|_1$.

**Theorem 3.2.3.** *Suppose $p \in [1, \infty)$ is finite. Then the following is true.*

(i) *Fix a mollifier $\rho$ and $f \in \mathcal{L}^p(\mathbb{R}^n)$, then $\|\rho_\delta * f - f\|_p \to 0$, as $\delta \to 0$.*

(ii) *$C^\infty_0(\Omega)$ is a dense subset of $L^p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open, <u>bounded</u> or not.*[4]

*Proof.* See e.g. [Bre11, (i) Thm. 4.22, (ii) Cor. 4.23] or [Sal16, (ii) Thm. 7.35]. □

**Exercise 3.2.4.** Check that the density result (ii) fails for $p = \infty$ on $\Omega = \mathbb{R}^n$.

---

[2] Note that $r = \frac{pq}{p+q-pq} \geq 1$ iff $\frac{pq}{p+q} \geq \frac{1}{2}$ iff $p \geq \frac{q}{2q-1}$ iff $q \geq \frac{p}{2p-1}$.
Also note that $r < \infty$ iff $p + q > pq$ iff $p < \frac{q}{q-1}$ iff $q < \frac{p}{p-1}$.

[3] a function $f$ is called **symmetric** if $f(-x) = f(x)$ for every $x$

[4] 'subset' actually refers to the image of the natural injection $C^\infty_0(\Omega) \to L^p(\Omega)$, $f \mapsto [f]$

**Remark 3.2.5.** Concerning part (i): Note that $\rho_\delta * f \in L^p(\mathbb{R}^n)$ by Young's inequality (A.1.1) with $r = p$ and $q = 1$. As mentioned earlier, if $f \in \mathcal{L}^p(\Omega)$ use the natural zero extension, still denoted by $f \in \mathcal{L}^p(\mathbb{R}^n)$, to get convergence $\rho_\delta * f \to f$ in the $L^p$-norm. If $f \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$, use the $\delta$-extension $\bar{f}$ in (3.3.4) below to get $\|\cdot\|_p$-convergence along pre-compacts $Q \Subset \Omega$, see 3.3.1.

Concerning part (ii): The density result *fails* for Sobolev spaces on bounded domains $Q$, whereas on $\mathbb{R}^n$ density *does generalize* to Sobolev spaces; see Remark 4.2.8 and Theorem 5.1.1 (iii), respectively.

## 3.3 Local Mollification

Convolution with a compactly supported smooth function can be used to smoothen out a locally integrable function on $\mathbb{R}^n$, see (3.1.2). Let us detail this in the general case of a locally integrable function that is only defined on $\Omega \subset \mathbb{R}^n$. Note that, in contrast to integrability, local integrability is not necessarily inherited by the natural zero extension – but by the zero $\delta$-extension.

As $\mathcal{L}^p_{\mathrm{loc}} \subset \mathcal{L}^1_{\mathrm{loc}}$ by Hölder, let us deal with the largest space $\mathcal{L}^1_{\mathrm{loc}}$ right away.

### 3.3.1 Locally integrable functions

Suppose $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ is a locally integrable function on a non-empty open subset $\Omega \subset \mathbb{R}^n$. For $\delta \geq 0$ consider the open subset of $\Omega$ defined by

$$\Omega^\delta := \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}, \qquad \Omega_i := \Omega^{1/i}, \quad i \in \mathbb{N}.$$

Observe that $\Omega_i \subset \Omega_{i+1}$ and $\cup_{i\in\mathbb{N}} \Omega_i = \Omega = \Omega^0$. By openness of $\Omega$ its subset $\Omega^\delta$ is non-empty whenever $\delta > 0$ is sufficiently small.

Note that the natural zero extension of $f$ to $\mathbb{R}^n$ is not necessarily locally integrable, as we do not have control how $f$ behaves near $\partial\Omega$. A way out is to restrict $f$ to a closed subset of $\Omega$ and extend this, still locally integrable, restriction to $\mathbb{R}^n$ by zero. To do this in a mollification compatible way we define the **zero $\boldsymbol{\delta}$-extension** of $f : \Omega \to \mathbb{R}$ for $\delta > 0$ by

$$\bar{f} := \bar{f}^{(\delta)} := \begin{cases} f & , \text{on } \overline{\Omega^\delta}, \\ 0 & , \text{on } \mathbb{R}^n \setminus \overline{\Omega^\delta}, \end{cases} \qquad \bar{f} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ if } f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega). \tag{3.3.4}$$

Cf. Fig. 3.1. If $\Omega = \mathbb{R}^n$, then $\bar{f} = f$. Define the **mollification** of $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ by

$$f^\delta := \rho_\delta * \bar{f} \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^n). \tag{3.3.5}$$

Smoothness and compact support are guaranteed by Theorem 3.1.6. Since $\bar{f} = 0$ outside $\overline{\Omega^\delta}$ we have $f^\delta = 0$ outside $\Omega$. So $E := E(f^\delta) := E(\rho_\delta, \bar{f})$, the bad set, is contained in $\Omega$. The value of the mollification is given by

$$f^\delta(x) := \left(\rho_\delta * \bar{f}\right)(x) = \begin{cases} \int_{B_\delta(x) \subset \Omega^\delta} \rho_\delta(x-y) f(y)\, dy & , x \in E^{\mathrm{C}} \cap \overline{\Omega^{2\delta}}, \\ 0 & , x \in E \quad \cap \overline{\Omega^{2\delta}}, \end{cases}$$

at points $x \in \overline{\Omega^{2\delta}}$. The bad set $E(f^\delta) \subset \Omega$ is of measure zero by Remark 3.1.5.

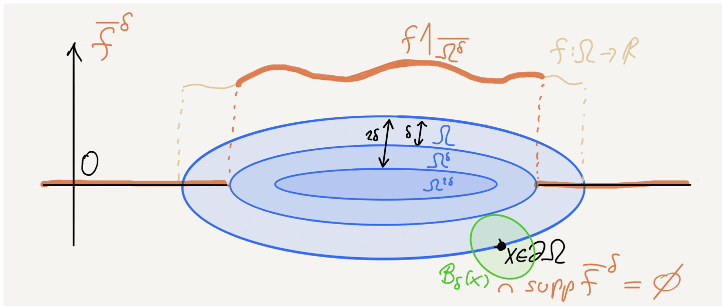


Figure 3.1: The zero $\delta$-extension $\bar{f} = \overline{f}^{(\delta)} \in \mathcal{L}^1_{\rm loc}(\Omega)$ of $f \in \mathcal{L}^1_{\rm loc}(\Omega)$

**Lemma 3.3.1** (Local $L^\infty$ convergence)**.** *Given $p \in [1,\infty]$ and $f \in \mathcal{L}^p_{\rm loc}(\Omega)$, then $\rho_\delta * \bar{f} \in C_0^\infty(\Omega)$ converges to $f$ in $\|\cdot\|_{L^\infty(Q)}$ along pre-compacts $Q \Subset \Omega$.*

*Proof.* Lemma 3.3.2 via approximation; see e.g. [GT01, Le. 7.2]. □

### 3.3.2 Continuous functions

Suppose $u$ is a continuous function on $\Omega \subset \mathbb{R}^n$. Since $C^0(\Omega) \subset \mathcal{L}^1_{\rm loc}(\Omega)$ we define the mollification $u^\delta = \rho_\delta * \bar{u} \in C_0^\infty(\Omega)$ by (3.3.5). It has empty bad set $E(u^\delta) = \emptyset$, since the extension $\bar{u} = \bar{u}^{(\delta)}$ is locally bounded.[5] So the **mollification of a continuous function**, also called a ***C*⁰ mollification**, is given by[6]

$$\begin{aligned} u^\delta(x) := \left(\rho_\delta * \bar{u}\right)(x) &= \int_{B_\delta(x)} \rho_\delta(x-y) u(y)\, dy \\ &= \int_{B_1(0)} \rho(z) u(x+\delta z)\, dz \end{aligned} \tag{3.3.6}$$

for any $x \in \Omega^{2\delta} = \{x \in \Omega \mid d(x, \partial\Omega) > 2\delta\}$.

**Lemma 3.3.2** (Local uniform convergence)**.** *For continuous functions $u$ on $\Omega$ the mollification $u^\delta := \rho_\delta * \bar{u}$ converges to $u$ uniformly on compact sets $K \subset \Omega$.*

*Proof.* Given $K \subset \Omega$, pick $\delta > 0$ sufficiently small such that $\Omega^{2\delta} \supset K$. Then the value of $u^\delta := \rho_\delta * \bar{u} \in C_0^\infty(\Omega)$ defined by (3.3.5) is given along $K$ by (3.3.6). So

$$\begin{aligned} \sup_K \left|u - u^\delta\right| &= \sup_{x\in K} \left| \int_{\{|z|\le 1\}} \rho(z) \left(u(x) - u(x+\delta z)\right) dz \right| \\ &\le \sup_{x\in K} \sup_{|z|\le 1} \underbrace{\left|u(x) - u(x+\delta z)\right|}_{\le c_x |\delta z|} \\ &\le c\delta. \end{aligned}$$

[5] Indeed $\int_{\mathbb{R}^n} |\rho_\delta(x-y)\bar{u}(y)| dy = \int_{B_\delta(x)} |\rho_\delta(x-y)\bar{u}(y)| dy \le \|\bar{u}\|_{C^0(B_\delta(x))} < \infty$.

[6] Change of variable $z(y) := -\frac{x-y}{\delta}$ with $dz = \delta^{-n} dy$ using the symmetry $\rho(-z) = \rho(z)$.

We used continuity of $u$ and compactness of the closure of the $\delta$-neighborhood $U_\delta(K)$ of $K$ to replace the continuity constant $c_x$ of $u$ at $x$ by a constant $c$ uniform in $K$. It enters that $K \subset \Omega^{2\delta}$ implies $\overline{U_\delta(K)} \subset \overline{\Omega^\delta} \subset \Omega = \operatorname{dom} u$. $\square$

## 3.4 Applications

**Lemma 3.4.1.** *Suppose $v \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ has the property that*

$$\int_\Omega v\phi = 0 \tag{3.4.7}$$

*for all $\phi \in C_0^\infty(\Omega)$. Then $v$ vanishes almost everywhere, in symbols $v = 0$ a.e.*

The lemma guarantees uniqueness of weak derivatives almost everywhere; cf. Lemma 4.1.6.
The lemma is easy to prove for continuous $v$ in which case the conclusion is that $v$ even vanishes pointwise. The lemma is already less easy to prove for non-negative $v \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$. Pick $\phi \in C_0^\infty(\Omega, [0,1])$ which is 1 over a given compact $K \subset \Omega$. That Lebesgue integral zero implies that a non-negative integrand is zero a.e. indeed requires a bit of work; see e.g. [Sal16, Le. 1.49 (iii) $\Rightarrow$(i)]. Consequently over $K$ we get $v = \phi v = 0$ a.e., but $K$ was arbitrary.

*Proof.* Pick a compact subset $K \subset \Omega$ of the open subset $\Omega$ of $\mathbb{R}^n$. Fix $\chi \in C_0^\infty(\Omega)$ with $\chi \equiv 1$ along $K$. Extending $\chi$ and $v$ to $\mathbb{R}^n$ by zero the product $v_\chi := \chi v \in \mathcal{L}^1(\mathbb{R}^n)$ is an integrable function on $\mathbb{R}^n$. Pick a mollifier $\rho$ on $\mathbb{R}^n$ with rescalings $\rho_\delta$. Now, on the one hand, Theorem 3.2.3 (i) asserts that

$$[\rho_\delta * v_\chi] \to [v_\chi] \quad \text{in } L^1(\mathbb{R}^n), \text{ as } \delta \to 0.$$

On the other hand, at any $x \in \mathbb{R}^n$ the function

$$(\rho_\delta * v_\chi)(x) = \int_{\mathbb{R}^n} \rho_\delta(x-y)\chi(y)v(y)\,dy = \int_\Omega \underbrace{\rho_\delta(x-y)\chi(y)}_{=:\phi_x(y)} v(y)\,dy = 0$$

vanishes by hypothesis (3.4.7) since $\phi_x \in C_0^\infty(\Omega)$: Indeed $\operatorname{supp}\phi_x \subset \operatorname{supp}\chi \subset \Omega$. By uniqueness of limits $[\chi v] = [0]$ in $L^1(\mathbb{R}^n)$. Thus along any compact subset $K \subset \Omega$ we get that $v = \chi v = 0$ a.e. on $K$, in symbols $m(\{v \neq 0\} \cap K) = 0$.

It remains to conclude that $v = 0$ a.e. on $\Omega$. To see this use the nested sequence of compact sets $K_i \subset K_{i+1} \subset \Omega = \cup_{i=1}^\infty K_i$ provided by Lemma 2.1.1. Set $A := \{v \neq 0\}$ and consider the Lebesgue null sets $A_i := A \cap K_i$ to obtain that $A_i \subset A_{i+1}$ and $A = \cup_{i=1}^\infty A_i$. Thus $m(\{v \neq 0\}) = \lim_{i\to\infty} m(A_i) = 0$ by [Sal16, Thm. 1.28 (iv)]. For a slightly different proof see [Bre11, Cor. 4.24]. $\square$

# Chapter 4

# Sobolev spaces

In typical applications, say in the analysis of PDEs, whereas the property of being continuously differentiable is of fundamental importance, it is often difficult to establish this property. Therefore its is desirable to introduce a weaker version of differentiability, thereby enlarging the spaces $C^k$, hence making it easier to establish membership in the larger space.
The idea comes from Lebesgue integration theory where the value at an individual point, in fact, along sets of measure zero (null sets), is not seen by the integral. But insensibility to null sets ruins non-degeneracy of the natural norm candidate. The way out is to identify functions that differ only along a null set.

In Section 4.1 we introduce the new weak concept of differentiability on the level of functions. A key tool is local mollification from Section 3.3.

In Section 4.2 we quotient out by equality up to null sets and look at the resulting Banach spaces, called Sobolev spaces, which come in the three flavours $W^{k,p}_{\mathrm{loc}}$, $W^{k,p}$, and $W^{k,p}_0$.

## 4.1 Weak derivatives of locally integrable fcts

### 4.1.1 The mother of all Sobolev spaces $\mathcal{L}^1_{\mathrm{loc}}$

Key observation: Given two smooth compactly supported functions on $\Omega \subset \mathbb{R}^n$, the formula for partial integration still makes sense if one of the functions is just a locally integrable function. Use the formula as criterion in a definition!

**Definition 4.1.1** (Weak derivatives of locally integrable functions)**.** Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of non-negative integers $\alpha_i \in \mathbb{N}_0$. A locally integrable function $u : \Omega \to \mathbb{R}$, in symbols $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$, is said to **admit a weak derivative corresponding to $\boldsymbol{\alpha}$** if there is some $u_\alpha \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\forall \phi \in C^\infty_0(\Omega): \qquad \int_\Omega u(D^\alpha \phi)\, dm = (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \, dm. \tag{4.1.1}$$

The locally integrable function $u_\alpha$ is called **a weak derivative of $u$ corresponding to $\alpha$**. By linearity of the integral $(u+v)_\alpha = u_\alpha + v_\alpha$ and $(cu)_\alpha = cu_\alpha$. We call $u$ **weakly differentiable** if it admits weak derivatives up to order 1.

**Definition 4.1.2** (Vector spaces of functions admitting weak derivatives)**.** For $k \in \mathbb{N}_0$ and $p \in (1, \infty]$ consider the vector spaces

$$\begin{aligned} \mathcal{W}^{k,1}_{\mathrm{loc}}(\Omega) &:= \{u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega) \mid \text{all weak derivatives } u_\alpha \text{ exist up to order } k\} \\ \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega) &:= \{u \in \mathcal{W}^{k,1}_{\mathrm{loc}}(\Omega) \mid Q \Subset \Omega,\ |\alpha| \le k \ \Rightarrow\ \|u_\alpha\|_{L^p(Q)} < \infty\}. \end{aligned}$$

We say that a sequence $u_\ell \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega)$ **converges in $\mathcal{W}^{k,p}_{\mathrm{loc}}$** if there is an element $u \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega)$ such that for any pre-compact $Q \Subset \Omega$

$$\|(u_\ell)_\alpha - u_\alpha\|_{L^p(Q)} \to 0, \quad \text{as } \ell \to \infty,$$

for any weak derivative of order $|\alpha| \le k$. In symbols **$u_\ell \to u$ in $\mathcal{W}^{k,p}_{\mathrm{loc}}$**.

**Definition 4.1.3** (Pre-Banach spaces)**.** For $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ define

$$\mathcal{W}^{k,p}(\Omega) := \{u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega) \mid \text{all } u_\alpha \text{ exist up to order } k \text{ and } \|u_\alpha\|_p < \infty\}.$$

**Remark 4.1.4** (Ordinary derivatives are weak derivatives)**.** If $u$ is of class $C^k$, then any partial derivative of order $\ell \le k$ is a weak derivative by the theorem of partial integration and compact support of the $\phi$'s.

**Remark 4.1.5** (Weak derivatives are not unique on the level of functions)**.** Suppose $u_\alpha$ is a weak derivative of $u$ corresponding to the multi-index $\alpha$. Then so is any $\tilde{u}_\alpha$ that differs from $u_\alpha$ on a set of measure zero (same RHS in (4.1.1)) and besides these there are no other weak derivatives of $u$ by Lemma 4.1.6. Furthermore, any $\tilde{u}$ that differs from $u$ on a set of measure zero has the same weak derivatives as $u$ (same LHS in (4.1.1)).

**Lemma 4.1.6** (Uniqueness almost everywhere)**.** *If $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ has weak derivatives $u_\alpha$ and $\tilde{u}_\alpha$, then $u_\alpha = \tilde{u}_\alpha$ a.e.*

*Proof.* Lemma 3.4.1 with $v = u_\alpha - \tilde{u}_\alpha$. □

### 4.1.2 Examples

**Exercise 4.1.7.** The Cantor function $c : [0, 1] \to [0, 1]$, cf. [Sal16, Exc. 6.24], does not admit a weak derivative, although the derivative of $c$ exists almost everywhere (namely along the complement of the Cantor set $C$) and is zero.

**Exercise 4.1.8.** Let $I = (-1, +1)$. For any $p \in [1, \infty]$ show the following:

(i) The absolute value function $u(x) = |x|$ is weakly differentiable. A weak derivative $u_1$ is given by the **sign function**

$$\mathrm{sign}\,(x) := \begin{cases} -1 & , x < 0, \\ 0 & , x = 0, \\ +1 & , x > 0. \end{cases}$$

(ii) The sign function does not have a weak derivative.

[Hint: In case you get stuck consult [Eva98, §5.2.1].]

Note that for weak differentiability the *corner* of $|\cdot|$ did not matter, but the *jump discontinuity* of sign was a problem. Check that sign is not absolutely continuous. What is the problem with the Cantor function $c$?

**Exercise 4.1.9.** Let $Q \subset \mathbb{R}^2$ be the open unit ball and consider the function $u(x) := |x|^{-\gamma}$ for $x \neq 0$ and $u(0) := 0$. For which values of $\gamma > 0$ and $p \in [1, \infty]$

- is $u \in \mathcal{L}^p(Q)$?
- does $u$ admit a weak derivative?

[Hint: In case you get stuck consult [Eva98, §5.2.2].]

### 4.1.3 ACL characterization

The notion of weak derivative is more general than partial derivative. This leads to the question if some property of classical differentiability survives. An excellent treatment of Sobolev spaces from a more geometric measure theory point of view emphasizing the role of absolute continuity is presented in Ziemer's book [Zie89].

**Theorem 4.1.10** (ACL characterization)**.** *Let $p \in [1, \infty]$ and $u \in \mathcal{L}^p(\Omega)$. Then a function $u$ lies in $\mathcal{W}^{1,p}(\Omega)$ iff it admits a version $u^*$, i.e. $u = u^*$ a.e., that has the **ACL property**, i.e. $u^*$ is absolutely continuous on almost all line segments in $\Omega$ parallel to the coordinate axis and whose partial derivatives $\partial_1 u^*, \dots, \partial_n u^*$ exist pointwise a.e. (extend by zero) and are $p$-integrable.*

*Proof.* [Zie89, Thm. 2.1.4] □

In the terminology of Section A.1.2 there are the inclusions[1]

$$\mathbf{LC} \subset \mathbf{AC} \subset \underbrace{\left(\mathbf{ACL} \cap \{\partial_1 u^*, \dots, \partial_n u^* \text{ } p\text{-integrable}\}\right)}_{W^{1,p}} \subset \textbf{Diff-a.e.}$$

For $u \in \mathcal{L}^p(\Omega)$ one can reformulate the ACL theorem as follows: $u \in \mathcal{W}^{1,p}(\Omega)$ iff $u$ has a version $u^*$ that lies in $\mathcal{W}^{1,p}(\Lambda)$ for almost all line segments $\Lambda$ in $\Omega$ parallel to the coordinate axes and the strong gradient satisfies $|Du| \in \mathcal{L}^p(\Omega)$.

To obtain an equivalent statement replace almost all line segments $\Lambda$ by almost all $k$-dimensional planes $\Lambda_k$ in $\Omega$ parallel to the coordinate $k$-planes.

For a characterization of $\mathcal{W}^{1,p}(\Omega)$ through approximation see Section 4.1.5. For a characterization of the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $p \in (1, \infty)$, in terms of difference quotients see Section 4.2.2.

[1] strictly speaking, the identification is $u^* \mapsto [u^*] \in W^{1,p} = \mathcal{W}^{1,p}/\sim$

### 4.1.4 Weak and partial derivatives

**Lemma 4.1.11.** *Suppose a continuous function $u : \Omega \to \mathbb{R}$ is weakly differentiable and $u_{e_1}, \dots, u_{e_n}$ are even continuous (thus unique). Show that $u \in C^1(\Omega)$ and the weak derivatives coincide with the partial ones, in symbols*

$$(u_{e_1}, \dots, u_{e_1}) = (\partial_{x_1} u, \dots, \partial_{x_n} u).$$

*Proof.* As differentiation is a local problem, fix any pre-compact subset $Q \Subset \Omega$ and pick a constant $\delta > 0$ such that $Q \Subset \Omega^{2\delta} := \{x \in \Omega \mid \operatorname{dist}(x, \Omega) > 2\delta\}$. Recall that $u^\delta := \rho_\delta * \bar{u} \in C_0^\infty(\Omega)$ is given along $\Omega^{2\delta}$ by formula (3.3.6). Along the closure of $\Omega^{2\delta}$ the formula does not see the choice of extension, so in that domain it is safe to use the notation $\rho_\delta * u \in C^\infty(\Omega^{2\delta})$. The partial derivative $\partial_i u^\delta$ is according to (3.1.2) and at $x \in \Omega^{2\delta}$ given by

$$\begin{aligned}
\partial_i u^\delta(x) = \left(\partial_i \rho_\delta * u\right)(x) &= \int_{B_\delta(x) \subset \Omega^\delta} \underbrace{\left(\partial_i \rho_\delta\right)(x-y)}_{=-\partial_i(\rho_\delta(x-y))} \cdot u(y)\, dy \\
&= (-1)^2 \int_{B_\delta(x) \subset \Omega^\delta} \rho_\delta(x-y) \cdot u_{e_i}(y)\, dy \\
&= \left(\rho_\delta * u_{e_i}\right)(x).
\end{aligned}$$

The second step is by definition (4.1.1) of a weak derivative. By assumption $u$ and its weak derivative $u_{e_i}$ are continuous, so by Lemma 3.3.2 we have that

$$u^\delta = \rho_\delta * u \to u, \quad \partial_i u^\delta = \rho_\delta * u_{e_1} \to u_{e_i}, \quad \text{as } \delta \to 0, \tag{4.1.2}$$

uniformly along the compact set $K := \bar{Q}$.

It remains to show that the partial derivatives of $u$ exist on $Q$ and are equal to the (continuous) weak derivatives. But this means that $u \in C^1(Q)$. Because $Q \Subset \Omega^{2\delta}$ and $\delta > 0$ were both arbitrary, this proves that $u \in C^1(\Omega)$.
To this end pick $z \in Q$, then for any $|h| > 0$ so small that $z + he_i \in Q$ we get

$$\begin{aligned}
u(z + he_i) - u(z) &= \lim_{\delta \to 0} \left(u^\delta(z + he_i) - u^\delta(z)\right) \\
&= \lim_{\delta \to 0} \int_0^h \partial_i u^\delta(z + se_i)\, ds \\
&= \int_0^h u_{e_i}(z + se_i)\, ds.
\end{aligned}$$

Step one is by pointwise convergence $u^\delta \to u$ on $Q$. Step two holds by the fundamental theorem of calculus. Step three holds by uniform convergence $\partial_i u^\delta \to u_{e_i}$ on $Q$.[2] Divide by $h$ and take the limit $h \to 0$ to get that the partial derivative of $u$ at $z \in Q$ exists and equals the weak one $\partial_i u(z) = u_{e_i}(z)$. □

[2] By uniform convergence along $Q$ the sequence $\partial_i u^\delta$, modulo finitely many members, is dominated by the integrable function $u_{e_i} + 1$, so by the Lebesgue dominated convergence Theorem 2.1.5 limit and integral commute.

### 4.1.5 Approximation characterization

**Theorem 4.1.12.** *Suppose $p \in [1, \infty]$ and $u, v \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$. Equivalent are*

$$v = u_\alpha \quad \Leftrightarrow \quad \exists (u_\ell) \subset C^\infty_0(\Omega) \,:\, u_\ell \to u,\, (u_\ell)_\alpha \to v \text{ in } \mathcal{L}^p_{\mathrm{loc}}(\Omega).$$

*Proof.* Proposition 4.1.13 and (4.1.2); cf. [GT01, Thm. 7.4]. □

A function $u \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega)$, with all weak derivatives up to order $k$, can be $L^p$ approximated along any pre-compact by a sequence of smooth functions $u_\ell$.

**Proposition 4.1.13** (Local approximation by smooth functions)**.** *Let $k \in \mathbb{N}$ and $p \in [1, \infty]$ and suppose $u \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega)$.*[3] *Then there is a sequence of smooth functions $u_\ell \in C^\infty_0(\Omega)$ such that $u_\ell \to u$ in $\mathcal{W}^{k,p}_{\mathrm{loc}}$, as $\ell \to \infty$.*

*Proof.* Recall from (3.3.5) the definition of the mollification $u^\delta := \rho_\delta * \bar{u} \in C^\infty_0(\Omega)$ of a function $u$ on $\Omega \subset \mathbb{R}^n$ using the $\delta$-extension $\bar{u} = \bar{u}^{(\delta)} : \mathbb{R}^n \to \mathbb{R}$, as defined by (3.3.4). Along the closure of $\Omega^{2\delta} := \{x \in \Omega \mid \operatorname{dist}(x, \Omega) > 2\delta\}$ the formula does not see the choice of extension, so in that domain it is safe to use the notation $\rho_\delta * u \in C^\infty(\Omega^{2\delta})$.

Pick $Q \Subset \Omega$ and let $\delta > 0$ be small such that $\Omega^{2\delta} \Supset Q$. Now consider any multi-index of order $|\alpha| \le k$ and let $x \in Q$. By Theorem 3.1.6 (iv) the derivative of the convolution can then be thrown on the smooth factor, hence

$$\begin{aligned}
D^\alpha(\rho_\delta * u)(x) &= (D^\alpha \rho_\delta * u)(x) \\
&= \int_\Omega D^\alpha_x \rho_\delta(x - y) \cdot u(y)\, dy \\
&= (-1)^{|\alpha|} \int_\Omega D^\alpha_y \underbrace{\rho_\delta(x - y)}_{=:\phi_x \in C^\infty_0} \cdot u(y)\, dy \\
&= (-1)^{|\alpha| + |\alpha|} \int_\Omega \rho_\delta(x - y) \cdot u_\alpha(y)\, dy \\
&= (\rho_\delta * u_\alpha)(x) \qquad \big(\boldsymbol{u} \in W^{k,p}(\Omega),\ x \in Q,\ Q \Subset \Omega\big).
\end{aligned} \tag{4.1.3}$$

Here Step four is by definition (4.1.1) of the weak derivative of $u$. Note that (4.1.3) proves that one can throw the derivative of a mollification onto the factor of class $W^{k,p}$ in which case it turns into a weak derivative. Also

$$D^\alpha u^\delta = \rho_\delta * u_\alpha \to u_\alpha \text{ in } L^\infty(Q), \text{ so in } L^p(Q), \text{ as } \delta \to 0, \tag{4.1.4}$$

by Lemma 3.3.1. The choice of extension is invisible along $Q$. For $\ell \in \mathbb{N}$ set $u_\ell := u^{1/\ell}$. This proves Proposition 4.1.13. □

[3] Without 'loc' the assertion fails for $p = \infty$, as $u \equiv 1 : \mathbb{R} \to \mathbb{R}$ shows.

### 4.1.6 Bounded weakly differentiable means Lipschitz

**Definition 4.1.14.** A function $u : \Omega \to \mathbb{R}$ is called **Lipschitz continuous** if

$$|u(x) - u(y)| \le C\,|x - y|$$

for some constant $C$ and all $x, y \in \Omega$. The smallest such constant, say $L$, is called the **Lipschitz constant** of $u$ on $\Omega$. It is called **locally Lipschitz continuous** if it is Lipschitz continuous on every pre-compact $Q \Subset \Omega$ or, equivalently, on every compact $K \subset \Omega$.

**Exercise 4.1.15.** Confirm "equivalently" above. [Hint: Figure A.1.]

**Proposition 4.1.16** ($C^{0,1}_{\mathrm{loc}} = \mathcal{W}^{1,\infty}_{\mathrm{loc}}$)**.** *A function $u : \Omega \to \mathbb{R}$ belongs to $C^{0,1}_{\mathrm{loc}}$ iff $u$ is weakly differentiable with locally bounded weak derivatives.*

*Proof.* See e.g. [Eva98, §5.8.2]. Cf. also [MS04, Exc. B.1.8]. □

### 4.1.7 Leibniz or product rule

For interesting different approaches and further information concerning the present subsection and the previous one we refer to [Zie89, §2.1 §2.2].

**Proposition 4.1.17** (Leibniz rule for weak derivatives)**.** *Given $p \in [1, \infty]$, let $u \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega)$ and $v \in \mathcal{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$. Then $uv$ lies in $\mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega)$ with weak derivatives*

$$(uv)_\alpha = u_\alpha v + u v_\alpha, \qquad |\alpha| \le 1. \tag{4.1.5}$$

**Corollary 4.1.18.** *Given $p \in [1, \infty]$, let $u \in \mathcal{W}^{1,p}(\Omega)$ and $v \in \mathcal{W}^{1,\infty}(\Omega)$. Then the product $uv$ lies in $\mathcal{W}^{1,p}(\Omega)$ and it satisfies Leibniz (4.1.5) and the estimate*

$$\|uv\|_{1,p} \le \|u\|_{1,p} \|v\|_{1,\infty}\,. \tag{4.1.6}$$

*Proof.* Since $\mathcal{W}^{1,q} \subset \mathcal{W}^{1,q}_{\mathrm{loc}}$ for $q \in [1, \infty]$ Leibniz (4.1.5) holds. Now use the estimate $\|(uv)_\alpha\|_p \le \|u_\alpha\|_p \|v\|_\infty + \|u\|_p \|v_\alpha\|_\infty$. □

*Proof of Proposition 4.1.17.* The following is true for $i = 1, \dots, n$. By Proposition 4.1.13 there are sequences $u_k, v_k \in C^\infty_0(\Omega)$ such that, as $k \to \infty$, it holds

$$\begin{aligned} &u_k \to u, \qquad \frac{\partial u_k}{\partial x_i} = (u_k)_{e_i} \to u_{e_i} \qquad \text{in } \mathcal{L}^p_{\mathrm{loc}}(\Omega) \\ &v_k \to v, \qquad \frac{\partial v_k}{\partial x_i} = (v_k)_{e_i} \to v_{e_i} \qquad \text{in } \mathcal{L}^\infty_{\mathrm{loc}}(\Omega), \text{ thus in } \mathcal{L}^p_{\mathrm{loc}}(\Omega). \end{aligned}$$

Note that $\mathcal{L}^p_{\mathrm{loc}}$ convergence for finite $p$ implies pointwise convergence almost everywhere for a subsequence (same notation); cf. Figure A.3. By partial integration and Leibniz for smooth functions we get for each $k$ that

$$\int_\Omega u_k v_k \frac{\partial \phi}{\partial x_i} = -\int_\Omega \left( \frac{\partial u_k}{\partial x_i} v_k + u_k \frac{\partial v_k}{\partial x_i} \right) \phi \qquad \forall \phi \in C^\infty_0(\Omega).$$

Apply $\phi$-wise the dominated convergence Theorem 2.1.5 to obtain in the limit[4]

$$\int_\Omega uv \frac{\partial \phi}{\partial x_i} = -\int_\Omega (u_{e_i} v + u v_{e_i})\,\phi \qquad \forall \phi \in C_0^\infty(\Omega).$$

This proves the formula (4.1.5) for the weak derivative of the product. The product $uv$ is of class $\mathcal{L}^p_{\mathrm{loc}}(\Omega)$ since $v \in \mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$. The right hand side of $(uv)_{e_i}$ is of the same type '$\mathcal{L}^p\mathcal{L}^\infty + \mathcal{L}^p\mathcal{L}^\infty$', hence of class $\mathcal{L}^p_{\mathrm{loc}}(\Omega)$. □

**Exercise 4.1.19.** Deal with terms 2 and 3 in the previous proof; cf. footnote.

**Remark 4.1.20** (Alternative hypotheses)**.** Suppose $p \in [1,\infty]$. Then the space $\mathcal{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$ is preserved under taking the product $uv$ of two elements $u$ and $v$. The product rule (4.1.5) still holds true; see [Bre11, Prop. 9.4].

### 4.1.8 Chain rule and change of coordinates

**Proposition 4.1.21** (Chain rule – composition)**.** *Let $p \in [1,\infty]$ and suppose $u \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega)$.[5] Then for every function $F \in C^1(\mathbb{R})$ with bounded derivative $F'$ the post-composition $F \circ u$ lies in $\mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega)$ and its weak derivatives are given by*

$$(F\circ u)_{e_i} = F'(u)\cdot u_{e_i}, \qquad i = 1,\dots,n.$$

For necessary and sufficient conditions for the chain rule in $\mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^d)$ see [LM07].

*Proof.* The following is true for every $i = 1,\dots,n$. By Proposition 4.1.13 there is a sequence $u_k \in C_0^\infty(\Omega)$ such that, as $k \to \infty$, there is convergence

$$u_k \to u, \qquad \frac{\partial u_k}{\partial x_i} = (u_k)_{e_i} \to u_{e_i} \qquad \text{in } \mathcal{L}^p_{\mathrm{loc}}(\Omega), \text{ thus in } \mathcal{L}^1_{\mathrm{loc}}(\Omega).$$

Note that $\mathcal{L}^p_{\mathrm{loc}}$ convergence for finite $p$ yields a.e. pointwise convergence of some subsequence (for which we use the same notation); cf. Figure A.3.

Instead of doing things again 'by hand' as in the previous proof, let us check out the comfort of Theorem 4.1.12. Following [GT01, Le. 7.5] let us show that

$$\begin{aligned} f_k := F\circ u_k \to F \circ u =: f \qquad & \text{in } \mathcal{L}^p_{\mathrm{loc}}(\Omega), \\ (f_k)_{e_i} := F'(u_k)\,(u_k)_{e_i} \to F'(u)\,u_{e_i} =: v \qquad & \text{in } \mathcal{L}^p_{\mathrm{loc}}(\Omega), \end{aligned}$$

and that $f, v \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$. Theorem 4.1.12 tells that $v = f_\alpha$ and we are done.

---

[4] Fix $\phi$. Pick $Q$ with $\operatorname{supp}\phi \Subset Q \Subset \Omega$. Replace everywhere $\int_\Omega$ by $\int_Q$. There are three terms. Term 1: The sequence $f_k := u_k v_k \partial_i \phi$ lies in $\mathcal{L}^p(Q)$, since $v_k, \phi \in \mathcal{L}^\infty(Q)$, and $f_k$ converges to $f := uv\partial_i\phi \in \mathcal{L}^p(Q)$ in $\mathcal{L}^p$, so a.e. Indeed along $Q$ we get adding zero ($\phi_i := \partial_i \phi$)

$$\|u_k v_k \phi_i - uv\phi_i\|_p \le \|u_k - u\|_p \|v_k\|_\infty \|\phi_i\|_\infty + \|u\|_p \|v_k - v\|_\infty \|\phi_i\|_\infty \to 0.$$

Now $g := |f| + 1 \in \mathcal{L}^1(Q)$ dominates $|f_k(x)| \le |f(x)| + 1$ a.e. (since $f_k \to f$ in $\mathcal{L}^p$, so a.e.).

[5] The notation $\mathcal{W}^1(\Omega)$ in [GT01, Le. 7.5] corresponds to our notation $\mathcal{W}^{1,1}_{\mathrm{loc}}(\Omega)$.

a) To this end pick $Q \Subset \Omega$. Norms are over $Q$ from now on. We obtain

$$|F(u(x))| \le |F(u(x)) - F(0)| + |F(0)| \le c' \, |u(x)| + |F(0)|$$

where $c' := \|F'\|_{L^\infty(\Omega)}$. So $f = F \circ u \in \mathcal{L}^p(Q)$ since $u$ is. Also $v \in \mathcal{L}^p(Q)$:

$$\|v\|_p = \|F'(u) \cdot u_{e_i}\|_p \le c' \, \|u_{e_i}\|_p < \infty.$$

b) Similarly as above we get that

$$\|f_k - f\|_p = \|F(u_k) - F(u)\|_p \le c' \, \|u_k - u\|_p \to 0.$$

c) For finite $p \in [1, \infty)$ we show that $\|F'(u_k)\,(u_k)_{e_i} - F'(u)\,u_{e_i}\|_p \to 0$. From $L^p$ convergence we get, after taking subsequences, that both $u_k \to u$ and $(u_k)_{e_i} \to u_{e_i}$ converge a.e. along $Q$, similarly for absolute values. Thus

$$g_k := c' \, |(u_k)_{e_i}| \to c' \, |u_{e_i}| =: g \quad \text{a.e. along } Q$$

Together with continuity of $F'$ we still get a.e. pointwise convergence

$$h_k := F'(u_k)\,(u_k)_{e_i} \to F'(u)\,u_{e_i} =: h \quad \text{a.e. along } Q.$$

Note that $|h_k| \le g_k$ pointwise along $Q$. Since $(u_k)_{e_i} \to u_{e_i}$ in $L^p(Q)$ the hypothesis in (2.2.8) is satisfied, hence the conclusion, namely that $\|h_k - h\|_1 = \|F'(u_k)\,(u_k)_{e_i} - F'(u)\,u_{e_i}\|_p \to 0$. This illustrates the usefulness of the generalized Lebesgue dominated convergence Theorem 2.2.8, doesn't it?

d) Case $p = \infty$. Pick $Q \Subset \Omega$. Note that $u \in \mathcal{W}^{1,\infty}(Q) \subset \mathcal{W}^{1,q}(Q)$ for any $q \in [1, \infty)$ by Hölder. So by b) the weak derivatives of $F \circ u$ exist and are given by $F'(u) \cdot u_{e_i}$. We verify that $F \circ u$ and $F'(u) \cdot u_{e_i}$ both lie in $\mathcal{L}^\infty(Q)$. Let all norms be over $Q$. Then $\|F'(u) \cdot u_{e_i}\|_\infty \le c' \, \|u_{e_i}\|_\infty < \infty$. And $F \circ u \in L^\infty(Q)$, because the sequence $F \circ u_k \in C^1(\bar{Q}) \subset \mathcal{L}^\infty(Q)$ converges to $F \circ u$. Indeed

$$\|F \circ u - F \circ u_k\|_\infty \le c' \, \|u_k - u\|_\infty \to 0.$$

But $\mathcal{L}^\infty(Q)$ is a Banach space, so it includes the limit $F \circ u$. □

**Remark 4.1.22** (Lipschitz chain rule)**.** At first glance it looks like the proof might go through for Lipschitz $F$ with bounded derivative (which exists a.e. by Rademacher's Theorem 7.3.2). In a) replace $c'$ by the Lipschitz constant. However, in b) continuity of $F'$ enters crucially to get a.e. convergence $h_k \to h$, doesn't it?

Slightly modifying the assumptions an a.e. chain rule for Lipschitz functions $F$ is given in [Zie89, Thm. 2.1.11].

**Proposition 4.1.23** (Change of coordinates)**.** *Let $U, V \subset \mathbb{R}^n$ be open and let $\psi : V \to U$ be a bijection which is Lipschitz continuous and so is its inverse. Let $p \in [1, \infty]$ and $u = u(x) \in \mathcal{W}^{1,p}(U)$. Write $x = \psi(y)$. Then $v = v(y) := u \circ \psi \in \mathcal{W}^{1,p}(V)$ and there is the identity for the weak derivatives*[6] *$v_{y_i}$ and $u_{x_j}$*

$$(u \circ \psi)_{y_i} = \sum_{j=1}^{n} (u_{x_j} \circ \psi) \cdot (\psi_j)_{y_i} \qquad \textit{a.e. on } V. \tag{4.1.7}$$

[6] for obvious reasons we use the notation $v_{y_i}$ and $u_{x_j}$, and not our usual one $v_{e_i}$ and $u_{e_j}$

*Proof.* Details of the proof are given in [Zie89, Thm. 2.2.2]. For finite $p \in [1, \infty)$ the proof is based on the **transformation law**

$$\int_Y f \circ \psi \cdot |\det d\psi| \; dy = \int_{\psi(Y)} f \, dx \tag{4.1.8}$$

valid for measurable functions $f : U \to \mathbb{R}$ and measurable subsets $Y \subset V$. In fact, bi-Lipschitz maps preserve (Lebesgue) measurability of sets. $\square$

### 4.1.9 Equivalence classes of locally integrable functions

**Definition 4.1.24** (Weak derivative of $\boldsymbol{u} = [u] \in L^1_{\mathrm{loc}} = \mathcal{L}^1_{\mathrm{loc}}/\sim$)**.** Suppose a representative $u$ of an equivalence class $[u] \in L^1_{\mathrm{loc}}(\Omega)$ admits a weak derivative $u_\alpha$ corresponding to a multi-index $\alpha$. In this case, the equivalence class

$$[u]_\alpha := [u_\alpha] \in L^1_{\mathrm{loc}}(\Omega)$$

is called **the weak derivative of $[u]$ corresponding to $\alpha$** and $|\alpha| = \alpha_1 + \dots + \alpha_n$ is called the **order** of the weak derivative.

**Notation.** Writing $[u]$ for equivalence classes is often, but not always, very convenient, e.g. when it comes to take derivatives. We shall alternatively use boldface $\boldsymbol{u} := [u]$ to denote equivalence classes. The above then reads

$$\boldsymbol{u}_\alpha := \boldsymbol{u}_{\boldsymbol{\alpha}} \in L^1_{\mathrm{loc}}(\Omega).$$

By Remark 4.1.5 the definition does not depend on the choice of the representative $u$ of $[u]$. By Lemma 4.1.6 any choice of a weak derivative $u_\alpha$ of $u$ provides the same equivalence class $[u_\alpha]$, i.e. the same element of $L^1_{\mathrm{loc}}(\Omega)$.

To summarize, *if some*, hence any, representative of an element $[u] \in L^1_{\mathrm{loc}}(\Omega)$ admits a weak derivative, then *there is* a *unique* element of $L^1_{\mathrm{loc}}(\Omega)$, denoted by $[u]_\alpha$, whose representatives are the weak derivatives of some, hence any, representative of $[u]$.

## 4.2 Definition and basic properties

Throughout $\Omega \subset \mathbb{R}^n$ is open and $Q \Subset \mathbb{R}^n$ pre-compact where $n \geq 1$. Unless mentioned otherwise, we suppose $p \in [1, \infty]$ and $k \in \mathbb{N}_0$. We write $\boldsymbol{u} = [u]$.

### 4.2.1 The Sobolev spaces $W^{k,p}$

By definition the Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable classes $[u]$ that admit weak derivatives $[u]_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ up to order $k$ and each of them is $p$-integrable on $\Omega$. In symbols,[7]

$$W^{k,p}(\Omega) := \left\{ \boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega) : \forall |\alpha| \leq k \;\; \exists \; \boldsymbol{u}_\alpha \in L^p(\Omega) \right\} \subset L^p(\Omega)$$

[7] writing $\exists \boldsymbol{u}_\alpha \in L^p(\Omega)$ means that the weak derivative exists and it is $p$-integrable

where $p \in [1,\infty]$ and $k \in \mathbb{N}_0$.[8] For finite $p \in [1,\infty)$ define the value of the $\boldsymbol{W^{k,p}}$**-norm** of a a class $\boldsymbol{u} \in W^{k,p}(\Omega)$ by the value

$$\|u\|_{k,p} := \left(\int_\Omega \sum_{|\alpha|\le k} |u_\alpha(x)|^p \, dx\right)^{1/p} = \left(\sum_{|\alpha|\le k} \|u_\alpha\|_p^p\right)^{1/p} \in [0,\infty)$$

where $u$ is any representative of the class and $u_\alpha : \Omega \to \mathbb{R}$ is the weak derivative corresponding to the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. For infinite $p = \infty$ the $\boldsymbol{W^{k,\infty}}$**-norm** of a class $\boldsymbol{u} \in W^{k,\infty}(\Omega)$ is defined as the maximum

$$\|u\|_{k,\infty} := \max_{|\alpha|\le k} \|u_\alpha\|_\infty$$

where $u$ is any representative of the class.

A representative $u$ of a class $\boldsymbol{u} \in W^{k,p}(\Omega)$ is called a $\boldsymbol{W^{k,p}}$ **function on** $\boldsymbol{\Omega}$. To indicate that a sequence of $W^{k,p}$ functions $u_\ell$ converges to $u$ we just write

$$u_\ell \to u \;\text{ in } W^{k,p}(\Omega) \quad :\Leftrightarrow \quad \|u_\ell - u\|_{W^{k,p}(\Omega)} \to 0, \text{ as } \ell \to \infty.$$

**Proposition 4.2.1** (Weak derivatives)**.** *Let $\boldsymbol{u}, \boldsymbol{v} \in W^{k,p}(\Omega)$ and $|\alpha| \le k$, then*

(i) $D^\alpha \boldsymbol{u} \in W^{k-|\alpha|,p}(\Omega)$ *and for all multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| \le k$*

$$D^\beta(D^\alpha \boldsymbol{u}) = D^\alpha(D^\beta \boldsymbol{u}) = D^\alpha(D^\beta \boldsymbol{u});$$

(ii) *for $\lambda, \mu \in \mathbb{R}$ the linear combination $\lambda\boldsymbol{u} + \mu\boldsymbol{v}$ also lies in $W^{k,p}(\Omega)$ and*

$$D^\alpha(\lambda\boldsymbol{u} + \mu\boldsymbol{v}) = \lambda D^\alpha\boldsymbol{u} + \mu D^\alpha\boldsymbol{v} \quad \text{, whenever } |\alpha| \le k;$$

(iii) *for open sets $U \subset \Omega$ one has the inclusion $W^{k,p}(U) \subset W^{k,p}(\Omega)$;*

(iv) *multiplication with a smooth compactly supported function $\phi \in C_0^\infty(\Omega)$ preserves $W^{k,p}(\Omega)$ and the weak derivatives of the product are given by*

$$D^\alpha(\phi\boldsymbol{u}) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta\phi \cdot D^{\alpha-\beta}\boldsymbol{u}.$$

*Proof.* [Eva98, §5.2.3 Thm. 1]. □

**Exercise 4.2.2.** For $k \in \mathbb{N}_0$ and $p \in [1,\infty]$, there is the equivalence

$$\boldsymbol{u} \in W^{k+1,p}(\Omega) \quad \Leftrightarrow \quad \boldsymbol{u} \in W^{1,p}(\Omega),\, \boldsymbol{u_{e_1}}, \dots, \boldsymbol{u_{e_n}} \in W^{k,p}(\Omega).$$

[Hints: '⇒' Proposition 4.2.1 (i).
'⇐' Write down (4.1.1) for the weak derivative $(u_{e_i})_\beta$ of $u_{e_i}$, then use that $u_{e_i}$ is the weak derivative of $u$ utilizing once more (4.1.1).]

[8] To see the inclusion '⊂' note that for $\alpha_0 = (0,\dots,0)$ one has $u_{\alpha_0} = u$ in (4.1.1). Hence $\boldsymbol{u} = \boldsymbol{u_{\alpha_0}} \in L^p(\Omega)$ in order for $\boldsymbol{u}$ to be element of $W^{k,p}(\Omega)$.

**Exercise 4.2.3.** Let $Q \subset \mathbb{R}^2$ be the open unit ball. Find an example of $p \geq 1$ and a $W^{1,p}$ function $u : Q \to \mathbb{R}$ which is unbounded with respect to the $L^\infty$-norm (2.2.4) on each open subset $U$ of $Q$. Conclude that your example is not Lipschitz continuous.

**Convention 4.2.4** (Natural inclusion)**.** Notation such as

$$C^\infty(Q) \subset W^{k,p}(Q)$$

makes sense if (and indicates that) we identify $C^\infty(Q)$ with its image in $W^{k,p}(Q)$ under the map $u \mapsto [u]$. Of course, whenever a class $[u] \in W^{k,p}$ contains a **continuous representative**, notation $u^*$, we tacitly choose $u^*$ to represent $[u]$.

**Theorem 4.2.5.** *For every integer $k \geq 0$ the following is true.*

(i) $W^{k,p}(\Omega)$ *is a Banach space for* $p \in [1, \infty]$.

(ii) $W^{k,p}(\Omega)$ *is separable for finite* $p \in [1, \infty)$.

(ii) $W^{k,p}(\Omega)$ *is reflexive for finite* $p \in (1, \infty)$ *larger* 1.

(iv) $C^\infty(\bar{D})$ *is a dense subset of* $W^{k,p}(D)$ *for finite* $p \in [1, \infty)$ *and Lipschitz domain* $D \Subset \mathbb{R}^n$*; cf. Remark 4.2.8 and Theorem 5.1.3.*

*Proof.* (i-iii) The idea is to isometrically embedd $W^{k,p}(\Omega)$ as a closed subspace of $L^p(\Omega \times \cdots \times \Omega)$ by mapping $u$ to the list that contains all weak derivatives $D^\alpha u$ up to order $k$. See e.g. [AF03, §3.5]. Completeness and reflexivity are inherited by closed subspaces of a Banach space. Furthermore, every subspace[9] $A$ of a separable *metric* space $X$ is separable.[10] Recall Theorem 2.3.1 (i-iii). All three properties are transferred back to $W^{k,p}(\Omega)$ via the isometry. For a direct proof of (i) see e.g. [Eva98, §5.2.3 Thm. 2]. □

### 4.2.2 Difference quotient characterization of $W^{1,p}$

Observe that by continuity of $u \in L^p(\mathbb{R}^n)$ under the shift map, see Exercise 2.3.3, we have for each $\xi \in \mathbb{R}^n$ that $\|u(\cdot + h\xi) - u\|_p \to 0$, as $h \to 0$.

**Theorem 4.2.6.** *Let $p \in (1, \infty)$. Then $u \in W^{1,p}(\mathbb{R}^n)$ iff $u \in L^p(\mathbb{R}^n)$ and the function $h \mapsto |h|^{-1} \|u(\cdot + h\xi) - u\|_p$ is bounded for any $\xi \in \mathbb{R}^n$.*

*Proof.* See e.g. [Zie89, Thm. 2.1.6] or [GT01, §7.11] or [Ste70, Ch. V §3.5]. □

[9] A **subspace** of a metric space is a subset equipped with the **induced metric**, namely, the restriction of the ambient metric.

[10] Consider the set of open balls of rational radii centered at the elements of the countable dense subset of $X$. Consider the collection of intersections of these balls with $A$. In each such intersection select one element. The set $S$ of selected elements is countable and dense in $A$.

### 4.2.3 The compact support Sobolev spaces $W_0^{k,p}$

By definition the compact support Sobolev space $W_0^{k,p}(\Omega)$ is the closure with respect to the norm $\|\cdot\|_{k,p}$ of the vector subspace $\iota(C_0^\infty(\Omega)) \subset W^{k,p}(\Omega)$ that consists of the compactly supported smooth functions on $\Omega$. Here $\iota(u) := [u]$ is the natural injection. In symbols,

$$W_0^{k,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{k,p} := \overline{\iota(C_0^\infty(\Omega))}^{\|\cdot\|_{k,p}}.$$

In other words, the space $W_0^{k,p}(\Omega)$ is the completion of the linear subspace $\iota(C_0^\infty(\Omega))$ of the Banach space $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ where $\iota(u) := [u]$.

**Remark 4.2.7.** Being a closed Banach subspace $W_0^{k,p}(\Omega)$ inherits properties (i) completeness, (ii) separability, and (iii) reflexivity in Theorem 4.2.5; see e.g. [AF03, Thm. 1.22].

**Remark 4.2.8** (One difference between $L^p$ and Sobolev spaces)**.** Part (ii) of Theorem 3.2.3 in case of non-empty *pre-compact* sets $Q \Subset \mathbb{R}^n$ is in sharp contrast to what happens for Sobolev spaces with $k \geq 1$. Namely, it says that

$$L_0^p(Q) := \overline{C_0^\infty(Q)}^p = L^p(Q)$$

for finite $p \in [1, \infty)$, whereas for Lipschitz $D$ the two Sobolev spaces

$$W_0^{k,p}(D) := \overline{C_0^\infty(D)}^{k,p} \subsetneq \overline{C^\infty(\bar{D})}^{k,p} = W^{k,p}(D)$$

are different; cf. Theorem 5.1.3. Indeed the characteristic function $\chi_D$ represents an element of $W^{k,p}(D) \subset L^p(D)$, but not of $W_0^{k,p}(D)$.

**Exercise 4.2.9.** For non-empty intervals $I \Subset \mathbb{R}$ and finite $p$ show that $\boldsymbol{\chi_I} \in W^{1,p}(I) \setminus W_0^{1,p}(I)$. Illustrate graphically why the characteristic function can be approximated through elements of $C_0^\infty(I)$ in the $L^p$, but not in the $W^{k,p}$ norm.

### 4.2.4 The local Sobolev spaces $W_{\mathrm{loc}}^{k,p}$

By definition the local Sobolev space $W_{\mathrm{loc}}^{k,p}(\Omega)$ consists of all locally integrable classes $[u]$ whose restriction to any pre-compact $Q \Subset \Omega$ lies in $W^{k,p}(Q)$, i.e.

$$W_{\mathrm{loc}}^{k,p}(\Omega) := \left\{[u] \in L_{\mathrm{loc}}^1(\Omega) \mid \forall Q \Subset \Omega : [u|_Q] \in W^{k,p}(Q)\right\} \subset L_{\mathrm{loc}}^p(\Omega).$$

To see the inclusion '$\subset$' note that $u \in [u] \in W_{\mathrm{loc}}^{k,p}(\Omega)$ restricted to any pre-compact $Q \Subset \Omega$ is an element $u|_Q \in \mathrm{L}^p(Q)$. In other words, any representative $u$ is $p$-integrable over all pre-compact subsets of $\Omega$.

### 4.2.5 How the spaces relate

To summarize, for $k \in \mathbb{N}$ and $p \in [1, \infty]$ there are the, in general strict, inclusions

$$\begin{array}{ccccc} L^p(\Omega) & \subset & L^p_{\mathrm{loc}}(\Omega) & \subset & L^1_{\mathrm{loc}}(\Omega) \\ \cup & & \cup & & \cup \\ W^{k,p}(\Omega) & \subset & W^{k,p}_{\mathrm{loc}}(\Omega) & \subset & W^{1,p}_{\mathrm{loc}}(\Omega) \\ \cup & & & & \\ W^{k,p}_0(\Omega) & & & & \end{array}$$

In each line the second inclusion holds by Hölder's inequality (2.2.5).

### 4.2.6 Basic properties – products and coordinate change

The elements of $L^\infty(\Omega)$ are represented by almost everywhere bounded functions. Let us call the elements of $W^{k,\infty}(\Omega)$ **$k$-bounded Sobolev functions**.

**Exercise 4.2.10** (Products with $k$-bounded Sobolev functions)**.** Let $k \in \mathbb{N}_0$ and $p \in [1, \infty]$. Then the following is true. If $\boldsymbol{u} \in W^{k,p}(\Omega)$ and $\boldsymbol{v} \in W^{k,\infty}(\Omega)$, then the product class $\boldsymbol{uv} := [uv]$ lies again in $W^{k,p}(\Omega)$ and

$$\|uv\|_{k,p} \le c \|u\|_{k,p} \|v\|_{k,\infty}$$

for some constant $c$ that depends only on $k$ and $n$.

[Hint: Induction based on Exercise 4.2.2 and Corollary 4.1.18.

Suppose $k \in \mathbb{N}$. A $\boldsymbol{C^{k-1,1}}$**-diffeomorphism** is a $C^{k-1}$-diffeomorphism $\psi$ such that the partial derivatives of $\psi$ and $\psi^{-1}$ up to order $(k-1)$ are Lipschitz continuous. See Definition 5.1.5 for Lipschitz domains.

**Exercise 4.2.11** (Change of coordinates)**.** Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. Let $U, V \subset \mathbb{R}^n$ be open and let $\psi : V \to U$ be a $C^{k-1,1}$-diffeomorphism. Then the following is true. Pull-back $\psi^* : W^{k,p}(U) \to W^{k,p}(V)$, $[u] \mapsto [u \circ \psi]$, is an isomorphism of Banach spaces. Indeed any $[u] \in W^{k,p}(U)$ satisfies the estimate

$$\|u \circ \psi\|_{W^{k,p}(V)} \le c \|u\|_{W^{k,p}(U)}$$

where $c > 0$ is independent of $[u]$. For weak derivatives of $u \circ \psi$ see (4.1.7).

Note that the change of coordinates transformation shows that one can define Sobolev spaces on manifolds.

# Chapter 5

# Approximation and extension

## 5.1 Approximation

Major application of density of smooth functions in a given Sobolev space:

- Avoid to work with weak derivatives in proofs
  - first approximate a Sobolev function $u$ by smooth functions $u_\ell$
  - so you can use usual derivatives on the smooth functions $u_\ell$
  - often a limit argument leads back to the original Sobolev function $u$

The quality of possible approximations depends on geometrical properties of the function domain, such as boundedness or having a smooth boundary. The present chapter follows very closely [Eva98, §5.3], including the proof of Theorem 5.1.7 which, however, we spell out for the Lipschitz case, not only $C^1$.

The key tool in this chapter is mollification. Whereas Section 5.1.1 is about local approximation, so we need to work with $\delta$-extensions as defined by (3.3.4), in Sections 5.1.2 and 5.1.3 we get away with natural zero extensions, because of the much stronger hypothesis on the functions to be of class $W^{k,p}$ on the whole domain, not just on compact subsets. (Thus no dangerous non-integrability can be lurking near the boundary. ;-)

### 5.1.1 Local approximation – any domain

The following generalization of Lemma 3.3.1 to Sobolev spaces is about "interior/local approximation by smooth functions with domain any open set".

**Theorem 5.1.1.** *Let $u \in \boldsymbol{u} \in W^{k,p}(\Omega)$ with $k \in \mathbb{N}_0$ and $p \in [1,\infty]$. Let*

$$u^\delta := \rho_\delta * u \quad \text{in } \Omega^{2\delta} := \{x \in \Omega \mid \operatorname{dist}(x,\Omega) > 2\delta\}$$

*be the convolution of $u$ with a mollifier $\{\rho_\delta\}_{\delta>0}$. Then $u^\delta \in C^\infty(\Omega^{2\delta})$ and*

$$u^\delta \to u \quad \text{in } W^{k,p}_{\mathrm{loc}}(\Omega), \text{ as } \delta \to 0.$$

Recall from (3.3.5) the definition of the mollification $u^\delta := \rho_\delta * \bar{u} \in C_0^\infty(\Omega)$ of a function $u$ on $\Omega \subset \mathbb{R}^n$ using the $\delta$-extension $\bar{u} = \bar{u}^{(\delta)} : \mathbb{R}^n \to \mathbb{R}$, as defined by (3.3.4). Along the closure of $\Omega^{2\delta} := \{x \in \Omega \mid \mathrm{dist}(x, \Omega) > 2\delta\}$ the formula does not see the choice of extension, so in that domain it is safe to use the notation $\rho_\delta * u \in C^\infty(\Omega^{2\delta})$.

*Proof.* Proposition 4.1.13. □

**Remark 5.1.2** (Case $\Omega = \mathbb{R}^n$)**.** Let $k \in \mathbb{N}_0$ and $p \in [1, \infty)$. Recall from Theorem 3.2.3 (i) that (4.1.4) is valid with $Q$ replaced by $\mathbb{R}^n$. Thus any $W^{k,p}(\mathbb{R}^n)$ function $u$ can be $W^{k,p}$ approximated by a family $u^\delta \in C^\infty(\mathbb{R}^n)$. Why does this not prove that $C^\infty(\mathbb{R}^n)$ is dense in the Banach space $W^{k,p}(\mathbb{R}^n)$? How about density of $C_0^\infty(\mathbb{R}^n)$ in $W^{k,p}(\mathbb{R}^n)$? In symbols, how about

$$W^{k,p}(\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}^n)}^{k,p} =: W_0^{k,p}(\mathbb{R}^n)?$$

### 5.1.2 Global approximation on bounded domains

The next theorem is about "global approximation by smooth functions on a bounded open domain." For $k = 0$ it is covered by Theorem 3.2.3 which asserts that already the subset $C_0^\infty$ of $C^\infty$ is dense in $L^p$. For $k \geq 1$ and pre-compact $Q \Subset \mathbb{R}^n$ this is not any more so, but $C^\infty$ is still dense.

**Theorem 5.1.3.** *Let $Q \Subset \mathbb{R}^n$ be pre-compact. Let $u \in \boldsymbol{u} \in W^{k,p}(Q)$ for $k \in \mathbb{N}$ and finite $p \in [1, \infty)$. Then there is a sequence of smooth functions $u_\ell \in C^\infty(Q)$ with $p$-integrable derivatives of all orders $|\alpha| \leq k$, i.e. $\boldsymbol{u_\ell} \in W^{k,p}(Q)$, such that*

$$C^\infty(Q) \ni u_\ell \to u \ \text{ in } W^{k,p}(Q).$$

*Equivalently, over pre-compact sets Sobolev spaces are closures of the form*

$$W^{k,p}(Q) = \overline{C^\infty(Q)}^{k,p}.$$

**Remark 5.1.4** (Wrong for $p = \infty$)**.** A counterexample for $k = 1$ and $\Omega = (-1, 1)$ is $u = |\cdot|$; cf. [AF03, Ex. 3.18].

Because the functions $u : Q \to \mathbb{R}$ are of class $W^{k,p}$ along their whole domain $Q$, as opposed to $W^{k,p}_{\mathrm{loc}}$, we can work in the mollification of $u$ with the natural zero extension to $\mathbb{R}^n$, still denoted by $u$.

*Proof.* Pick a function $u \in \boldsymbol{u} \in W^{k,p}(Q)$ and let $\varepsilon > 0$. There are three steps. Firstly, we construct a locally finite open cover $(V_i)_{i=0}^\infty$ of $Q$ and fix a subordinated partition of unity $(\chi_i)$. Secondly, we enlarge each set $V_i$ to some $W_i$, still in $Q$ but containing an $\varepsilon_i$-neighborhood of $V_i$ for some $\varepsilon_i(\varepsilon) > 0$, in order to have space for the convolution of $\chi_i u$ along $V_i$ by a mollifier $\rho_{\varepsilon_i} \in C_0^\infty(B_{\varepsilon_i})$.

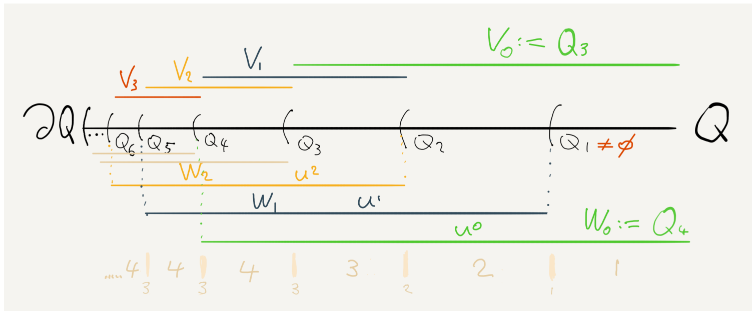


Figure 5.1: Locally finite covers $V_i := Q_{i+3} \setminus \bar{Q}_{i+1}$ and $W_i := Q_{i+4} \setminus \bar{Q}_i$ of $Q$

This provides the smooth functions $u^i := \rho_{\varepsilon_i} * (\chi_i u) \in C_0^\infty(W_i)$. Thirdly, we globalize by considering the sum of smooth functions $v = \sum u^i$ which is itself smooth since the sum will be locally finite. By construction along any given $V \Subset Q$ one gets $\|v - u\|_{W^{k,p}(V)} \le \varepsilon$. Now take the sup over all such $V$.

I. For $i \in \mathbb{N}$ set $Q_i := \{x \in Q \mid \operatorname{dist}(x, \partial Q) > \frac{1}{i}\} \Subset Q_{i+1}$ to get a nested open cover of $Q$. (If $Q_1 = \emptyset$, replace $\frac{1}{i}$ by $\frac{1}{i+c}$ for some large constant $c$ so $Q_1 \neq \emptyset$.) As we illustrated in Figure 5.1 define $V_i := Q_{i+3} \setminus \operatorname{cl} Q_{i+1}$ for $i \ge 1$ and $V_0 := Q_3$. Let $(\chi_i)_{i=0}^\infty$ be a **partition of unity subordinate to the open cover** of $Q$ by the $V_i$'s, that is $\chi_i \in C_0^\infty(V_i)$ and $\sum_{i=0}^\infty \chi_i \equiv 1$. Enlarge the $V_i$'s by setting $W_i := Q_{i+4} \setminus \operatorname{cl} Q_i$ for $i \ge 1$ and $W_0 := Q_4$. Obviously $\operatorname{supp}(\chi_i u) \subset V_i \subset W_i$ and $[\chi_i u] \in W^{k,p}(Q)$; cf. Proposition 4.2.1 (iv).

II. Let $\rho$ be a mollifier. Fix $i \in \mathbb{N}_0$. By Theorem 3.2.3 (i) there is convergence

$$u^\delta := \rho_\delta * (\chi_i u) \to \chi_i u \quad \text{in } L^p(Q), \text{ as } \delta \to 0,$$

and analogously by (iv) for $D^\alpha u^\delta = \rho_\delta * (\chi_i u)_\alpha$, see (4.1.3), and the weak derivative $(\chi_i u)_\alpha$ whenever $|\alpha| \le k$. Suppose $\delta = \delta(i) > 0$ is so small that

a) $U_\delta(V_i) \subset W_i$, so convolution is well defined and $u^\delta \in C_0^\infty(W_i)$;

b) $\|\rho_\delta * (\chi_i u) - \chi_i u\|_{W^{k,p}(Q)} \le \frac{\varepsilon}{2^{i+1}}$.

Set $\varepsilon_i := \delta(i)$ and $u^i := u^{\varepsilon_i} \in C_0^\infty(W_i)$.

III. The function $v := \sum_{i=0}^\infty u^i$ on $Q$ is well defined and smooth, because about any point of $Q$ there is an open neighborhood which meets at most five of the $W_i$'s, hence the support of at most five $u^i$'s; cf. Figure 5.1. Note that any of the $Q_j$'s from I. is pre-compact in $Q$. Writing $u = \sum \chi_i u$ we get that

$$\|v - u\|_{W^{k,p}(Q_j)} \le \sum_{i=0}^\infty \|\rho_{\varepsilon_i} * (\chi_i u) - \chi_i u\|_{W^{k,p}(Q_j)} \le \varepsilon$$

where the final step uses the inclusion $Q_j \subset Q$ and b).

With $f := \sum_{|\alpha| \le k} |D^\alpha (v - u)|^p$ the function

$$A \mapsto m_f(A) := \int_A f \, dm$$

is a measure on the Lebesgue $\sigma$-algebra on $Q$. Indeed the axioms of $\sigma$-additivity and non-triviality hold by $\sigma$-additivity of the integral and as $m_f(Q_j) \le \varepsilon < \infty$. Since the sequence of sets $Q_j \subset Q_{j+1}$ is ascending with union $Q$, Theorem 1.28 part (iv) in [Sal16] guarantees the second identity in the following

$$\|v-u\|^p_{W^{k,p}(Q)} = m_f(Q) = \lim_j m_f(Q_j) = \lim_j \|v-u\|^p_{W^{k,p}(Q_j)} \le \varepsilon^p.$$

This completes the proof of Theorem 5.1.3. □

### 5.1.3 Approximation even up to $\partial$ on Lipschitz domains

Imposing mild smoothness assumptions on the boundary of a pre-compact $D \Subset \mathbb{R}^n$, namely, Lipschitz continuity (thus differentiable almost everywhere; cf. Figure A.1), approximation of a $W^{k,p}(D)$ function is possible by smooth functions $u_\ell$ whose domain is the compact set $\overline{D}$, i.e. their domain even includes the boundary. This is in sharp contrast to Theorem 5.1.3 which provides approximations $u_\ell$ defined on the open set $D$ only, so finiteness of $\|u_\ell\|_{C^k(D)}$ is not guaranteed at all by Theorem 5.1.3.

**Definition 5.1.5** (Lipschitz domain $D$)**.** A pre-compact $D \Subset \mathbb{R}^n$ is called a **Lipschitz domain** [1] if for each point $x^0 \in \partial D$ there is a radius $r > 0$ and a Lipschitz continuous map $\gamma : \mathbb{R}^{n-1} \supset \omega \to \mathbb{R}$ with Lipschitz constant, say $M$, such that,[2] firstly, the part of the boundary $\partial D$ inside the open ball $B_r(x^0)$ is the graph of $\gamma$ and, secondly, the part of $D$ inside the ball is of the simple form

$$D \cap B_r(x^0) = \{x \in B_r(x^0) \mid x_n > \gamma(x_1, \dots, x_{n-1})\},$$

as illustrated in Figure 5.2. The Lipschitz constant $M$ of $\gamma$ is called the **Lipschitz bound** of the local parametrization $(\gamma, \omega)$ of the Lipschitz domain $D$.

**Definition 5.1.6** (Smooth domain $D$)**.** A pre-compact $D \Subset \mathbb{R}^n$ is called a $\boldsymbol{C^k}$**-domain** if all local graph maps $\gamma$ can be chosen of class $C^k$. In case $k = \infty$ we call $D$ a **smooth domain**.

The next theorem is about "global approximation by functions smooth up to the boundary of Lipschitz domains". It is wrong for $p = \infty$; cf. Remark 5.1.4.

**Theorem 5.1.7.** *Let $D \Subset \mathbb{R}^n$ be a Lipschitz domain. Suppose $u \in \boldsymbol{u} \in W^{k,p}(D)$ with $k \in \mathbb{N}$ and finite $p \in [1, \infty)$. Then there is a sequence $u_\ell \in C^\infty(\overline{D})$ with*[3]

$$C^\infty(\overline{D}) \ni u_\ell \to u \;\text{ in } W^{k,p}(D).$$

*Equivalently, over Lipschitz domains Sobolev spaces are closures of the form*

$$W^{k,p}(D) = \overline{C^\infty(\overline{D})}^{k,p}.$$

[1] These domains are called *special Lipschitz domains* in [Ste70, Ch. VI §3.2]. If one relaxes Lipschitz to $\alpha$-Hölder with $\alpha \in (0,1)$, then the Extension Theorem 5.2.1 fails as shown in [Ste70, Ch. VI §3.2].

[2] after possibly renaming and reorienting some coordinate axes, here $\omega \subset \mathbb{R}^{n-1}$ is open

[3] Observe that any given derivative $D^\alpha u_\ell$ extends continuously to the compact set $\overline{D}$, so it is bounded on $D$, hence $\|u_\ell\|_{C^k(D)} < \infty$.

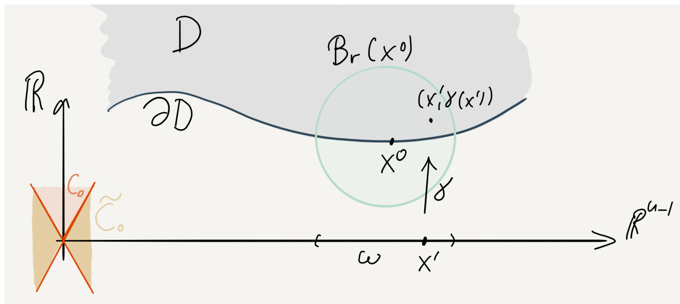


Figure 5.2: Lipschitz domain $D \subset \mathbb{R}^n$ and upward cone $C_0$

*Proof.* The proof takes three steps. Firstly, we generate room for mollification by shifting $u : D \to \mathbb{R}$ locally near $\partial D$ 'upward', meaning inside $D$. Here the Lipschitz condition enters which tames the geometry of $\partial D$ in that $\partial D$ is forced to stay in the horizontal part of one and the same double cone $\tilde{C}_0$ translated along $\partial D$, notation $\tilde{C}_x = x + \tilde{C}_0$ for $x \in \partial D$. The vertical upward cone $C_x$ will then be available for mollification.

STEP 1: UPWARD SHIFT TO GENERATE ROOM FOR MOLLIFICATION WITHIN $D$

Given $x^0 \in \partial D$, pick a Lipschitz continuous map $\gamma : \mathbb{R}^{n-1} \supset \omega \to \mathbb{R}$ whose graph is the part of $\partial D$ inside the open ball $B_r(x^0)$. By compactness of $\partial D$ the map $\gamma : \omega \to \mathbb{R}$ is uniformly Lipschitz with Lipschitz constant, say $c > 0$. The closed horizontal double cone $\tilde{C}_0$ and the **open upward cone** $C_0$

$$\tilde{C}_0 := \{(x', x_n) \mid \pm |x_n| \le c\,|x'|\} \qquad C_0 := \{(x', x_n) \mid \pm |x_n| \le c\,|x'|\}$$

are illustrated by Figure 5.2. The significance of the horizontal double cone $\tilde{C}_0$ is that wherever you put it along $\partial D$ – consider the translate $\tilde{C}_y := y + \tilde{C}_0$ at $y \in \partial D$ – *that translated horizontal double cone $\tilde{C}_y$ contains the graph of $\gamma$ and therefore $\partial D$ (locally near $y$).* Hence the open upward cone $C_y$ lies in $D$, at least its part that lies within some small radius from $y$, say $r(y)$. See Figure 5.3 in which $y$ is denoted by $\tilde{x}$.

Let $V := D \cap B_{r/2}(x^0)$ be the open region strictly above the graph but still inside the ball of radius $r/2$; see Figure 5.3. For any $x \in V$ define the by $\varepsilon > 0$ **upward shifted point**

$$x^\varepsilon := x + \varepsilon\lambda e_n \qquad , x \in V,\, \varepsilon > 0.$$

As illustrated by Figure 5.3, let us put the unit ball $B_1$ sufficiently far out, centered at $x+\lambda e_n$ where $\lambda = \lambda(x) >> 1$ is sufficiently large such that eventually the ball fits into the upward cone. But then the whole family of balls $B_\varepsilon(x + \varepsilon\lambda e_n)$ for $\varepsilon \in (0, 1)$ stays in the upward cone. Moreover, for all $\varepsilon > 0$ sufficiently small the family is located near $x$, hence in the open neighborhood $V$ of $x$. Choose $\lambda$ larger, if necessary, so it works for all $x \in \gamma(\omega) \subset \partial D$.

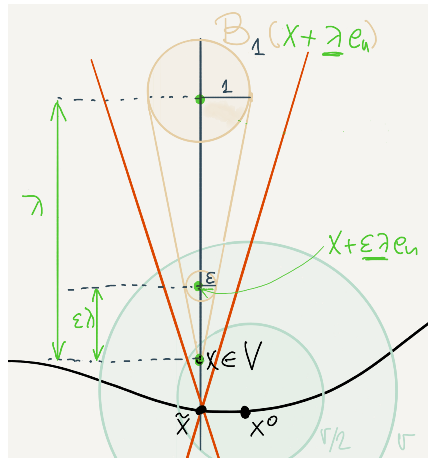


Figure 5.3: Fitting a ball $B_\varepsilon(x+\varepsilon\lambda e_n)$ into $C_{\tilde{x}} \cap B_r(x^0)$: $\lambda$ large, $\varepsilon$ small

Define the **translate** $u_\varepsilon(x) := u(x^\varepsilon)$, for $x \in V$. This is the function $u$ moved by distance $\varepsilon\lambda$ in the $e_n$ direction. Pick a mollifier $\rho = \{\rho_\varepsilon\}$. The convolution[4]

$$v^\varepsilon = \rho_\varepsilon * u_\varepsilon \in C^\infty(\overline{V})$$

is well defined on $V$ since by moving up we have created space for mollification. To see that $v^\varepsilon \in C^\infty(\overline{V})$, as opposed to $C^\infty(V)$, let us check that $v^\varepsilon$ is defined not only on the pre-compact $V$, but even on a small neighborhood. Let us illustrate this by considering the case $\tilde{x} \in \partial V \cap \partial D$, cf. Figure 5.3. In this case

$$v^\varepsilon(\tilde{x}) = \int_{B_\varepsilon(\tilde{x})} \rho_\varepsilon(\tilde{x}-y) \cdot u_\varepsilon(y)\, dy = \int_{B_\varepsilon(\tilde{x})} \rho_\varepsilon(\tilde{x}-y) \cdot u(\underbrace{y+\varepsilon\lambda e_n}_{\in B_\varepsilon(\tilde{x}+\varepsilon\lambda e_n)})\, dy.$$

But $B_\varepsilon(\tilde{x}+\varepsilon\lambda e_n)$ is contained in the upward cone $C_{\tilde{x}}$, so $v^\varepsilon(\tilde{x})$ is well defined. Argument and conclusion remain valid for points $x^*$ slightly below $\tilde{x}$, i.e. slightly outside $\overline{V}$, because $B_\varepsilon(x^*+\varepsilon\lambda e_n)$ still remains in $C_{\tilde{x}}$, thus in the domain of $u$.

Step 2: Convergence to $u$ along $V$

To see that $v^\varepsilon \to u$ in $W^{k,p}(V)$ pick a multi-index of order $|\alpha| \le k$, then

$$\begin{aligned}\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} &\le \|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} + \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)} \\ &\le \|\rho_\varepsilon * D^\alpha u_\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} + \|(u_\alpha)_\varepsilon - u_\alpha\|_{L^p(V)}\end{aligned}$$

where the second inequality holds by (4.1.3) and since weak derivative commutes with the linear operation of translation, in symbols $(u_\varepsilon)_\alpha = (u_\alpha)_\varepsilon$.[5] Now

[4] Use the natural zero extension of $u$, so $u_\varepsilon \in L^p(\mathbb{R}^n)$ and $\rho_\varepsilon * u_\varepsilon$ makes sense by (3.1.2).

[5] To see that $\int_V u_\varepsilon(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_V (u_\alpha)_\varepsilon(x)\phi(x)\, dx$ introduce the new variable $y := x+\varepsilon\lambda e_n$, take weak derivative of $u(y)$ according to (4.1.1), then go back to $x := y-\varepsilon\lambda e_n$.

consider the two terms in line two. As $\varepsilon \to 0$, term one goes to zero by Theorem 3.2.3 (i) and term two by Lemma 2.3.2. The lemma tells that the path $\gamma : \varepsilon \mapsto \tau_\varepsilon u_\alpha - u_\alpha$ is continuous with respect to the $L^p$ norm. Clearly $\gamma(0) = 0$.

Step 3: Globalization via partition of unity
Pick $\delta > 0$. By compactness of $\partial D$ there are finitely many points $x_i^0 \in \partial D$, radii $r_i > 0$, corresponding set $V_i = D \cap B_{r_i/2}(x_i^0)$, and functions $v^i \in C^\infty(\overline{V}_i)$, where $i = 1, \dots, N$, such that the balls $B_{r_i/2}(x_i^0)$ cover $\partial D$ and (by Step 2)

$$\left\| v^i - u \right\|_{W^{k,p}(V_i)} \le \delta. \tag{5.1.1}$$

Pick $V_0 \Subset D$ such that $(V_i)_{i=0}^N$ is an open cover of $D$. Utilize Theorem 5.1.1 to get a function $v^0 \in C^\infty(\overline{V}_0)$ that also satisfies (5.1.1).

Pick a smooth partition of unity $(\chi_i)_{i=0}^N$ subordinate to the open cover $(V_i)_{i=0}^N$. The finite sum of smooth functions $v := \sum_{i=0}^N \chi_i v_i$ is smooth and defined on some neighborhood of $D$. Thus $v \in C^\infty(\overline{D})$. Suppose $|\alpha| \le k$ and use the Leibniz rule Proposition 4.2.1 (iv) to get that

$$\begin{aligned}
\|D^\alpha v - D^\alpha u\|_{L^p(D)} &\le \sum_{i=0}^N \left\| D^\alpha(\chi_i v^i) - D^\alpha(\chi_i u) \right\|_{L^p(V_i)} \\
&= \sum_{i=0}^N \left\| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left( D^\beta \chi_i \cdot D^{\alpha-\beta} v^i - D^\beta \chi_i \cdot D^{\alpha-\beta} u \right) \right\|_{L^p(V_i)} \\
&\le C \sum_{i=0}^N \left\| v^i - u \right\|_{W^{k,p}(V_i)} \\
&\le C(N+1)\delta
\end{aligned}$$

where the constant $C = C(k,p) > 0$ also incorporates the $C^k$ norms of the cutoff functions $\chi_i$. This completes the proof of Step 3 and Theorem 5.1.7. □

## 5.2 Extensions and traces

### 5.2.1 Extension

Whereas in the realm of $L^p$ spaces extending an $L^p$ function on a domain $\Omega \subset \mathbb{R}^n$ to all $\mathbb{R}^n$ within $L^p$ is trivial, just extend naturally by zero. This does not work for Sobolev spaces, already not for those of first order $W^{1,p}$. Roughly speaking, jump singularities obstruct existence of weak derivatives, while corners are still digestible; cf. Exercise 4.1.7. The following extension theorem works for any $p \in [1, \infty]$, finite or not. The same smoothness assumption on the boundary as below will be required in both Sobolev inequalities, the sub- and the super-dimensional one, Theorems 6.1.3 and 6.2.3, respectively, because both proofs build on Theorem 5.2.1. Therefore we spell out details in the Lipschitz case, as opposed to the common $C^1$ assumption; cf. [Eva98, §5.4]. The $C^1$ condition allows to deal with the open unit ball – but not even the open unit cube in $\mathbb{R}^n$.

The extension theorem for Lipschitz domains is due to Calderon [Cal61, $1<p<\infty$] and Stein [Ste70, Ch. VI §3.4, $p=1,\infty$].

**Theorem 5.2.1** (Extension)**.** *Suppose $p\in[1,\infty]$ and let $D\Subset\mathbb{R}^n$ be a Lipschitz domain. Then the following is true. For any pre-compact $Q\Subset\mathbb{R}^n$ that contains the closure of $D$, in symbols $D\Subset Q\Subset\mathbb{R}^n$, there is a bounded linear operator*

$$E: W^{1,p}(D)\to W^{1,p}_0(Q)\hookrightarrow W^{1,p}(\mathbb{R}^n),\quad \boldsymbol{u}\mapsto E\boldsymbol{u}=\bar{\boldsymbol{u}}, \tag{5.2.2}$$

*such that $\bar{\boldsymbol{u}}|_D=\boldsymbol{u}$, equivalently $\bar{u}|_D=u$ a.e. along $D$ for representatives, and* [6]

$$\|\bar{u}\|_{W^{1,p}(Q)}\le c\,\|u\|_{W^{1,p}(D)}$$

*whenever $\bar{u}\in\bar{\boldsymbol{u}}$ and $\boldsymbol{u}\in W^{1,p}(D)$; the constant $c>0$ only depends on $n,D,Q$. In the case $p=\infty$ the constant is actually independent of $n$.*

The function $Eu$ is called an **extension** of $u\in\boldsymbol{u}\in W^{1,p}(D)$ to $Q$. For Lipschitz domains of class $C^{k-1,1}$ the extension operators are of the form $E: W^{k,p}(D)\to W^{k,p}_0(Q)$ and the constant $c$ also depends on $k$; see [GT01, Thm. 7.25] and [MS04, Prop. B.1.9].

*Idea of proof.* For details see e.g. [Eva98, §5.4]. The idea is to deal first with the model case of a function, say $v$, defined on an open upper half ball $V$; see Figure 5.4. One extends $v$ of class $C^1$ up to the boundary to the closure of the whole unit ball $B$ by explicit formulas (called reflections and depending on the desired degree of regularity). Secondly, "flatten out" the boundary through a coordinate change given by the natural **bi-Lipschitz map**[7] that comes from the Lipschitz graph map $\gamma$ of $\partial D$; see Figure 5.5. Here $D$ being a Lipschitz domain enters. Thirdly, construct local extensions along the boundary. Fourthly, globalize via partition of unity and cut off between $D$ and $Q$. □

*Proof.* The proof is in three steps. For finite $p$ we follow largely [Eva98, §5.4].

Step 1 (Extension operator in the half ball model). Suppose $B\subset\mathbb{R}^n$ is an open ball with center $y^0$ in the plane $\{y_n=0\}$. Let $B_+$ be the open upper half-ball of $B$. Assume $p\in[1,\infty]$. Then there is a linear map

$$E_0: W^{1,p}(B_+)\to W^{1,p}(B),\quad \boldsymbol{v}\mapsto E_0\boldsymbol{v}=\bar{\boldsymbol{v}}, \tag{5.2.3}$$

that satisfies the identity $\bar{\boldsymbol{v}}|_{B_+}=\boldsymbol{v}$, i.e. acts by extension. The estimates hold

$$\|\bar{v}\|_{W^{1,p}(B)}\le c_n\,\|v\|_{W^{1,p}(B_+)}\,,\quad \|\bar{v}\|_{W^{1,\infty}(B)}=\|v\|_{W^{1,\infty}(B_+)}\,, \tag{5.2.4}$$

for every $v\in\boldsymbol{v}$ and finite $p$. The constant is given by $c_n=16\cdot 2^{2n-1}$.

*Proof of Step 1.* Let $\Sigma=B\cap\{y_n=0\}$ be the $(n-1)$ ball that divides $B$ into $B_+$ and $B_-$; see Figure 5.4.

[6] The notation $\bar{\boldsymbol{u}}:=E\boldsymbol{u}$ is useful. It helps to avoid misleading notation such as writing $Eu\in E\boldsymbol{u}$ for an element of the extension equivalence class. There is no naturally defined operator of the form $u\mapsto Eu$ on the level of Sobolev functions, only on subcategories. In contrast, the theorem provides an operator $\boldsymbol{u}\mapsto E\boldsymbol{u}$ on the level of equivalence classes.

[7] Lipschitz homeomorphism whose inverse is Lipschitz

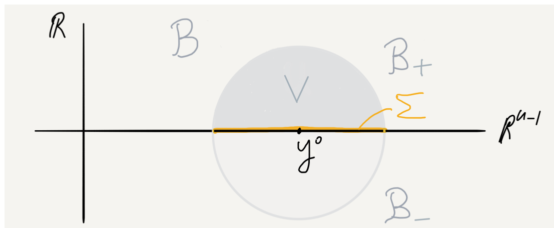


Figure 5.4: Open upper half $B_+$ of open ball $B$

**Finite $p \in [0,\infty)$.** Consider the inclusions $C^\infty(\overline{B}_+) \subset C^1(\overline{B}_+) \subset W^{1,p}(B_+)$. By Theorem 5.1.7, here finite $p$ enters, the first set is dense in $W^{1,p}$, hence so is the second one, abbreviated $C^1$. For $v \in C^1$ we define the extension $E_0 v := \bar{v}$ pointwise and show that it lies in $C^1(\overline{B})$, thus in $W^{1,p}(B)$. We then prove the estimate (5.2.4) for any $v$ in the dense set $C^1$. By density $E_0$ extends uniquely to a bounded linear operator with domain $W^{1,p}(B)$ and we are done.

To this end we define the extension of $v \in C^1(\overline{B}_+)$ to $\overline{B}$ by[8]

$$\bar{v}(y) := \begin{cases} v(y) & , y \in \overline{B}_+, \\ -3v(y_*, -y_n) + 4v(y_*, -\frac{y_n}{2}) & , y = (y_*, y_n) \in \overline{B}_-. \end{cases}$$

Here and throughout $y_* = (y_1, \dots, y_{n-1})$ and $y = (y_*, y_n) \in \mathbb{R}^n$. To see that $\bar{v} \in C^1(\overline{B})$ write $v^+ := \bar{v}|_{\overline{B}_+} = v$ and $v^- := \bar{v}|_{\overline{B}_-}$. So $v^- = v^+$ along $\Sigma$. Now

$$\partial_n v^-(y) = 3\partial_n v(y_*, -y_n) - 2\partial_n v(y_*, -\tfrac{y_n}{2}).$$

Along $\{y_n = 0\}$ this becomes $(3-2)\partial_n v(y_*, 0) = \partial_n v^+(y_*, 0)$. So $\partial_n v^- = \partial_n v^+$ along $\{y_n = 0\}$. As $v^- = v^+$ along $\{y_n = 0\}$, we also have $\partial_i v^- = \partial_i v^+$ along $\{y_n = 0\}$ for $i = 1, \dots, n-1$. Hence $D^\alpha v^- = D^\alpha v^+$ along $\{y_n = 0\}$ for $|\alpha| \le 1$. This proves that $\bar{v} \in C^1(\overline{B})$. Straightforward calculation[9] proves (5.2.4).

**Infinite $p = \infty$.** Consider the linear map ($W^{1,\infty} \simeq C^{0,1}$ by Theorem 7.2.1)

$$E_0 : C^{0,1}(B_+) \to C^{0,1}(B), \quad v \mapsto E_0 v := \bar{v},$$

where the extension $\bar{v}$ of $v$ from $\overline{B}_+$ to $\overline{B}$ is given by simple horizontal reflection

$$\bar{v}(y) := \begin{cases} v(y) & , y \in \overline{B}_+, \\ v(y_*, -y_n) & , y = (y_*, y_n) \in \overline{B}_- \subset \mathbb{R}^{n-1} \times \mathbb{R}. \end{cases}$$

Check that $\bar{v}$ is indeed Lipschitz, even with the same Lipschitz constant as $v$.[10] Check that $\|\bar{v}\|_{W^{1,\infty}(B)} = \|v\|_{W^{1,\infty}(B_+)}$ for every $v \in W^{1,\infty}(B_+)$. □

---

[8] Try to pointwise define $\bar{v}$ for $v \in \boldsymbol{v} \in W^{1,p}(B_+)$. Which values to assign along $\Sigma$?

[9] With $|D\bar{v}|^2 = |\partial_1 \bar{v}|^2 + \dots + |\partial_n \bar{v}|^2$ get $\|\bar{v}\|_{L^p(B)} \le (1+8^p)^{1/p} \|v\|_{L^p(B_+)} \le 16 \|v\|_{L^p(B_+)}$ and $\|D\bar{v}\|_{L^p(B)} \le (\kappa(1+8^p))^{1/p}\, 2^{n-1} \|Dv\|_{L^p(B_+)} \le 16 \cdot 2^{2n-2} \|Dv\|_{L^p(B_+)}$. By Exercise A.1.2 $\kappa^{1/p} \le 2^{n-1}$. Thus $c_n = 16 \cdot 2^{2n-1}$ (depends on $n$ as we used $\ell^2$ norm to define $|Dv|$).

[10] Hints: This is clear if $x, y$ are in the same half ball. If one is in $B_+$, the other in $B_-$, consider the unique point $z$ of the line segment between $x$ and $y$ that lies in $\Sigma$. In $|\bar{v}(x) - \bar{v}(y)|$ add $0 = -\bar{v}(z) + \bar{v}(z)$ and note that $|x - z| + |z - y| = |x - y|$.

Step 2 (Boundary flattening coordinates).
To "flatten out" the boundary of $D$ locally near $x^0 \in \partial D$ recall that by Definition 5.1.6 the part of $\partial D$ in a small open ball $B_r(x^0)$ is the graph of a Lipschitz map $\gamma : \mathbb{R}^{n-1} \supset \omega \to \mathbb{R}$ with Lipschitz constant $M$. The domain $D$ lies above the graph of $\gamma$; see Figure 5.5.

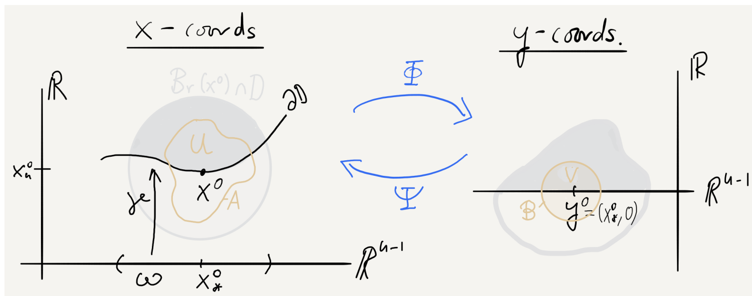


Figure 5.5: Coordinates that locally flatten out $\partial D$

Consider the open neighborhoods $X := \omega \times \mathbb{R}$ of $x^0 = (x_*^0, x_n^0)$ and $Y := \omega \times \mathbb{R}$ of $y^0 = (x_*^0, 0)$. Define a bi-Lipschitz map $\Phi : X \to Y$ by the formula

$$y = \Phi(x) \qquad \begin{cases} y_i := x_i = \Phi^i(x) & , \; i = 1, \dots, n-1, \\ y_n := x_n - \gamma(x_*) =: \Phi^n(x) & . \end{cases} \tag{5.2.5}$$

Note that $|\Phi(x) - \Phi(z)| = |x - z|$ and $\Phi(x^0) = y^0$. The inverse $\Psi : Y \to X$ is

$$x = \Psi(y) \qquad \begin{cases} x_i := y_i = \Psi^i(y) & , \; i = 1, \dots, n-1, \\ x_n := y_n + \gamma(y_*) =: \Psi^n(y) & . \end{cases}$$

Note that $|\Psi(y) - \Psi(z)| = |y - z|$ and $\Phi = \Psi^{-1}$. Observe that the map $x \mapsto \Phi(x)$ "flattens out" $\partial D$ near $x^0$ in the sense that the boundary image is the open subset $\omega$ of $\mathbb{R}^{n-1}$. By Rademacher's Theorem 7.3.2 the graph map $\gamma$ is differentiable for a.e. $x_* \in \omega$. Hence the linearizations of $\Phi$ and $\Psi$ exist pointwise a.e. and, furthermore, they are triangular matrizes with diagonal elements 1. Thus $\det d\Phi = 1 = \det d\Psi$ pointwise a.e.

Step 3 (Local extension $(x^0, \bar{u} := \bar{v} \circ \Psi^{-1})$ of $u|_U$ to $A$ about $x^0 \in \partial D$).
Choose $u \in \boldsymbol{u} \in W^{1,p}(D)$. Pick a ball $B$ centered at the point $y^0 = \Phi(x^0)$ and contained in the open neighborhood $\Phi(B_r(x^0))$ of $y^0$, as illustrated by Figure 5.5. Let $V := B_+$ be the upper open half ball of $B$. Now consider the restriction of $u : D \to \mathbb{R}$ to the open set $U := \Psi(V)$, still denoted by $u : U \to \mathbb{R}$. Observe that $u \in \mathcal{W}^{1,p}(U)$; just restrict the weak derivatives.
Next pull back $u : U \to \mathbb{R}$ to the $y$ coordinates to obtain the function $v := u \circ \Psi : V \to \mathbb{R}$ which lies in $\mathcal{W}^{1,p}(V)$ by the change of coordinates Proposition 4.1.23. The identity $\|v\|_{W^{1,p}(V)} = \|u\|_{W^{1,p}(U)}$ holds since $d\Psi$ is of unit determinant; use (4.1.7) in case $p = \infty$ and the transformation law (4.1.8) in case of finite $p$.

Next employ the extension operator constructed in Step 1 to pick an extension $\bar{v} \in \bar{\boldsymbol{v}} := E_0\boldsymbol{v}$ of $v = u \circ \Psi$ from the upper half ball $V = B_+$ to the whole ball $B$. The extension of $u$ from $U = \Psi(V)$ to $A := \Psi(B)$ is defined by

$$\bar{u} := \bar{v} \circ \Psi^{-1} = \overline{u \circ \Psi} \circ \Psi^{-1} \in \mathcal{W}^{1,p}(A), \qquad \bar{\boldsymbol{u}} := [\bar{u}] \in W^{1,p}(A).$$

By the argument five lines above $\|\bar{u}\|_{W^{1,p}(A)} = \|\bar{v}\|_{W^{1,p}(B)}$. The estimate (5.2.4) for the extension operator $E_0$ in the half ball model of Step 1 shows that

$$\|\bar{u}\|_{W^{1,p}(A)} = \|\bar{v}\|_{W^{1,p}(B)} \leq c_n \|v\|_{W^{1,p}(B_+)} = c_n \|u\|_{W^{1,p}(U)} \tag{5.2.6}$$

for all $u \in \boldsymbol{u} \in W^{1,p}(D)$ and finite $p$. For $p = \infty$ equality holds and $c_n = 1$.

Step 4 (Globalising via finite partition of unity).
Let $p \in [1, \infty]$ and $u \in \boldsymbol{u} \in W^{1,p}(D)$. By Step 3 and compactness of $\partial D$ there are finitely many boundary points $x^1, \dots, x^N \in \partial D$ and local extensions $(x^i, \bar{u}^i = \bar{v}^i \circ \Psi^{-1} : A_i \to \mathbb{R})_{i=1}^N$ covering $\partial D$, that is $\partial D \subset A_1 \cup \ldots \cup A_N$. Recall from Step 3 that $U_i = \Psi(V_i)$ is the part of $A_i = \Psi(B_i)$ in $D$. Pick $U_0 \Subset D$ in order to get an open cover $(U_i)_{i=0}^N$ of $D$. Set $A_0 := U_0$ in order to get a pre-compact set $A := \cup_{i=0}^N A_i$ containing the closure of $D$, in symbols $D \Subset A \Subset \mathbb{R}^n$. Pick a smooth partition of unity $\chi = (\chi_i)_{i=0}^N$ subordinate to the open cover $(A_i)_{i=0}^N$ of $A$. Extend $u \in \boldsymbol{u} \in W^{1,p}(D)$ to $A$ by

$$\bar{u} := \sum_{i=0}^N \chi_i \bar{u}_i : A \to \mathbb{R}, \qquad \bar{\boldsymbol{u}} := [\bar{u}] \in W^{1,p}(A), \tag{5.2.7}$$

where $\bar{u}_0 := u : A_0 = U_0 \to \mathbb{R}$ and where as in Step 3

$$\bar{u}^i := \bar{v}^i \circ \Psi^{-1} = \overline{u^i \circ \Psi} \circ \Psi^{-1} \in \mathcal{W}^{1,p}(A_i), \quad A_i = \Psi(B_i).$$

Indeed $\bar{u} \in \mathcal{W}^{1,p}(A)$ by Proposition 4.2.1 (iv). There are the estimates

$$\begin{aligned}
\|\bar{u}\|_{W^{1,p}(A)} &\leq \sum_{i=0}^N \|\chi_i \bar{u}_i\|_{W^{1,p}(A_i)} \\
&\leq \sum_{i=0}^N \|\chi_i\|_{W^{1,\infty}(A_i)} \|\bar{u}_i\|_{W^{1,p}(A_i)} \\
&\leq c_n \left( \max_{i=0,\dots,N} \|\chi_i\|_{1,\infty} \right) \sum_{i=0}^N \left\| u^i \right\|_{W^{1,p}(U_i)} \\
&\leq c_n (N+1) \left( \max_{i=0,\dots,N} \|\chi_i\|_{1,\infty} \right) \|u\|_{W^{1,p}(D)} \\
&=: C \|u\|_{W^{1,p}(D)}
\end{aligned}$$

where $C$ depends only on $n$ and $D$ for finite $p$ and only on $D$ in case $p = \infty$. Inequality two uses the product rule (4.1.6). Inequality three holds by (5.2.6) for $\bar{u}^i$; recall that $c_n = 16 \cdot 2^{2n-1}$ for finite $p$ and $c_n = 1$ for $p = \infty$. The final inequality four uses that $u^i : U_i \to \mathbb{R}$ is just the restriction of $u : D \to \mathbb{R}$ and this happens $N + 1$ times.

Given $\boldsymbol{u} \in W^{1,p}(D)$ and $Q$ such that $D \Subset Q \Subset \mathbb{R}^n$, then $D \Subset (Q \cap A) \Subset \mathbb{R}^n$. Pick a cutoff function $\chi \in C_0^\infty(Q \cap A, [0,1])$ with $\chi \equiv 1$ on $D$. Then $\chi \bar{\boldsymbol{u}} = [\chi \bar{u}] \in W^{1,p}(Q)$ by Proposition 4.2.1 (iv) where $\bar{\boldsymbol{u}} = [\bar{u}] \in W^{1,p}(Q)$ is the restriction to $Q$ of (5.2.7). That $\chi \bar{\boldsymbol{u}}$ even lies in a subspace, namely

$$E\boldsymbol{u} := \chi \bar{\boldsymbol{u}} \in W_0^{1,p}(Q) := \overline{C_0^\infty(Q)}^{1,p},$$

follows by approximating $\bar{u} \in \bar{\boldsymbol{u}} \in W^{1,p}(Q)$ by smooth functions $v_\ell \in C^\infty(Q)$ according to Theorem 5.1.3. So the classes of $\chi v_\ell \in C_0^\infty(Q)$ approximate $\chi \bar{\boldsymbol{u}}$. As for the estimate, since $\operatorname{supp} \chi \subset (Q \cap A)$ and using the previous estimate

$$\begin{aligned}
\|\chi \bar{u}\|_{W^{1,p}(Q)} &= \|\chi \bar{u}\|_{W^{1,p}(Q \cap A)} \\
&\le \|\chi \bar{u}\|_{W^{1,p}(A)} \\
&\le \|\chi\|_{W^{1,\infty}(A)} \|\bar{u}\|_{W^{1,p}(A)} \\
&\le C \|\chi\|_{W^{1,\infty}(A)} \|u\|_{W^{1,p}(D)} .
\end{aligned}$$

This concludes the proof of Theorem 5.2.1. □

### 5.2.2 Trace

While the usual notion of differentiability is hereditary – the restriction of a $C^1$ function to a codimension one submanifold is $C^1$ again – for weak derivatives the situation is more subtle: The values of a function on a set of measure zero are irrelevant, but a codimension one submanifold is of measure zero itself. The following trace theorem works for finite $p \in [1, \infty)$.

**Theorem 5.2.2** (Trace). *Let $p \in [1, \infty)$ be finite and $D \Subset \mathbb{R}^n$ be Lipschitz. Then there is a bounded linear operator $T : W^{1,p}(D) \to L^p(\partial D)$, $\boldsymbol{u} \mapsto T\boldsymbol{u}$, such that for any $u \in \boldsymbol{u} \in W^{1,p}(D)$ one has*[11]

$$\|Tu\|_{L^p(\partial D)} \le c \|u\|_{W^{1,p}(D)}$$

*where the constant $c > 0$ only depends on $p$ and $D$. In case $\boldsymbol{u}$ has a uniformly continuous representative $u^*$, in symbols $u^* \in C^0(\overline{D})$, the trace $T\boldsymbol{u}$ has a (uniformly) continuous representative, denoted by $(Tu)^*$, which is given by restricting $u^*$ to the boundary, in symbols $(Tu)^* = u^*|_{\partial D}$, likewise of $u \in \boldsymbol{u}$.*

Any representative function $Tu \in T\boldsymbol{u} \in L^p(\partial D)$ is called a **trace** of $\boldsymbol{u} \in W^{1,p}(D)$ on $\partial D$.

[11] We use the abusive notation $Tu \in T\boldsymbol{u} = T[u]$, abusive since $Tu$ is not an operator applied to $u$, but $Tu$ just denotes an element of $L^p(\partial D)$. We should really write $f \in T\boldsymbol{u} \in L^p(\partial D)$.

*Proof.* Details are given in [Eva98, §5.5] for $C^1$ domains; just use the Lipschitz coordinate change (5.2.5) and the Lipschitz approximation Theorem 5.1.7. The skeleton of the proof is the same as the one of the Extension Theorem 5.2.1. One starts with the model case where $\partial D$ is locally flat. $\square$

**Theorem 5.2.3** (Trace zero functions in $W^{1,p}$)**.** *Let $p \in [1, \infty)$ and let $D \Subset \mathbb{R}^n$ be Lipschitz. For $[u] \in W^{1,p}(D)$ there is the equivalence*

$$[u] \in W_0^{1,p}(D) \qquad \Leftrightarrow \qquad Tu = 0 : \partial D \to \mathbb{R}.$$

*Proof.* '⇒' Theorem 5.2.2. '⇐' Harder, see [Eva98, §5.5] for details. $\square$

# Chapter 6

# Sobolev inequalities

## 6.1 Sub-dimensional case $kp < n$

### 6.1.1 Gagliardo-Nirenberg-Sobolev inequality ($p < n$)

Throughout this section $k = 1$, so **sub-dimensional case** means that

$$1 \le p < n \qquad \text{(thus } n \ge 2\text{).}$$

A powerful tool, as we shall see, would be an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \le c \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Compact support is necessary as $u \equiv 1$ shows. For which $q$ and $c$ can such estimate be expected? Suppose $0 \not\equiv u \in C_0^\infty(\mathbb{R}^n)$, pick a constant $\lambda > 0$ and insert the rescaled function $u_\lambda(x) := u(\lambda x)$ in the estimate above. Details are given in [Eva98, §5.6.1]. The upshot is that validity of the estimate implies the identity $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Now for this particular $q$ the estimate indeed holds true – it is the Gagliardo-Nirenberg-Sobolev inequality.

**Definition 6.1.1.** If $p \in [1, n)$, the **Sobolev conjugate of $p$** is

$$p^* := \frac{np}{n-p} \in (p, \infty)$$

or equivalently

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \qquad p \in [1, n), \quad p^* \in (p, \infty). \tag{6.1.1}$$

**Theorem 6.1.2** (Gagliardo-Nirenberg-Sobolev inequality)**.** *If $p \in [1, n)$, then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le \gamma_p \|Du\|_{L^p(\mathbb{R}^n)} \tag{6.1.2}$$

*for every compactly supported $u \in C_0^1(\mathbb{R}^n)$ and with $\gamma_p = \gamma_p(n) := \frac{p(n-1)}{n-p}$.*

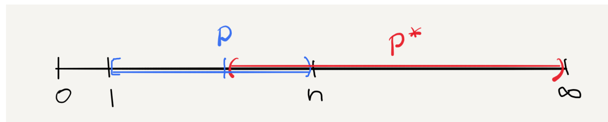


Figure 6.1: Sobolev conjugate $p^* := \frac{np}{n-p} \in (p, \infty)$ of $p \in [1, n)$

Note that $\gamma_1 = 1$ and $\gamma_{p>1} > 1$ explodes for $p \to n$. Compact support is necessary as $u \equiv 1$ shows. But the constant $\gamma_p$ doesn't see the size of $\operatorname{supp} u$.

*Proof.* There are two steps I ($p = 1$) implying II ($1 < p < n$); cf. [Eva98, §5.6.1].

**I. Case $\boldsymbol{p = 1}$.** By the fundamental theorem of calculus and compact support

$$\begin{aligned} |u(x)| &= \left| \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \dots, y_i, \dots, x_n \, dy_i) \right| \\ &\le \int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n \, dy_i)| \, dy_i. \end{aligned}$$

Multiply the inequalities for $i = 1, \dots, n$ and take the power $\frac{1}{n-1}$ to get

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n \, dy_i)| \, dy_i \right)^{\frac{1}{n-1}}$$

Integrate the variable $x_1$ over $\mathbb{R}$, then extract the term corresponding to $i = 1$ (it does not depend on $x_1$) to obtain that

$$\begin{aligned} &\int_{x_1=-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} \, dx_1 \\ &\le \int_{x_1=-\infty}^{\infty} \prod_{i=1}^{n} \left( \int_{y_i=-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)| \, dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &= \left( \int_{y_1=-\infty}^{\infty} |Du(y_1, x_2 \dots, x_n)| \, dy_1 \right)^{\frac{1}{n-1}} \\ &\qquad \int_{x_1=-\infty}^{\infty} \underbrace{\prod_{i=2}^{n} \overbrace{\left( \int_{y_i=-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)| \, dy_i \right)^{\frac{1}{n-1}}}^{f_i(x_1)}}_{f_2(x_1) \dots f_n(x_1), \quad \frac{1}{n-1} + \dots + \frac{1}{n-1} = 1} dx_1 \\ &\le \left( \int_{y_1=-\infty}^{\infty} |Du(y_1, x_2 \dots, x_n)| \, dy_1 \right)^{\frac{n}{n-1}} \\ &\qquad \prod_{i=2}^{n} \Bigg( \int_{x_1=-\infty}^{\infty} \underbrace{\int_{y_i=-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)| \, dy_i \, dx_1}_{|f_i(x_1)|^{n-1}} \Bigg)^{\frac{n}{n-1}} \end{aligned}$$

where the last step is by the generalized Hölder inequality (2.2.7).

Now integrate this estimate with respect to $x_2$ over $\mathbb{R}$, integrate the resulting estimate with respect to $x_3$ and so on until finally you have integrated the $x_n$ variable over $\mathbb{R}$ and have gotten to the desired estimate (6.1.2) for $p=1$, namely

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx &\le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} |Du|\, dx_1\dots dy_i\dots dx_n\right)^{\frac{1}{n-1}} \\ &= \left(\int_{\mathbb{R}^n} |Du|\, dx\right)^{\frac{n}{n-1}}.\end{aligned} \tag{6.1.3}$$

For a few more details of the iteration we refer to [Eva98, §5.6.1 Thm. 1].

**II. Case $p \in (1,n)$.** Apply (6.1.3) to $v := |u|^\gamma$ to determine $\gamma$, namely

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} |D\,|u|^\gamma| = \gamma\int_{\mathbb{R}^n} \underbrace{|u|^{\gamma-1}}_{\frac{p-1}{p}}\, \underbrace{|Du|}_{+\ \frac{1}{p}=1} \\ &\le \gamma\left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n} |Du|^p\right)^{\frac{1}{p}}\end{aligned}$$

where we used Hölder (2.2.5). Set $\gamma := \frac{p(n-1)}{n-p}$ to equalize the two exponents. Note that $(\gamma-1)\frac{p}{p-1} = \gamma\frac{n}{n-1} = \frac{pn}{n-p} = p^*$, hence $\frac{n-1}{n} = \frac{\gamma}{p^*}$ and $\frac{p-1}{p} = \frac{\gamma-1}{p^*}$. So the above estimate becomes $\|u\|_{p^*}^{\gamma} \le \gamma \|u\|_{p^*}^{\gamma-1}\|Du\|_p$ which is (6.1.2). $\square$

The proof of the following theorem is based on the Extension Theorem 5.2.1 and therefore requires a Lipschitz domain $D$.

**Theorem 6.1.3** (Sub-dimensional $W^{1,p}(D)$ estimates)**.** *Suppose $D \Subset \mathbb{R}^n$ is Lipschitz. If $p \in [1,n)$ and $\boldsymbol{u} \in W^{1,p}(D)$, then $\boldsymbol{u} \in L^{p^*}(D)$ with the estimate*

$$\|u\|_{L^{p^*}(D)} \le c\,\|u\|_{W^{1,p}(D)}$$

*where the constant $c$ depends only on $p$, $n$, and $D$.*

The idea of proof is to extend $\boldsymbol{u} \in W^{1,p}(D)$ to a compact support Sobolev class $\bar{\boldsymbol{u}} \in W_0^{1,p}(Q)$ where $D \Subset Q \Subset \mathbb{R}^n$. Approximate a representative $\bar{u} \in \bar{\boldsymbol{u}}$ in $W^{1,p}$ by a Cauchy sequence $u_m \in C_0^\infty(Q)$. Use the Gagliardo-Nirenberg-Sobolev inequality to see that the $u_m$, extended to $\mathbb{R}^n$ by zero, also form a Cauchy sequence in $L^{p^*}$ with the same limit $\bar{u}$. Hence $\bar{u} \in L^{p^*}$ and the inequality $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le \gamma_p\,\|Du_m\|_{L^p(\mathbb{R}^n)}$ also holds for the limit element $\bar{u}$.

*Proof.* Extend $\boldsymbol{u} \in W^{1,p}(D)$ to $\bar{\boldsymbol{u}} := E\boldsymbol{u} \in W_0^{1,p}(Q) := \overline{C_0^\infty(Q)}^{1,p}$ for any pre-compact $Q$ with $D \Subset Q \Subset \mathbb{R}^n$ using the Extension Theorem 5.2.1. Pick a representative $\bar{u} \in \bar{\boldsymbol{u}}$ and approximate it by a Cauchy sequence $u_m \in C_0^\infty(Q)$ that converges to $\bar{u}$ in the $W^{1,p}$-norm, in symbols

$$u_m \to \bar{u}, \qquad Du_m \to D\bar{u}, \qquad \text{both in } L^p(Q).$$

Extend the $u_m$ to $\mathbb{R}^n$ by zero, then the Gagliardo-Nirenberg-Sobolev inequality $\|u_m - u_\ell\|_{p^*} \le \gamma_p \|Du_m - Du_\ell\|_p$ shows that the $u_m$ also form a Cauchy sequence in $L^{p^*}(Q)$ with limit, say $v \in L^{p^*}(Q)$. By compactness of $Q$ and since $p^* > p$ Hölder shows that $L^{p^*}$ convergence implies $L^p$ convergence; cf. Figure 6.2. Hence $v = \bar{u}$ by uniqueness of the limit. Taking limits on both sides of the Gagliardo-Nirenberg-Sobolev inequality $\|u_m\|_{L^{p^*}(Q)} \le \gamma_p \|Du_m\|_{L^p(Q)}$ and, in the first step, monotonicity of the integral shows that

$$\|\bar{u}\|_{L^{p^*}(D)} \le \|\bar{u}\|_{L^{p^*}(Q)} \le \gamma_p \|D\bar{u}\|_{L^p(Q)} \le \gamma_p \|\bar{u}\|_{W^{1,p}(Q)} \le \gamma_p C \|u\|_{W^{1,p}(D)}$$

where $C = C(p, D, Q)$ is the constant in the Extension Theorem 5.2.1. □

The difference between Theorem 6.1.3 and the next estimate is that only the gradient of $u$ appears on the right hand side if we work with compact support Sobolev spaces. Lipschitz boundary is not needed, but pre-compactness $Q \Subset \mathbb{R}^n$ since the Gagliardo-Nirenberg-Sobolev inequality requires compact support.

**Theorem 6.1.4** (Poincaré inequality – sub-dimensional $W_0^{1,p}(Q)$ estimate). *Let $Q \Subset \mathbb{R}^n$. If $p \in [1, n)$ and $\boldsymbol{u} \in W_0^{1,p}(Q)$, then $\boldsymbol{u} \in L^q(Q)$ with the estimate*

$$\|u\|_{L^q(Q)} \le c \|Du\|_{L^p(Q)}$$

*whenever $q \in [1, p^*]$ and where the constant $c$ depends only on $p$, $q$, $n$, and $Q$.*

The Poincaré inequality tells that on compact support Sobolev spaces the norm $\|u\|_{W^{1,p}(Q)}$ is equivalent to $\|Du\|_{L^p(Q)}$.

*Proof.* Given $\boldsymbol{u} \in W_0^{1,p}(Q) := \overline{C_0^\infty(Q)}^{1,p}$, pick a Cauchy sequence $(u_m) \subset C_0^\infty(Q)$ that converges in the $W^{1,p}$-norm to a representative $u$ of $\boldsymbol{u}$. Extend the $u_m$ to $\mathbb{R}^n$ by zero. Then the Gagliardo-Nirenberg-Sobolev inequality (6.1.2)

$$\|u_m\|_{L^{p^*}(Q)} = \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le \gamma_p \|Du_m\|_{L^p(\mathbb{R}^n)} = \gamma_p \|Du_m\|_{L^p(Q)}$$

tells that $(u_m)$ is a Cauchy sequence in $L^{p^*}(Q)$. Thus $u_m$ converges in $L^{p^*}$ to some $v \in \mathcal{L}^{p^*}(Q)$, but also in $L^p$ to $u \in \mathcal{L}^p(Q)$. Hence $u_m$ converges in $L^1_{\mathrm{loc}}$ to $v$ and also to $u$. Thus $u = v$ a.e. and therefore

$$\|u\|_{L^{p^*}(Q)} = \|v\|_{L^{p^*}(Q)} \le \gamma_p \|Du\|_{L^p(Q)}.$$

But $q$-Hölder (2.2.6) for $\frac{1}{q} = (\frac{1}{q} - \frac{1}{p^*}) + \frac{1}{p^*}$ tells that $u \in \mathcal{L}^q(Q)$ and

$$\|u\|_{L^q(Q)} \le |Q|^{\frac{1}{q} - \frac{1}{p^*}} \|u\|_{L^{p^*}(Q)}.$$

This proves Theorem 6.1.4 with $c = |Q|^{\frac{1}{q} - \frac{n-p}{np}} \frac{p(n-1)}{n-p}$. □

### 6.1.2 General Sobolev inequalities ($kp < n$)

The following hypotheses require dimension $n \geq 2$.

**Theorem 6.1.5** (General Sobolev inequalities $k < \frac{n}{p}$)**.** *Let $D \Subset \mathbb{R}^n$ be Lipschitz and $k \in \{1, \dots, n-1\}$ and $p \in [1, \frac{n}{k})$. Then any $\boldsymbol{u} \in W^{k,p}(D)$ lies in $L^q(Q)$ for*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \qquad q = p \cdot \frac{n}{n - kp} \; (> p),$$

*and there is the estimate*

$$\|u\|_{L^q(D)} \leq c \, \|u\|_{W^{k,p}(D)}$$

*where the constant $c$ depends only on $k$, $p$, $n$, and $D$.*

Before entering the proof let us see what the theorem actually means. Since $p < q$ and the domain $D$ is pre-compact Hölder's inequality (2.2.5) tells that $L^p \supset L^q$ whereas the Sobolev inequality tells $W^{k,p} \subset L^q$. In other words, the sub-dimensional Sobolev inequalities invert the Hölder inclusions - at the cost of asking existence of some derivatives.

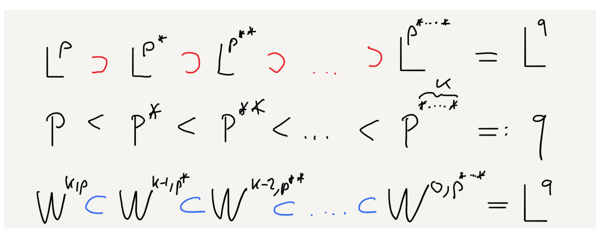


Figure 6.2: Sub-dimensional Sobolev inclusions run opposite to Hölder

*Proof.* If $k = 1$ we are done by Theorem 6.1.3 and $q = p^*$. Thus suppose $k \geq 2$. The proof is a $k \geq 2$ step inclusion process as illustrated by Figure 6.2. Given $\boldsymbol{u} \in W^{k,p}(D)$ with $kp < n$, the idea is to give away one order of derivative in exchange of enlarging $p$ to its Sobolev exponent $p^*$ defined by (6.1.1). More precisely, reduce $1/p$ by $1/n$ to get $1/p^*$. After $k$ steps all derivatives are gone, corresponding to subtracting $k/n$ from $1/p$. Due to the assumption $kp < n$ the difference $\frac{1}{p} - \frac{k}{n} > 0$ is still positive and its inverse will be $q$.

Let us abbreviate $W^{\ell,p} := W^{\ell,p}(D)$ and $\|\cdot\|_{\ell,p} := \|\cdot\|_{W^{\ell,p}(D)}$. Proposition 4.2.1 (i) tells that if $|\alpha| = \ell$, then $D^\alpha \boldsymbol{u} \in W^{k-\ell,p}$.

**Step 1.** Whenever $|\alpha| \leq k - 1$ we have $D^\alpha \boldsymbol{u} \in W^{1,p}$ and since $p < n/k \leq n$ is sub-dimensional Theorem 6.1.3 tells that $D^\alpha \boldsymbol{u} \in L^{p^*}$ with the estimate

$$\|D^\alpha u\|_{L^{p^*}} \leq c_1 \, \|D^\alpha u\|_{W^{1,p}} \leq c_1 \left( \|D^\alpha u\|_p + \|D D^\alpha u\|_p \right)$$

for some constant $c_1 = c_1(p, n, D)$. So $\boldsymbol{u} \in W^{k-1,p^*}$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. So

$$p^* = p \cdot \tfrac{n}{n-p} > p, \qquad p^* = n \cdot \tfrac{p}{n-p} < n \cdot \tfrac{n/k}{n-n/k} = n \cdot \tfrac{1}{k-1} \le n.$$

If $k = 2$ we are done by Theorem 6.1.3 and $q = p^{**}$. Suppose $k \ge 3$.

**Step 2.** By Step 1 whenever $|\beta| \le k-2$ we have $D^\beta \boldsymbol{u} \in W^{1,p^*}$ and since $p^* < n$ is sub-dimensional Theorem 6.1.3 tells that $D^\beta \boldsymbol{u} \in L^{p^{**}}$ with the estimate

$$\left\|D^\beta u\right\|_{p^{**}} \le c_2 \left\|D^\beta u\right\|_{1,p^*} \le c_2 \left(\left\|D^\beta u\right\|_{p^*} + \left\|DD^\beta u\right\|_{p^*}\right)$$

for some constant $c_2 = c_2(p^*(p,n), n, D) = c_2(p, n, D)$. Hence $\boldsymbol{u} \in W^{k-2,p^{**}}$ where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$. Thus

$$p^{**} = p^* \cdot \tfrac{n}{n-p^*} > p^*, \qquad p^{**} = n \cdot \tfrac{p^*}{n-p^*} < n \cdot \tfrac{n/(k-1)}{n-n/(k-1)} = n \cdot \tfrac{1}{k-2} \le n.$$

If $k = 3$ we are done by Theorem 6.1.3 and $q = p^{***}$. Suppose $k \ge 4$.

$\vdots$

Let us abbreviate $p_2 := p^{**}$ and $p_3 := p^{***}$ and so on.

$\vdots$

**Step $\boldsymbol{k-1}$.** By Step $k-2$ whenever $|\gamma| \le 1$ we have $D^\gamma \boldsymbol{u} \in W^{1,p_{k-2}}$ and as $p_{k-2} < n$ is sub-dimensional Theorem 6.1.3 tells that $D^\gamma \boldsymbol{u} \in L^{p_{k-1}}$ and

$$\|D^\gamma u\|_{p_{k-1}} \le c_{k-1} \|D^\gamma u\|_{1,p_{k-2}} \le c_2 \left(\|D^\gamma u\|_{p_{k-2}} + \|DD^\gamma u\|_{p_{k-2}}\right)$$

for some $c_{k-1} = c_{k-1}(p, n, D)$. So $\boldsymbol{u} \in W^{1,p_{k-1}}$ where $\frac{1}{p_{k-1}} = \frac{1}{p} - \frac{k-1}{n}$ and

$$p_{k-1} = p_{k-2} \cdot \tfrac{n}{n-p_{k-2}} > p_{k-2}, \quad p_{k-1} = n \tfrac{p_{k-2}}{n-p_{k-2}} < n \tfrac{n/2}{n-n/2} = n \tfrac{1}{2-1} = n.$$

**Step $\boldsymbol{k}$.** By Step $k-1$ we have $\boldsymbol{u} \in W^{1,p_{k-1}}$ and $p_{k-1} < n$ is sub-dimensional. Thus Theorem 6.1.3 tells that $\boldsymbol{u} \in L^q$, where $\frac{1}{q} := \frac{1}{p_k} = \frac{1}{p} - \frac{k}{n} > 0$ since $kp < n$, and with the estimate

$$\|u\|_q \le c_k \|u\|_{1,p_{k-1}} \le c_k \left(\|u\|_{p_{k-1}} + \|Du\|_{p_{k-1}}\right)$$

for some constant $c_k = c_k(p, n, D)$. Note that $p_k = p_{k-1} \cdot \frac{n}{n-p_{k-1}} > p_{k-1}$.

To conclude the proof set $q := p_k$ and consider the $L^q$ estimate. Estimate its RHS by the estimate in Step $k-1$ whose RHS is estimated by Step $k-2$ and so on until we get to Step 1 whose RHS involves terms $DD^\alpha u$ of order $k$ and in the $L^p$ norm. This concludes the proof of Theorem 6.1.5. $\square$

### 6.1.3 Compactness (Rellich-Kondrachov)

## 6.2 Super-dimensional case $kp > n$

### 6.2.1 Morrey's inequality ($p > n$) – continuity

Throughout this subsection $k = 1$, so **super-dimensional finite case** means

$$1 \le n < p < \infty.$$

For *infinite* $p = \infty$ see Section 7.2 on the relation to Lipschitz continuity.

**Theorem 6.2.1** (Morrey's inequality). *Given a finite $p > n$, every $u \in C^\infty(\mathbb{R}^n)$ satisfies the estimates*

$$[u]_{C^{0,\mu}(\mathbb{R}^n)} := \sup_{x \ne y} \frac{|u(x) - u(y)|}{|x-y|^\mu} \le c \left\| Du \right\|_{L^p(\mathbb{R}^n)}$$

*where $\mu := 1 - \frac{n}{p}$ and*

$$\left\| u \right\|_{C^0(\mathbb{R}^n)} := \sup |u| \le c \left( \left\| u \right\|_{L^p(\mathbb{R}^n)} + \left\| Du \right\|_{L^p(\mathbb{R}^n)} \right)$$

*where in both estimates $c := \frac{2^{n+1}}{{\sigma_n}^{1/p}} \left( \frac{p-1}{p-n} \right)^{1-\frac{1}{p}}$. Here $\sigma_n = n\beta_n$ is the area of the unit sphere in $\mathbb{R}^n$ and $\beta_n$ denotes the volume of the unit ball.*

Finiteness of the right hand sides is guaranteed for compactly supported $u$'s. To name the theorem we followed [Eva98, §5.6.2]. Since the relation between weak differentiability and continuity is one, if not the, fundamental pillar of the theory of Sobolev spaces we include the proof following [MS04, Le. B.1.16].

**Corollary 6.2.2.** *Theorem 6.2.1 asserts that for finite $p > n$ it holds that*

$$\left\| u \right\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le 3c \left\| u \right\|_{W^{1,p}(\mathbb{R}^n)} < \infty \tag{6.2.4}$$

*for every $u \in C_0^\infty(\mathbb{R}^n)$.*

*Proof of Theorem 6.2.1.* The proof takes three steps.
STEP 1. Suppose $B \Subset \mathbb{R}^n$ is a non-empty and convex set. Then every smooth function $u$ on $B$ of **mean value zero**, in symbols

$$(u)_B := \int_B u = 0,$$

satisfies at every point $x$ of $B$ the estimate

$$|u(x)| \le \frac{d^n {\sigma_n}^{1-\frac{1}{p}}}{n\,|B|} \left( \frac{p-1}{p-n} \right)^{1-\frac{1}{p}} d^{1-\frac{n}{p}} \left\| Du \right\|_{L^p(B)} \tag{6.2.5}$$

where $d := \operatorname{diam} B$ denotes the diameter of $B$ and $|B| := \operatorname{vol} B$ the volume.

To see this pick $x, y \in B$. For $t \in [0,1]$ the path $\gamma(t) := x + t(y-x)$ takes values in the convex domain $B$ of $u$. As $\dot\gamma = y - x$ and $\gamma(1) = y$ and $\gamma(0) = x$, we get

$$-\int_B \int_0^1 \underbrace{Du|_{x+t(y-x)}\,[y-x]}_{=\frac{d}{dt} u\circ\gamma(t)}\,dt\,dy = -\underbrace{\int_B u(y)\,dy}_{=(u)_B=0} + \int_B u(x)\,dy = |B|\cdot u(x).$$

Without changing notation extend the gradient $Du$ from $B$ to $\mathbb{R}^n$ by zero. So

$$\begin{aligned}
|B|\cdot|u(x)| &\le \int_B \int_0^1 |Du(x+t(y-x))|\cdot|y-x|\;dt\,dy\\
&\le \int_{B_d(x)} \int_0^1 |Du(x+t(y-x))|\cdot|\underbrace{y-x}_{z(y)}|\,dt\,dy\\
&= \int_{\{|z|\le d\}} \underbrace{\int_0^1 |Du(x+tz)|\cdot|z|\;dt}_{=:f(z)}\,dz.
\end{aligned}$$

Step two uses that $x \in B \subset B_d(x) := \{|y| < d\}$ since $d = \operatorname{diam} B$. Observe that $dy = dy_1 \wedge \dots \wedge dy_n$. In step three we introduced the new variable $z(y) := y - x$.

Next let $dS(r)$ denote the area form on the radius $r$ sphere $\{|z| = r\}$ and introduce *polar coordinates*[1] $r = |z|$ and $\eta = |z|^{-1}z$ with $r\eta = z$ to get that

$$\begin{aligned}
&= \int_0^d r^{n-1}\left(\int_{\{|\eta|=1\}} \overbrace{\left(\int_0^1 |Du(x+\underbrace{tr}_{\rho(t)}\eta)|\,r\,dt\right)}^{=f(r\eta)} dS(1)\right) dr\\
&= \int_0^d r^{n-1}\left(\int_0^r \rho^{n-1}\left(\int_{\{|\eta|=1\}} |Du(x+\underbrace{\rho\eta}_{y})|\cdot\underline{\rho^{1-n}}\,dS(1)\right) d\rho\right) dr\\
&= \int_0^d r^{n-1}\left(\int_{\{|y|\le r\}} |Du(x+y)|\cdot\underline{|y|^{1-n}}\,dy\right) dr.
\end{aligned}$$

To get line two we chose as a new variable the dilation $\rho(t) := rt$, so $r\,dt = d\rho$, we interchanged the order of integration, and we multiplied by 1. To obtain line three we interpreted $(\rho, \eta)$ as the polar coordinates of $y = \rho\eta \in \{|y| \le r\}$.

Next use the inclusion $\{|y| \le r\} \subset B_d(0)$ and monotonicity of the integral

[1] For $(r,\eta) : \{0 < |z| \le d\} \to (0,d] \times \mathbb{S}^{n-1}$, $z \mapsto (|z|, |z|^{-1}z)$, Fubini implies the formula

$$\int_{B_d(0)} f(z)\,dz = \int_0^d \left(\int_{\{|z|=r\}} f(z)\,dS(r)\right) dr = \int_0^d r^{n-1}\left(\int_{\{|\eta|=1\}} f(r\eta)\,dS(1)\right) dr.$$

Concerning **polar coordinates** we recommend [Fol99, §2.7].

to obtain a new $dy$ integral which is independent of $r$. Thus

$$\begin{aligned}
&\leq \left(\int_0^d r^{n-1}\,dr\right)\int_{B_d(0)} |Du(\underbrace{x+y}_{z(y)})|\cdot|y|^{1-n}\,dy\\
&= \frac{d^n}{n}\int_{B_d(x)} \underbrace{|Du(z)|\cdot|z-x|^{1-n}}_{1/p \quad + \quad 1/q \quad =1} dz.
\end{aligned}$$

Now apply Hölder (2.2.5) to get that

$$\begin{aligned}
&\leq \frac{d^n}{n}\left\|Du\right\|_{L^p(B_d(x))}\left(\int_{B_d(x)} |\underbrace{z-x}_{=:y}|^{(1-n)q} dz\right)^{\frac{1}{q}}\\
&= \frac{d^n}{n}\left\|Du\right\|_{L^p(B)}\left(\int_{\{|y|\leq d\}} |y|^{(1-n)q}\,dy\right)^{1-\frac{1}{p}}
\end{aligned}$$

To obtain the last line we used the inclusion $B\subset B_d(x)$ and the fact that we have defined $Du=0$ on the complement of $B$. Straightforward calculation[2] then proves (6.2.5) and thus Step 1.

Step 2. Let $p\in(n,\infty)$. Given $u\in C^\infty(\mathbb{R}^n)$ and $x,y\in\mathbb{R}^n$, set $x_0 := \frac{1}{2}(x+y)$ and $B:=B_r(x_0)$ where $2r:=|x-y|=\operatorname{diam} B =: d$. Then

$$|u(x)-u(y)|\leq \frac{2^{n+1}}{{\sigma_n}^{\frac{1}{p}}}\left(\tfrac{p-1}{p-n}\right)^{1-\frac{1}{p}}|x-y|^{1-\frac{n}{p}}\left\|Du\right\|_{L^p(B)}.$$

Step 2 proves the first estimate in Theorem 6.2.1. To see Step 2 use the triangle inequality to get $|u(x)-u(y)|\leq|u(x)-(u)_B|+|(u)_B-u(y)|$. Now apply Step 1 with $\frac{d^n\sigma_n}{n|B|}=\frac{(2r)^n n\beta_n}{nr^n\beta_n}=2^n$ to each of the two terms in the sum.

Step 3. Let $p\in(n,\infty)$. Given $u\in C^\infty(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, set $B:=B_1(x)$. Then

$$|u(x)|\leq \frac{2^n}{{\sigma_n}^{\frac{1}{p}}}\left(\tfrac{p-1}{p-n}\right)^{1-\frac{1}{p}}\left\|Du\right\|_{L^p(B)}+{\beta_n}^{1-\frac{1}{p}}\left\|u\right\|_{L^p(B)}.$$

[2] introduce polar coordinates, the radial coordinate being $r:=|y|$, to obtain that

$$\int_{\{|y|\leq d\}}|y|^{(1-n)q}\,dy=\sigma_n\int_0^d \underbrace{r^{n-1}r^{q-nq}}_{r^{(n-1)(1-q)=\frac{1-n}{p-1}}}\,dr=\frac{\sigma_n}{\frac{1-n}{p-1}+1}d^{\frac{1-n}{p-1}+1}=\sigma_n\frac{p-1}{p-n}d^{\frac{p-n}{p-1}}$$

Step 3 proves the second estimate in Theorem 6.2.1.[3] Apply Step 1 to get

$$\begin{aligned}|u(x)| &\le |u(x)-(u)_B| + |(u)_B| \\ &\le \frac{2^n}{{\sigma_n}^{\frac{1}{p}}} \left(\tfrac{p-1}{p-n}\right)^{1-\frac{1}{p}} \underbrace{2^{1-\frac{n}{p}}}_{\in(1,2)} \|Du\|_{L^p(B)} + {\beta_n}^{1-\frac{1}{p}} \|u\|_{L^p(B)} .\end{aligned}$$

where we used that $|B| := \mathrm{vol} B = \beta_n$ and $d := \operatorname{diam} B = 2$. Moreover, we applied Hölder to get the estimate $|(u)_B| \le \int_B |1 \cdot u| \le |B|^{1-1/p} \|u\|_{L^p(B)}$.
This completes the proof of Theorem 6.2.1. $\square$

**Theorem 6.2.3** (Super-dimensional $W^{1,p}(D)$ estimates)**.** *Suppose $D \Subset \mathbb{R}^n$ is Lipschitz. For super-dimensional $p \in (n,\infty)$ any equivalence class $\boldsymbol{u} \in W^{1,p}(D)$ admits a $\mu$-Hölder continuous representative $u^* \in C^{0,\mu}(D)$ where $\mu = 1-\frac{n}{p}$ and*

$$\|u^*\|_{C^{0,\mu}(D)} \le c \|u^*\|_{W^{1,p}(D)}$$

*where the constant $c$ depends only on $p$, $n$, and $D$.*

For infinite $p=\infty$ see Section 7.2 on Lipschitz continuity.

*Proof.* The idea of proof is precisely the same as the one for Theorem 6.1.3, have a look there, just replace Gagliardo-Nirenberg-Sobolev by Morrey.

Extend $\boldsymbol{u} \in W^{1,p}(D)$ to $\bar{\boldsymbol{u}} := E\boldsymbol{u} \in W_0^{1,p}(Q) := \overline{C_0^\infty(Q)}^{1,p}$ for any pre-compact $Q$ with $D \Subset Q \Subset \mathbb{R}^n$ using the Extension Theorem 5.2.1. Pick a representative $\bar{u} \in \bar{\boldsymbol{u}}$ and approximate it by a Cauchy sequence $u_m \in C_0^\infty(Q)$ that converges to $\bar{u}$ in the $W^{1,p}$, hence the $L^p$, norm. The Morrey inequality (6.2.4) for the $u_m$ extended to $\mathbb{R}^n$ by zero, namely $\|u_m - u_\ell\|_{C^{0,\mu}} \le 3c\|u_m - u_\ell\|_{W^{1,p}}$ where $\mu = 1-n/p$, tells that the $u_m$ also form a Cauchy sequence in the Hölder Banach space $C^{0,\mu}(Q)$. Thus it admits a limit, say $u^* \in C^{0,\mu}(Q)$.
Note that $\|\cdot\|_{L^p(Q)} \le |Q|^{1/p} \|\cdot\|_{C^0(Q)}$. Hence $C^{0,\mu}(Q)$ convergence implies $L^p(Q)$ convergence. Thus $u^* = \bar{u}$ a.e. by uniqueness of limits in $L^p(Q)$. But along $D$ we have that $u^* = \bar{u} = u$ a.e. and therefore $[u^*] = \boldsymbol{u} \in W^{1,p}(D)$.
Taking limits on both sides of the Morrey inequality $\|u_m\|_{C^{0,\mu}} \le 3c\|u_m\|_{W^{1,p}}$ and, in the first step, monotonicity of the supremum shows that

$$\|u^*\|_{C^{0,\mu}(D))} \le \|u^*\|_{C^{0,\mu}(Q)} \le 3c \|\bar{u}\|_{W^{1,p}(Q)} \le \gamma_p C \|u\|_{W^{1,p}(D)}$$

where $C = C(p,D,Q)$ is the constant in the Extension Theorem 5.2.1. $\square$

[3] The coefficient of $\|Du\|_p$ is obviously smaller than $c$. For the one of $\|u\|_p$ we get that $c := \frac{2^{n+1}}{{\sigma_n}^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p} > \frac{2^{n+1}}{{\sigma_n}^{1/p}} = \frac{2^{n+1}}{n^{1/p}{\beta_n}^{1/p}} \ge \frac{2^{n+1}}{n \cdot {\beta_n}^{1/p}} > \frac{\beta_n}{{\beta_n}^{1/p}} = {\beta_n}^{1-1/p}$.
The final inequality is equivalent to $\beta_n < 2^{n+1}/n$ which is true: For even $n=2k$ Wiki tells $\beta_{2k} = \pi^k/k!$ which is indeed smaller than $4^k/k = 2^{2k+1}/2k$. For odd $n = 2k+1$ one has that $\beta_{2k+1} = \frac{2}{1}\frac{2}{3}\frac{2}{5}\cdots\frac{2}{2k+1}\pi^k < \frac{4}{2k+1}4^k = \frac{2^{(2k+1)+1}}{2k+1}$.

### 6.2.2 General Sobolev inequalities ($kp > n$)

Given a non-integer real $r \in \mathbb{R} \setminus \mathbb{Z}$, then $\ell < r < \ell + 1$ for some $\ell \in \mathbb{Z}$. Let us call $\ell$ and $\ell + 1$ the **floor** and **ceiling integer** of $r$, respectively, in symbols

$$\lfloor r \rfloor := \ell, \qquad \lceil r \rceil := \ell + 1.$$

For an integer real $r \in \mathbb{Z}$ set $\lfloor r \rfloor := r =: \lceil r \rceil$.

**Theorem 6.2.4** (General Sobolev inequalities $k > \frac{n}{p}$)**.** *Let $D \Subset \mathbb{R}^n$ be Lipschitz and $k \in \mathbb{N}$ and $p \in [1, \infty)$. If $kp > n$ then any $\boldsymbol{u} \in W^{k,p}(D)$ has a $k - \lceil \frac{n}{p} \rceil$ times $\gamma$-Hölder continuously differentiable representative $u^* \in C^{k-\lceil \frac{n}{p} \rceil, \gamma}(D)$ where*

$$\gamma := \begin{cases} \lceil \frac{n}{p} \rceil - \frac{n}{p} & , \frac{n}{p} \notin \mathbb{N}, \\ \text{any } \mu \in (0,1) & , \text{otherwise (in this case } k - \lceil \frac{n}{p} \rceil \geq 1). \end{cases}$$

*Moreover, the Hölder representative $u^*$ satisfies the estimate*

$$\|u^*\|_{C^{k-\lceil \frac{n}{p} \rceil, \gamma}(D)} \leq c \, \|u^*\|_{W^{k,p}(D)} \tag{6.2.6}$$

*where the constant $c$ depends only on $k$, $p$, $n$, $\gamma$, and $D$.*

For infinite $p = \infty$ see Section 7.2 on Lipschitz continuity.

*Proof.* **Case $\boldsymbol{n/p} \notin \mathbb{N}$.** The idea is to employ the general sub-dimensional Sobolev inequalities to get from the largest sub-dimensional case $\ell p < n$ to the smallest super-dimensional one $1 \cdot r > n$ where $\frac{1}{r} = \frac{1}{p} - \frac{\ell}{n}$ and then contract the super-dimensional Sobolev $W^{1,r}$ estimate that we just proved.

More precisely, let $\lfloor \frac{n}{p} \rfloor := \ell < \frac{n}{p} < \ell + 1 \leq k$ be the first integer below $\frac{n}{p}$. So we are in the sub-dimensional case $\ell p < n$ and from our hypothesis $\boldsymbol{u} \in W^{k,p}(D)$ we conclude that $\boldsymbol{u} \in W^{k-\ell,r}(D)$ by the same iteration process as in the proof of the general sub-dimensional Sobolev inequalities, Theorem 6.1.5. Note that $n/p < \ell + 1$ iff $\ell p > n - p$. Thus

$$r = n \cdot \frac{p}{n - \ell p} > n \cdot \frac{p}{n - (n-p)} = n$$

is super-dimensional and $D^\alpha \boldsymbol{u} \in W^{1,r}(D)$ whenever $|\alpha| \in \{0, \dots, k - \ell - 1\}$. Such $D^\alpha \boldsymbol{u}$ admits a representative $(D^\alpha u)^* \in C^{0,\mu}(D)$ by Theorem 6.2.3 and

$$\|(D^\alpha u)^*\|_{C^{0,\mu}(D)} \leq c \, \|(D^\alpha u)^*\|_{W^{1,r}(D)}$$

where the constant $c$ depends only on $p$, $n$, $D$ and $\mu = 1 - \frac{n}{r} = 1 - \frac{n}{p} + \ell = \lceil \frac{n}{p} \rceil - \frac{n}{p}$. Hence all weak derivatives up to order $k - \ell - 1 = k - \lceil \frac{n}{p} \rceil$ are $\mu$-Hölder continuous by Lemma 4.1.11. This shows that $u^* \in C^{k-\lceil \frac{n}{p} \rceil, \lceil \frac{n}{p} \rceil - \frac{n}{p}}(D)$ as we had to prove. Take the maximum over all these estimates to obtain (6.2.6).

**Case $\boldsymbol{n/p} \in \mathbb{N}$.** Suppose $k > \frac{n}{p}$, i.e. $k - 2 \geq \frac{n}{p} - 1 \geq 0$, and $\boldsymbol{u} \in W^{k,p}(D)$. To apply the general sub-dimensional Sobolev inequalities, Theorem 6.1.5, we need

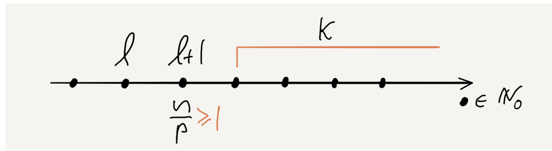


Figure 6.3: Case $\frac{n}{p} \in \mathbb{N}$ - choice of $\ell \in \mathbb{N}_0$

an integer $\ell < \frac{n}{p}$. For best result let us choose the largest one $\ell := \frac{n}{p} - 1$ as illustrated by Figure 6.3 Observe that $\ell \in \{0, \dots, k-2\}$ and $D^\alpha \boldsymbol{u} \in W^{\ell,p}(D)$ whenever $|\alpha| \le k - \ell$ $(\ge 2)$.

Case $\ell \ge 1$. a) In this case $D^\alpha \boldsymbol{u} \in W^{1,p}(D)$. So by sub-dimensionality ($\ell p < n$) Theorem 6.1.5 tells that $D^\alpha \boldsymbol{u} \in L^q(D)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\ell}{n}$ $(= \frac{1}{n})$ and it provides the constant $c = c(\ell(p,n), p, n, D)$ and the second of the two inequalities

$$\frac{\|D^\alpha u\|_{L^r(D)}}{|D|^{\frac{1}{r}-\frac{1}{n}}} \le \|D^\alpha u\|_{L^n(D)} \le c \|D^\alpha u\|_{W^{\ell+1,p}(D)}, \quad |\alpha| \le k - \ell \ (\ge 2).$$

Here the first inequality is by Hölder (2.2.6) and holds for any $r \in [1, n)$. This shows that $D^\beta \boldsymbol{u} \in W^{1,r}(D)$ whenever $|\beta| \le k - \ell - 1$ $(\ge 1)$ and $r \in [1, n)$.
b) Hence the consequence of Gagliardo-Nirenberg-Sobolev, the sub-dimensional $W^{1,r}$ estimate Theorem 6.1.3, asserts that $D^\beta \boldsymbol{u} \in L^{r^*}(D)$ with the estimate

$$\left\|D^\beta u\right\|_{L^{r^*}(D)} \le C \left\|D^\beta u\right\|_{W^{1,r}(D)}, \qquad |\beta| \le k - \ell - 1 \ (\ge 1), \quad r^* \in (n, \infty),$$

where $C = C(r, n, D)$.[4]
But this means that $D^\beta \boldsymbol{u} \in W^{1,r^*}(D)$ for every super-dimensional $r^* \in (n, \infty)$. Thus $D^\beta \boldsymbol{u}$ admits a representative $(D^\beta u)^* \in C^{0,\mu}(D)$ by Theorem 6.2.3 and

$$\left\|(D^\beta u)^*\right\|_{C^{0,\mu}(D)} \le c' \left\|(D^\beta u)^*\right\|_{W^{1,r}(D)}. \tag{6.2.7}$$

Here the constant $c'$ depends only on $r^*$, $n$, $D$ and

$$\mu : (n, \infty) \to (0, 1), \quad r^* \mapsto 1 - \frac{1}{r^*},$$

is a bijection. Hence all weak derivatives of $u^* = (D^0 u)^*$ up to order $k - \ell - 1 = k - \lceil \frac{n}{p} \rceil$ are $\gamma$-Hölder continuous by Lemma 4.1.11, in symbols $u^* \in C^{k - \lceil \frac{n}{p} \rceil, \gamma}$, and this is true for any $\gamma \in (0, 1)$.
Take the maximum over $|\beta| \le k - \ell - 1$ of the estimates (6.2.7) to obtain that

$$\|u^*\|_{C^{k-\lceil \frac{n}{p} \rceil, \gamma}(D)} \le C' \|u^*\|_{W^{k-\ell,r}(D)} \le C' \|u^*\|_{W^{k,r}(D)} \tag{6.2.8}$$

for all $\gamma \in (0, 1)$ and $r \in [1, n)$ and where $C' = C'(r^*, n, D, k)$. But $p < \frac{n}{\ell} \le n$, so we can choose $r := p$ to finish the proof of (6.2.6) in the case $\ell \ge 1$.

[4] The map $r^* : (\frac{n}{2}, n) \to (n, \infty)$, $r \mapsto \frac{nr}{n-r}$, is a bijection.

CASE $\ell = 0$ $(p = n)$. In this case we have $\boldsymbol{u} \in W^{k,n}(D)$ with $k \geq 2$. Then $D^\beta \boldsymbol{u} \in W^{1,n}(D)$ whenever $|\beta| \leq k - 1$. By Hölder (2.2.6) and finite measure of $D$ (compact closure) we get that $D^\beta \boldsymbol{u} \in W^{1,r}(D)$ whenever $|\beta| \leq k - 1$ and any sub-dimensional $r \in [1, n)$. Now continue as in part b) of case $\ell \geq 1$ above just setting $\ell = 0$ throughout. Only in the very last sentence arises a difference, because in the case at hand $p = n$ is not sub-dimensional, so we can't set $r := p$ to finish the proof. However, again by Hölder (2.2.6) we can estimate the RHS of (6.2.8) by the $W^{k,p}(D)$ norm since $r < n = p$.

This concludes the proof of Theorem 6.2.4. $\square$

# Chapter 7

# Applications

## 7.1 Poincaré inequalities

As an application of the compactness theorem in Section 6.1.3 one obtains

**Theorem 7.1.1** (Poincaré's inequality)**.** *Suppose $D \Subset \mathbb{R}^n$ is Lipschitz and connected. Let $p \in [1, \infty]$. Then there is a constant $c = c(n, p, D)$ such that*

$$\|u - (u)_D\|_{L^p(D)} \le c \, \|Du\|_{L^p(D)}$$

*for every $u \in W^{1,p}(D)$.*

*Proof.* [Eva98, §5.8.1] □

## 7.2 Lipschitz functions

**Theorem 7.2.1** (Identification of $W^{1,\infty}(D)$ with $C^{0,1}(D)$)**.** *Let $D \Subset \mathbb{R}^n$ be Lipschitz. Then every class $\boldsymbol{u} \in W^{1,\infty}(D)$ admits a (unique) Lipschitz continuous representative, notation $u^*$. Vice versa, any Lipschitz continuous map is weakly differentiable with a.e. bounded weak derivatives. In symbols, the map*

$$C^{0,1}(D) \rightarrowtail\!\!\!\twoheadrightarrow W^{1,\infty}(D), \quad u \mapsto [u],$$

*is a bijection with inverse $\boldsymbol{u} \mapsto u^*$.*

*Proof.* See e.g. [Eva98, §5.8.2]. □

For finite $p$ there is no such bijection. For super-dimensional finite $p \in (n, \infty)$ there is the injection provided by the Sobolev inequality Theorem 6.2.3, namely

$$W^{1,p}(D) \rightarrowtail C^{0,1-n/p}(D), \quad [u] \mapsto u^*.$$

**Exercise 7.2.2.** Find examples that show non-surjectivity of this injection.

## 7.3 Differentiability almost everywhere

**Theorem 7.3.1** (Super-dimensional differentiability a.e.)**.** *Given $\Omega \subset \mathbb{R}^n$, let $p \in (n, \infty]$ be super-dimensional. Then the following is true. Let $\boldsymbol{u} \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then its $(1-n/p)$-Hölder continuous representative*[1] *$u^* \in \boldsymbol{u}$, also any other representative $u$, is differentiable almost everywhere and strong and weak gradient coincide almost everywhere, in symbols $(\partial_1 u, \dots, \partial_n u) = (u_{e_1}, \dots, u_{e_n})$ a.e.*

*Proof.* See e.g. [Eva98, §5.8.3] □

**Theorem 7.3.2** (Rademacher's Theorem)**.** *Suppose $u$ is locally Lipschitz continuous in $\Omega \subset \mathbb{R}^n$. Then $u$ is differentiable almost everywhere in $\Omega$.*

*Proof.* Theorem 4.1.16 and Theorem 7.3.1. □

[1] see Theorem 6.2.3 for $p \in (n, \infty)$ and Theorem 7.2.1 for $p = \infty$

# Appendix A

# Appendix

## A.1 Allerlei

### A.1.1 Inequalities via convexity and concavity

**Lemma A.1.1** (Young's inequality). *Let $a, b \geq 0$ and $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{A.1.1}$$

*Proof.* If one of $a, b$ is zero, the inequality holds trivially. Assume $a, b > 0$. Observe that

$$\log(ab) = \log a + \log b = \frac{1}{p} \log a^p + \frac{1}{q} \log b^q \leq \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right)$$

where the last step holds by concavity of the logarithm. Since the logarithm is strictly increasing the result follows. □

Add $(a+b)^2 = a^2 + 2ab + b^2$ and $0 \leq (a-b)^2 = a^2 - 2ab + b^2$, or use Young (A.1.1), to get that

$$(a+b)^2 \leq 2a^2 + 2b^2 \quad a, b, \in \mathbb{R}. \tag{A.1.2}$$

More generally, for $a, b \in \mathbb{R}$ and $p \in [1, \infty)$ it is true that

$$|a+b|^p \leq (|a| + |b|)^p \leq 2^{p-1} \left(|a|^p + |b|^p\right). \tag{A.1.3}$$

This follows from convexity of $t^p$ on $(0, \infty)$. Hence $c^p + d^p \leq (c+d)^p$ for $c, d \geq 0$.[1] It is useful to keep in mind the following consequences, namely

$$\begin{aligned} \left\|u\right\|_{1,p} &:= \left(\left\|u\right\|_p^p + \left\|Du\right\|_p^p\right)^{\frac{1}{p}} \leq \left\|u\right\|_p + \left\|Du\right\|_p \\ \left\|u\right\|_p + \left\|Du\right\|_p &\leq \left(2^{p-1}\left(\left\|u\right\|_p^p + \left\|Du\right\|_p^p\right)\right)^{\frac{1}{p}} = \underbrace{2^{1-\frac{1}{p}}}_{\leq 2} \left\|u\right\|_{1,p}. \end{aligned} \tag{A.1.4}$$

[1] Indeed by convexity $\frac{1}{2^p}(c+d)^p = \left(\frac{c+d}{2}\right)^p \leq \frac{1}{2}\left(c^p + d^p\right)$; cf. [Rud87, Thm. 3.5 Pf.].

**Exercise A.1.2.** For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ let $|x|^2 := x_1^2 + \dots + x_n^2$. Then

$$|x| \le |x_1| + \dots + |x_n| .$$

More generally, for $p \in [1, \infty)$ there is the estimate

$$|x|^p \le (|x_1| + \dots + |x_n|)^p \le \kappa \left(|x_1|^p + \dots + |x_n|^p\right) \tag{A.1.5}$$

where $\kappa = 2^{(p-1)(n-1)}$ and $\kappa^{1/p} \le 2^{n-1}$. For $p = 2$ we get that

$$|x_1| + \dots + |x_n| \le 2^{\frac{n-1}{2}} |x| \le 2^{n-1} |x| . \tag{A.1.6}$$

### A.1.2 Continuity types and their relations

For a function $f : I \to \mathbb{R}$ on a compact interval, say $I = [0, a]$ with $a > 0$ in view of the examples, there are various types of continuity (Figures A.1 and A.2)

- **(C)** continuous
- **(UC)** uniformly continuous
- **($\alpha$-HC)** $\alpha$-Hölder continuous where $\alpha \in (0, 1)$
- **(LC)** Lipschitz continuous
- **(Diff)** differentiable
- **(AC)** absolutely continuous
- **(BV)** bounded variation
- **(Diff-a.e.)** differentiable almost everywhere

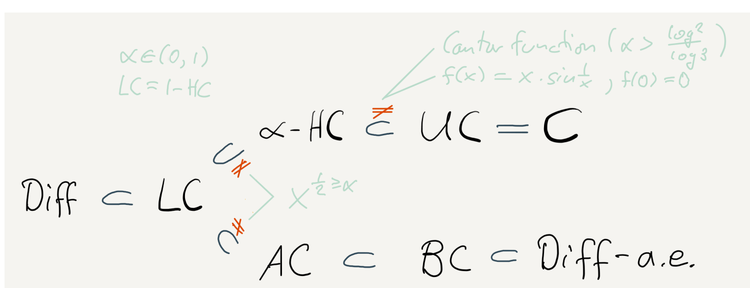


Figure A.1: Relations among continuity types (compact domain)

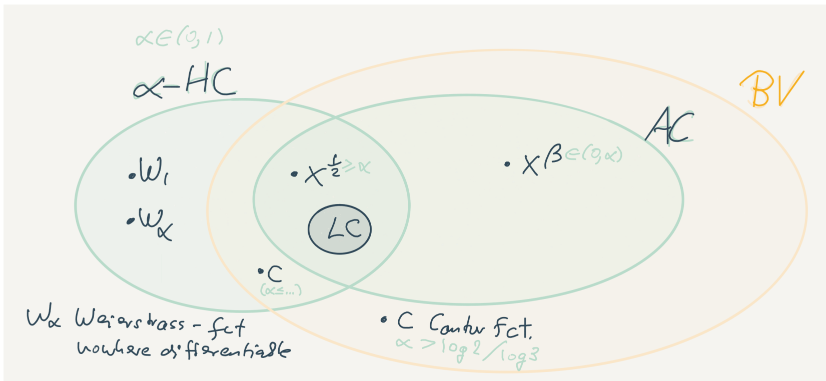


Figure A.2: Relations among continuity types

### A.1.3 Distance function

Saying that $Y$ is a **metric space** means that $Y$ is a set equipped with a metric[2] $d : Y \times Y \to [0, \infty)$ and the metric topology $\mathcal{T}_d$ whose basis are the open balls about the points of $Y$.

**Exercise A.1.3.** Let $Y$ be a metric space and equipp $Y \times Y$ with the product topology. Then the metric is continuous as a function $d : Y \times Y \to [0, \infty)$.

The **distance of a point** $y \in Y$ to a subset $A \subset Y$ is the infimum $d_A(y)$ of the distances from $y$ to the points $a$ of $A$, in symbols

$$d_A(y) := d(y, A) := \inf_{a \in A} d(y, a).$$

The map $d_A : Y \to [0, \infty]$ is called the **distance function** of $A$. We use the convention that the infimum over the empty set is infinite, in symbols

$$\inf_{\emptyset} := \infty.$$

**Lemma A.1.4.** *Suppose $Y$ is a metric space. Then for* any *subset $A \subset Y$ the distance function $d_A : Y \to [0, \infty]$ is continuous.*

*Proof.* Following [Dug66, Ch. IX Thm. 4.3] pick elements $x, y \in Y$. Then

$$d_A(x) := \inf_{a \in A} d(x, a) \le d(x, y) + \inf_{a \in A} d(y, a) = d(x, y) + d_A(y)$$

which shows that $d_A(x) - d_A(y) \le d(x, y)$. Interchange $x$ and $y$ to see that

$$|d_A(x) - d_A(y)| \le d(x, y).$$

Given $x \in Y$ and $\varepsilon > 0$, then for every $y \in Y$ with $d(x, y) < \delta := \varepsilon$ we get that $|d_A(x) - d_A(y)| \le d(x, y) < \varepsilon$. This shows that $d_A$ is continuous. $\square$

[2] axioms for a metric: non-degeneracy, symmetry, triangle inequality

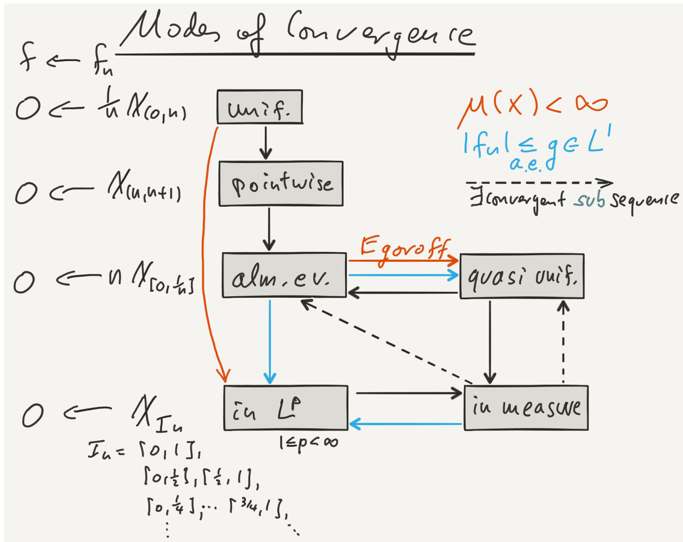


Figure A.3: Modes of convergence

### A.1.4 Modes of convergence

See [Fol99, §2.4] for details.

## A.2 Banach space valued Sobolev spaces

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