Lecture Notes on Morse and Conley Theory

– MANUSCRIPT IN PROGRESS –

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Chapter 1

Introduction

To put it in a nutshell, Morse and Conley theory connect the areas of analysis and topology through the theory of dynamical systems. Morse theory lives in the well known realm of hyperbolic dynamics and can be viewed as a special case of Conley theory.

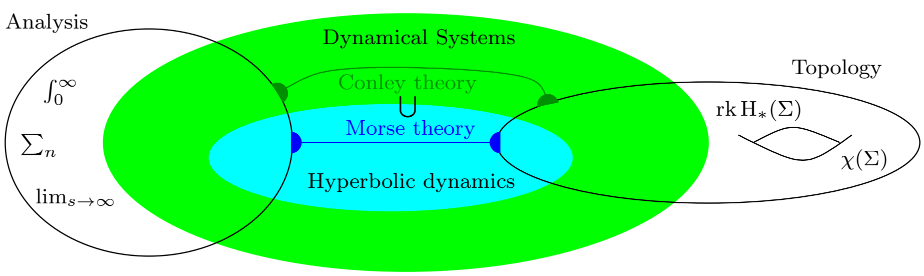

Figure 1.1: Morse and Conley theory connect analysis and topology

Morse theory

The early days 1920s-40s – Topology of sublevel sets

Consider a smooth manifold M. In the 20s of the last century it was the insight of Marston Morse [Mor34] that the topology of M is related to non-degenerate critical points of smooth functions $f : M \to \mathbb{R}$. By definition a **critical point** p of f satisfies the identity $df_p = 0$, that is it is an extremum of f. By Crit we denote the **set of critical points** of f. A critical point p is called **non-degenerate** if, given a local coordinate system $\phi = (x^1, \ldots, x^n) : M \supset U \to \mathbb{R}^n$ around p, the (symmetric) **Hessian matrix**

$$\mathrm{H}_p f := \left[\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right]_{i,j=1,\ldots,n} \tag{1.0.1}$$

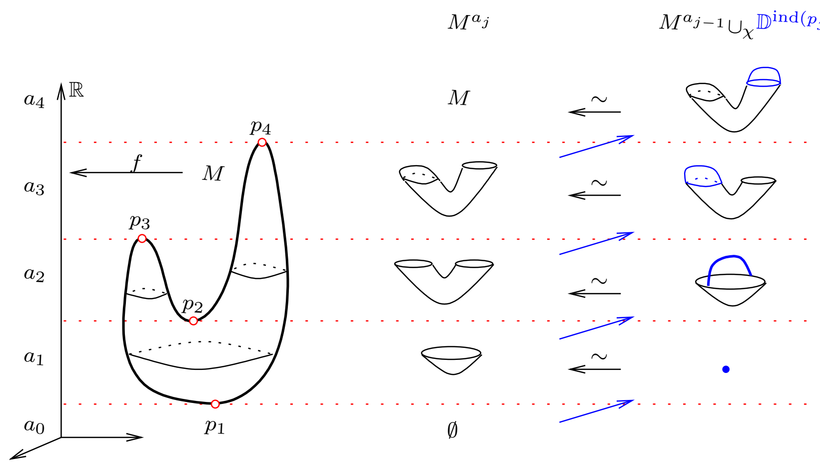

Figure 1.2: Height function f and attaching of disks $\mathbb{D}^k$ where $k = \mathrm{ind}(p_j)$

of f at p is **non-singular**, that is invertible or, equivalently, zero is not an eigenvalue of $\mathrm{H}_p f$. Hence critical points are isolated by the inverse function theorem 2.3.1. The number $\mathrm{ind}(x)$ of negative eigenvalues of $\mathrm{H}_p f$, counted with multiplicities, is called the **Morse index** of a non-degenerate critical point. The key idea to relate topology and analysis is to study the family of **sublevel sets**, also called **half-spaces**, which are the closed subsets of M defined by

$$M^a := \{f \le a\} := \{q \in M \mid f(q) \le a\}, \quad a \in \mathbb{R}. \tag{1.0.2}$$

A **regular value** of f is a constant b such that its **pre-image** $f^{-1}(b) = \{f = b\}$ contains no critical point. All other constants are called **critical values**. Note that according to this definition any constant whose pre-image is empty, that is which lies outside the range of f, is a regular value. The geometric significance of regular values lies in the fact that by the regular value theorem 2.4.1 their pre-images under f are **hypersurfaces** in M, that is codimension-1 submanifolds. Consequently, if a is a regular value, then M^a is a codimension-0 submanifold of M whose boundary is the hypersurface $f^{-1}(a)$.

Main results of the epoch are the following; cf. the beautyful survey [Bot88]. For compact M the first theorem tells that the submanifolds M^a vary smoothly as long as a does not cross a critical value. The second theorem asserts that the topological change which occurs when a crosses a simple critical level c, that is $f^{-1}(c)$ contains precisely one critical point p and p is non-degenerate, amounts to attaching a disk whose dimension k is the Morse index of p; see Figure 1.2.

Theorem 1.0.1 (Regular interval theorem)**.** *Assume $f : M \to \mathbb{R}$ is of class C^2 and the pre-image $f^{-1}[a, b]$ is compact*[1] *and contains no critical points of f.*

[1] Compactness is necessary: In the situation of Theorem 1.0.1 fix $q \in f^{-1}(b)$, restrict f to the manifold $N := M \setminus \{q\}$. Then $N^a = M^a \cong M^b = N^b \cup \{q\} \not\cong N^b$. Removing points from the interior $f^{-1}(a, b)$ quickly leads to examples where N^b cannot deformation retract to N^a.

Then M^b and M^a are diffeomorphic, that is $M^b \cong M^a$. Furthermore, the sublevel set M^a is a strong deformation retract of M^b.

The regular interval theorem not only predicts existence of critical points whenever two sublevel sets are of different homotopy type, the technique of proof immediately establishes a version of Birkhoff's famous minimax principle [BH35a, BH35b]; see Theorem 3.2.3.
In 1956 John Milnor [Mil56] constructed the first example of two manifolds which are homeomorphic, but not diffeomorphic. This was very much unexpected. Milnor was awarded the Fields medal in 1962 [Whi62]. Existence of a homeomorphism he obtained from a consequence of Theorem 1.0.1 known as the Reeb sphere theorem; see Theorem 3.2.6.

By a ***k*-cell** e^k in M we mean the homeomorphic image of the open unit k-disk in M. Its topological boundary $\dot{e}^k$ is given by $\operatorname{cl} e^k \setminus e^k$.

Theorem 1.0.2 (Attaching a cell)**.** *Assume $f : M \to \mathbb{R}$ is of class C^2 and $a, b \in \mathbb{R}$ are regular values of f such that the pre-image $f^{-1}[a,b]$ is compact and contains precisely one critical point x of f. If x is non-degenerate and k denotes its Morse index, then there exists a k-cell e^k in $f^{-1}(a,b)$ whose boundary $\dot{e}^k$ sits in $f^{-1}(a)$ and such that M^b deformation retracts onto M^a with a k-cell attached, that is onto $M^a \cup e^k$. So both spaces are of the same homotopy type:*

$$M^b \sim M^a \cup e^k.$$

In view of these results it seems a promising idea that if we wish to control the topology of the whole manifold M we should, first of all, require that all critical points of f be non-degenerate. A function f is called a **Morse function** if it is of class C^2 and all critical points are non-degenerate. For such functions Marston Morse proved his famous inequalities [Mor34, VI Thm. 1.1] relating analysis and topology. For instance, the **weak Morse inequalities** assert that

$$c_k \ge b_k := \operatorname{rank} \mathrm{H}_k(M;\mathbb{F}), \qquad \sum_{j=0}^n (-1)^j c_j = \sum_{j=0}^n (-1)^j b_j,$$

for $k = 0, \dots, n = \dim M < \infty$ where $c_k = c_k(M,f)$ is the number of critical points of Morse index k and $\mathbb{F}$ is any field. The **rank** $b_k = b_k(M;\mathbb{Q})$ of singular homology with rational coefficients, that is the number of elements of a basis of this vector space, is called the ***k*-th Betti number** of M. For instance, for the example in Figure 1.2 we get that

$$c_k(M,f) = \begin{cases} 2 & , k=2, \\ 1 & , k=1, \\ 1 & , k=0, \end{cases} \qquad b_k(M) = \begin{cases} 1 & , k=2, \\ 0 & , k=1, \\ 1 & , k=0. \end{cases}$$

1950s/60s – Dynamical systems and critical manifolds

Dynamical systems entered the stage inconspicuously in 1949 through René Thom's note [Tho49]. Given any smooth function $f : M \to \mathbb{R}$, pick a Riemannian metric g on M and consider the corresponding **downward gradient vector field** $-\nabla f$ on M determined by the identity of 1-forms $df = g(\nabla f, \cdot)$ since g is non-degenerate. The **downward gradient flow** associated to (M, f, g) is the 1-parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ of diffeomorphisms of M defined by $\varphi_t q := \gamma_q(t)$ where γ_q denotes the unique solution of the initial value problem

$$\dot\gamma(t) = -(\nabla f) \circ \gamma(t), \qquad \gamma(0) = q, \tag{1.0.3}$$

for smooth curves $\gamma : \mathbb{R} \to M$. Obviously critical points of f are fixed points of the flow. The **stable manifold** of a critical point x of f consists of all points $q \in M$ which under the flow asymptotically approach x in infinite forward time:

$$W^s(x) = W^s(x; -\nabla f) := \left\{ q \in M \;\Big|\; \lim_{t\to\infty} \varphi_t q \text{ exists and is equal to } x \right\}. \tag{1.0.4}$$

The **unstable manifold** of x is defined correspondingly by taking the limit over infinite *backward* time $t \to -\infty$. For non-degenerate x the **(un)stable manifold theorem** asserts that $W^u(x)$ is an embedded open disk of dimension $k = \mathrm{ind}(x)$ and $W^s(x)$ is an embedded open disk of complementary dimension. Thom observed that in case of a Morse function the unstable manifolds decompose $M = \cup_{x\in\mathrm{Crit}} W^u(x)$ into open disjoint disks, one for each critical point x, where the dimension of each disk is given by the corresponding Morse index. Of course, the stable manifolds decompose M as well.
Thom was awarded the Fields medal in 1958 [Hop58] for his cobordism theory which opened the field of differential topology.

The yet hidden enormous power of dynamical systems methods was only discovered, not to say unleashed, about 10 years later by Stephen Smale. In [Sma60] he proved strong Morse inequalities for dynamical systems more general than gradient flows based on the insight that *it is the intersections of unstable and stable manifolds that encode the topology of* M. He introduced the – ever since ubiquitous – **Morse-Smale (MS) condition** which asks that all unstable and stable manifolds have transverse intersection, in symbols

$$W^u(x) \pitchfork W^s(y), \qquad \forall x, y \in \mathrm{Crit}.$$

By definition **two submanifolds of *M* intersect transversely** if at each point of their intersection the two tangent spaces together span the whole tangent space to M, that is

$$T_q W^u(x) + T_q W^s(y) = T_q M, \qquad \forall q \in W^u(x) \cap W^s(y).$$

In this case it is common to say that the intersection is **cut out transversely** and differential topology, see e.g. [GP74, Hir76], asserts that it is a smooth manifold whose codimension is the sum of the two codimensions. Thus

$$\dim M_{xy} = \mathrm{ind}(x) - \mathrm{ind}(y), \qquad M_{xy} := W^u(x) \pitchfork W^s(y), \tag{1.0.5}$$

and M_{xy} is called the **connecting manifold** of x and y. We will see below that these ideas lead to a description (known as the Morse-Witten complex) of singular homology of M in terms of the 1-dimensional connecting manifolds. On the other hand, Smale extended Theorem 1.0.1 to the smooth category, namely, he proved that in fact M^b is *diffeomorphic* to M^a with a so-called thickened handle $\mathbb{D}^k \times \mathbb{D}^{n-k}$ attached where $k = \operatorname{ind}(x)$ and $n = \dim M$. This brought him to his handlebody theory and led to proofs of the h-cobordism theorem and the generalized Poincaré conjecture in dimensions ≥ 5, see [Sma61a, Sma62, Sma67], earning him the Fields medal in 1966 [Tho66].

Critical manifolds – Morse-Bott theory

The other, again very influential, strang in these years was Raoul Bott's idea [Bot54] to replace the concept of non-degenerate critical point by non-degenerate critical manifold and extend Morse theory to the more general situation; for applications see the survey [Bot82]. A **critical manifold** is a connected component of the set Crit of critical points of a function $f : M \to \mathbb{R}$ which carries the structure of a submanifold of M. It is called **non-degenerate** if at each point of the critical manifold the part of the Hessian matrix in the normal direction is non-singular. Motivation for and applications of these ideas came from the study of the length and energy functionals on the space of closed loops in a Riemannian manifold (M, g). Critical points are closed geodesics, that is 1-periodic curves $\gamma : \mathbb{R} \to M$ satisfying an ODE of second order which does not depend on time. Thus once you have one closed geodesic γ, shifting the initial time, say $\gamma_\tau(\cdot) := \gamma(\tau + \cdot)$, gives you a whole circle worth of geodesics. Actually this problem involves another, quite serious, generalization of Morse theory, namely, the transition from finite dimensional manifolds M to infinite dimensional ones such as the free loop space of M. Among others it was Richard Palais [Pal63] who paved the road in this respect, for instance, by generalizing the Morse-Lemma [Pal69a].

1980s/90s – Connecting trajectories

Remark 1.0.3. Assume M is a compact manifold equipped with a Morse function f. It is an exercise in algebra to see that the strong Morse inequalities (3.4.9) are equivalent to existence of a chain complex[2] $\mathcal{C} = (\mathrm{C}_k, \partial_k)_{k\in\mathbb{Z}}$ such that, firstly, each chain group C_k is the free abelian group generated by the set Crit_k of critical points of f of Morse index k and, secondly, the Betti numbers $b_k(\mathcal{C})$ of the corresponding homology groups $\mathrm{H}_*\mathcal{C}$ coincide with the Betti numbers $b_k(M) := \mathrm{rank}\mathrm{H}_*(M; \mathbb{Z})$ of singular integral homology of M.

While in the early 1980s Remark 1.0.3 was well known in mathematics and at least a topological model for such a chain complex was known, see [Mil65a, Thm. 7.4], it was a physicist, Edward Witten [Wit82], to discover a geometric model. He received the Fields medal in 1990 [Ati90], the only physicist so far, for a multitude of achievements one of which is the discovery of the geometric chain complex.

[2] The $\partial_k : \mathrm{C}_k \to \mathrm{C}_{k-1}$ are group homomorphisms such that $\partial_{k-1} \circ \partial_k = 0$ for all k.

Witten considered a supersymmetric quantum mechanical system which involves a deformed Hodge differential $d_s := e^{-sf} d e^{sf}$ depending on a real parameter s and where d is the usual exterior derivative acting on differential forms on M. He observed that for large parameter values the states of the system concentrate near the critical points of the Morse function f and that there are "tunneling" effects, called instantons, among these ground states. For very large s these instantons localize very close to downward gradient flow trajectories (1.0.3) which connect critical points of *Morse index difference one*. The mathematical rigorous construction of Witten's Hodge theory approach is due to Bernard Helffer and Johannes Sjöstrand [HS85].

Physics therefore suggests the following geometric construction assuming (MS). A downward gradient flow trajectory which connects two critical points whose Morse indices differ by one is called an *isolated flow line*. Counting these appropriately defines the boundary operator of the sought for geometric chain complex. Its chain groups are generated by the critical points of the Morse function $f : M \to \mathbb{R}$. This chain complex, graded by the Morse index, is called the (geometric) **Morse-Witten complex** and denoted by $\mathcal{CM} = (\mathrm{CM}_k, \partial_k)_{k\in\mathbb{Z}}$. The resulting homology theory is called **Morse homology**.

Consider again the example in Figure 1.2. For simplicity we count modulo two using $\mathbb{Z}_2$-coefficients. Observe that there are precisely four isolated flow lines, one from each local maximum to the saddle point $s := p_2$ and two from s to the minimum p_1. Thus Morse homology of M with $\mathbb{Z}_2$-coefficients is given by $\mathrm{HM}_2(M;\mathbb{Z}_2) = \mathbb{Z}_2$, generated by the cycle $p_3 + p_4$, and $\mathrm{HM}_0(M;\mathbb{Z}_2) = \mathbb{Z}_2$ is generated by p_1. Note that the saddle $s = p_2$ is a cycle since $\partial_1 s = 2p_1 = 0$, but s is also a boundary, namely of either of the two local maxima. Thus $\mathrm{HM}_0(M;\mathbb{Z}_2) = 0$. Obviously M is homeomorphic to the 2-sphere and we have just calculated $\mathrm{H}_*(\mathbb{S}^2;\mathbb{Z}_2)$.

The mathematical rigorous construction of the geometric Morse-Witten complex has been established by Matthias Schwarz [Sch93], who used an infinite dimensional functional analytic framework in the spirit of Floer theory, and the present author [Web93], who notized that the key tools in finite dimensional hyperbolic dynamics – the λ-Lemma of Jacob Palis [Pal69b] and the Grobman-Hartman theorem [Gro59,Har60] – serve to show that counting isolated flow lines indeed satisfies $\partial^2 = 0$, that is ∂ is a boundary operator. Marzin Poźniak [Poź91] contributed an elegant method to show that Morse-Witten homology does not depend on the choice of the Morse function f and the Riemannian metric g. Andreas Floer [Flo89b] and Dietmar Salamon [Sal90] utilized Conley index theory to show that Morse homology is isomorphic to singular homology of M.

2000s – Infinite dimensions and semi-flows

In a series of papers Alberto Abbondandolo and Pietro Majer, see e.g. [AM06], generalized the construction of the Morse-Witten complex from finite dimensional manifolds M to Banach manifolds equipped with a gradient-like vector field which generates a genuine flow. Recently, the present author [Web13] generalized the theory from flows towards *semi*-flows, more precisely, towards the forward semi-flow known as the heat flow on the Hilbert manifold given by the free loop space ΛM of M. In the course of calculating heat flow homology [Web17] the need for a *backward* λ-lemma for the heat flow emerged. Rather surprisingly – given that the heat flow does not admit a backward flow – a *backward* λ-lemma for the *forward* heat flow was indeed discovered recently [Web14b]. A rich source of semi-flows appears in the mathematical field of geometric analysis. It is yet to be explored if **semi-flow Morse homology** can be established in interesting cases and if it can answer some open problems.

Of course, **many more people contributed to the many facets of Morse theory**. Needless to say that the above historical account exclusively reflects experience and knowledge of the author.

Conley theory

Comments and outline of these notes

Exercises

Morse theory

Exercise 1.0.4. Show that the definition of a *non-degenerate* critical point using as criterium non-singularity of the Hessian matrix (1.0.1) does not depend on the choice of coordinates (see [GP74] if you get stuck), neither does $\mathrm{ind}(p)$.

Exercise 1.0.5. Show that the non-degeneracy condition (1.0.1) implies that a non-degenerate critical point is **isolated** or, in other words, **discrete**. This means that there is a neighborhood which contains no other critical points. [Hint: Transversality theory.]

Exercise 1.0.6. Assume $f : M \to \mathbb{R}$ is a smooth function on a manifold M.

(i) $\dim M = 1$: Give an example of an isolated degenerate critical point. Give an example of a non-isolated, thus by Exercise 1.0.5 degenerate, critical point such that f is not constant near the critical point.

(ii) $\dim M = 2$: Give an example of a *degenerate* critical point x for each of the three cases:

a) $\mathrm{Crit} = \{x\}$;

b) Crit is a 1-dimensional submanifold of M;

c) Crit is not a submanifold of M.

Exercise 1.0.7. Give two proofs, one of analytic and one of algebraic nature, that the time-0-map φ_0 determined by (1.0.3) is the identity map on M.

Exercise 1.0.8. Show the dimension formula (1.0.5).

Exercise 1.0.9. If you are familiar with CW-complexes, construct a CW-complex X homotopy equivalent to the surface M in Figure 1.2. Calculate the CW-homology of X.

Exercise 1.0.10. Prove the asserted equivalence in Remark 1.0.3; see [Web95, Thm. 2.4] in case you get stuck.

Conley theory

Chapter 2

Preliminaries

In this short chapter we collect a number of basic definitions and results in topology, algebra, analysis, and geometry. We also fix some notation.

2.1 Topology

We follow a common convention in topology, namely that a **map** *between topological spaces* is automatically understood continuous. The symbol '$\cong$' denotes isomorphism. Hence, for instance, in the smooth category '$\cong$' means diffeomorphism. It is an exception that in the topological category we denote isomorphisms, that is homeomorphisms, by the "weaker" symbol '$\simeq$'. The "weakest" symbol '$\sim$' denotes homotopy when applied to maps and homotopy equivalence when applied to topological spaces.

Definition 2.1.1. A **topological space** or simply a **space** is a pair $(X, \mathcal{O})$ where X is a set and $\mathcal{O}$ is a **topology** on X, that is a collection $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ of subsets $\mathcal{O}_\lambda \subset X$, called the **open sets** of the topology, which satisfy the following three axioms.

`(TOP-1)` Arbitrary unions of open sets are open.

`(TOP-2)` Finite intersections of open sets are open.

`(TOP-4)` The two natural subsets $\emptyset, X \subset X$ are open.

Complements of open sets are called **closed sets** and a set which is both open and closed is called **clopen**. For simplicity we usually denote $(X, \mathcal{O})$ by X.
A **subspace of a topological space** X consists of a subset $A \subset X$ equipped with the **subspace topology** $\mathcal{O}_A = \{A \cap \mathcal{O}_\lambda\}_{\lambda\in\Lambda}$.

Definition 2.1.2. Suppose X is a topological space. The **topological boundary** of a subset $A \subset X$ is the set $\dot{A}$ obtained by taking the closure of A, denoted by $\bar{A}$ or $\operatorname{cl} A$, and then removing the interior of A, denoted by $\mathring{A}$ or $\operatorname{int} A$.

Definition 2.1.3. A **neighborhood of a point q** of a topological space X is a subset $A \subset X$ which contains an open subset $U \subset X$ that contains q. Any subset $A \subset X$ which is a neighborhood of some point, that is which has non-empty interior $\operatorname{int} A$, is called a **neighborhood**.

Definition 2.1.4. Given topological spaces X and Y, then calling $f : X \to Y$ a **map** implicitely means that f is also continous. If f happens not to be continuous we call it application or function. In general, if the domain A or the target B of $g : A \to B$ is not a topological space to start with, then we call g likewise map, application, or function. In this case there is no continuity involved, of course. Two maps $f, g : X \to Y$ between topological spaces are called **homotopic**, in symbols $f \sim g$, if there is a **homotopy** between them, that is a map

$$h : I \times X \to Y, \qquad I := [0, 1],$$

such that $h_0 := h(0, \cdot) = f$ and $h_1 := h(1, \cdot) = g$.

Definition 2.1.5. A topological space X is called a **contractible space** if the identity map $\mathrm{id} : X \to X$ is homotopic to a constant map, that is to a map of the form $\iota_{x_0} : X \to X$, $x \mapsto x_0$, for some $x_0 \in X$.

Definition 2.1.6. Two topological spaces X and Y are called **homotopy equivalent**, in symbols $X \sim Y$, if there exist **reciprocal homotopy inverses**, that is maps $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\beta \circ \alpha \sim id_X$ and $\alpha \circ \beta \sim id_Y$. We say that two homotopy equivalent spaces are of the same **homotopy type**.

Definition 2.1.7. Assume X is a topological space and A is a subspace. By $\iota : A \hookrightarrow X$ we denote the inclusion.

(i) A **retraction** is a map $r : X \to A$ such that $r \circ \iota = \mathrm{id}_A$, that is $r(a) = a$ for every $a \in A$.

(ii) A **deformation retraction of X onto A** is a homotopy $h : I \times X \to X$ between the identity map $h_0 = \mathrm{id}_X$ on X and a retraction $h_1 = r : X \to A$. In this case we say that **A is a deformation retract of X**.

(iii) A deformation retract is called a **strong deformation retract** if, in addition, the points of A stay fixed throughout the homotopy, that is $h_t(a) = a$ for all $t \in I$ and $a \in A$.

Definition 2.1.8.

(i) A **pair of topological spaces** (X, A) consists of a topological space X and a subspace A. The **cartesian product of pairs** is defined by

$$(X, A) \times (Y, B) := (X \times Y, A \times Y \cup X \times B).$$

(ii) A **map of pairs**

$$f : (X, A) \to (Y, B)$$

is a (continuous) map $f : X \to Y$ which maps A into B, that is $f(A) \subset B$.

(iii) A **homotopy between maps of pairs** $f, g : (X, A) \to (Y, B)$ is a map of pairs

$$h : (I \times X, I \times A) \to (Y, B)$$

such that $h_0 = f$ and $h_1 = g$. Consequently $h_t(A) \subset B$ for every $t \in I$.

2.2 Algebra

Definition 2.2.1. (i) A **group** $(G, *)$ is a set G together with a **binary operation**, that is a map[1] $* : G \times G \to G$ such that the following axioms hold:

`(associativity)` For all $a, b, c \in G$ one has $a * (b * c) = (a * b) * c$;

`(neutral element)` There is an element $e \in G$ such that $e * a = a = a * e$ for every $a \in G$;

`(inverses)` For every $a \in G$ there is an element a' in G with $a * a' = e = a' * a$.

Additive notation: If the group operation is denoted by $+$, then one denotes the neutral element e by 0, inverses by $-a$, and k times combining the same element a by ka.
Multiplicative notation: If the group operation is denoted by $\cdot$, then one denotes the neutral element e by 1 called unit or identity, inverses by a^{-1}, and k times combining the same element a by a^k.

(ii) It is common to denote a group $(G, *)$ simply by the underlying set G. If G is a finite set, the group is called a **finite group**.

(iii) A subset H of a group G is called a **subgroup of G**, in symbols $H \le G$, if H also forms a group under the binary operation of G. This is the case if and only if H is non-empty and preserved under the binary operation and under taking inverses.

(iv) A subgroup $H \le G$ is called a **normal subgroup**, in symbols $H \trianglelefteq G$, if H is invariant under conjugation by the elements of G, that is $gHg^{-1} = H$ for each $g \in G$. Precisely the class of normal subgroups can be used to construct quotient groups G/N from a given group.

Definition 2.2.2. A group G is called **abelian** if combining elements does not depend on the order, that is if

`(commutativity)` $a * b = b * a$ for all $a, b \in G$.

An abelian group that admits a **basis** – a generating and linearly independent set – is called a **free abelian group**. It is called a **finitely generated** free abelian group, if it admits a finite basis. Any two bases of a free abelian group have the same cardinality called the **rank of the free abelian group**. Any

[1] A binary operation is often formulated as an **operation** which not only combines any two elements $a, b \in G$ to form an element $a * b$, but which is **closed** in the sense that $a * b$ always lies in G. A binary operation is a special case of a **binary function**, that is a map $A \times B \to C$ where A, B, C are sets. *Binary* refers to the fact that the domain is the cartesian product of *two* sets.

two maximal linearly independent subsets of an abelian group have the same cardinality called the **rank**, or **torsion-free rank**, **of the abelian group**; cf. Remark B.3.75.

Remark 2.2.3. [Abelian groups] (i) Every subgroup F of an abelian group G is normal (which is precisely what is needed to form a quotient group).

(ii) The rank of an abelian group G coincides with the dimension of the $\mathbb{Z}$-module tensor product $G \otimes_{\mathbb{Z}} \mathbb{Q}$ viewed as $\mathbb{Q}$-vector space; cf. Exercise 2.2.26.

(iii) *Theorem of Dedekind*: Every subgroup of a free abelian group G is itself a free abelian group. If G is not finitely generated, the proof uses the axiom of choice in the form of Zorn's Lemma; see [Lan93a, Thm. I.7.3 p.41, pf. p.880].[2]

Definition 2.2.4. A commutative **ring** R consists of a set R equipped with two maps $+, \cdot : R \times R \to R$, addition and multiplication, such that

– under addition R is an abelian group with neutral element 0;

– multiplication is associative, commutative, and has a unit element 1;

– both operations are compatible in the sense that

$$(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma, \qquad \alpha(\beta+\gamma) = \alpha\beta + \beta\gamma, \quad \alpha, \beta, \gamma \in R,$$

which is called **distributivity** and where $\alpha\beta := \alpha{\cdot}\beta$.

The integers $\mathbb{Z}$ carry naturally the structure of a ring. Every ring contains the two neutral elements 0 and 1, but they are not necessarily different. The smallest ring, the zero ring $R = \{0\}$ has one element, the neutral element of addition. In this case the neutral element of multiplication, the unit 1, necessarily coincides with 0.

Definition 2.2.5. An **integral domain** is a non-zero commutative ring R which admits no **zero divisors**, that is the product of any two non-zero elements is non-zero. A **principal ideal domain** or **PID** is an integral domain in which every ideal is principal, that is can be generated by a single element.

With respect to divisilibity PID's behave similar like the archetype PID $\mathbb{Z}$ in the sense that one has unique decomposition into prime elements and any two elements have a greatest common divisor.

Definition 2.2.6. A **field** is a ring $(\mathbb{F}, +, \cdot)$ such that

- under multiplication $\mathbb{F}^* := \mathbb{F} \setminus \{0\}$ is an abelian group with neutral element 1, called **unit**.

In other words, a field is a commutative ring in which every non-zero element admits a multiplicative inverse. Thus $\mathbb{Z}$ is not a field, but for example $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ with p prime are. Every field contains the two neutral elements 0 and 1 and they are necessarily different. The smallest field is $\mathbb{Z}_2 = \{0, 1\}$.

[2] In fact, the proof is given for a free R-module over a PID (principal ideal domain) R.

Definition 2.2.7. An ***R*-module** M, more precisely, a module $(M, +, \cdot)$ over a ring $(R, +, \cdot)$ consists of

- an abelian group $(M, +)$ with neutral element 0 and
- a map called **scalar multiplication**[3]

$$R \times M \to M, \quad (\alpha, m) \mapsto \alpha \cdot m,$$

which satisfies $1 \cdot m = m$ and is compatibel with the other three operations in the sense that

$$(\alpha + \beta) \cdot m = \alpha \cdot m + \beta \cdot m, \qquad \alpha \cdot (m + n) = \alpha \cdot m + \alpha \cdot n,$$

and

$$(\alpha \cdot \beta) \cdot m = \alpha \cdot (\beta \cdot m)$$

for all elements of R and M.

The additive group $(\{0\}, +)$ is a module over any ring. Any ring R is a module over itself. Any abelian group is a $\mathbb{Z}$-module (and trivially vice versa).

Theorem 2.2.8. *Every submodule of a free R-module over a PID R is free.*

Proof. [Lan93a, p.880]. See also Remark 2.2.3 (iii). □

Definition 2.2.9. A module over a field is called a **vector space**.

Definition 2.2.10. Let R be a commutative ring. An ***R*-algebra** consists of

- an R-module $(\mathcal{A}, +, \cdot)$ and
- an R-bilinear map $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

We require also that the product $\circ$ is *associative* and admits a *unit* element $\mathbb{1}$.

Remark 2.2.11. a) What we call R-module is strictly speaking a left R-module.
b) Associativity and unit are usually not part of the algebra definition. We include them for simplicity of presentation.
c) If one drops from an R-algebra $\mathcal{A}$ (as defined above) the coefficient ring R, including scalar multiplication, and replaces bilinearity by distributivity, then $\mathcal{A}$ becomes precisely a ring. So the notion of algebra generalizes the one of ring.

Categories and Functors

Following [Dol95, Ch. I §1] we recollect categories and functors.

Definition 2.2.12 (Category)**.** A **category** $\mathcal{C}$ consists of

(i) a *collection of objects*, denoted by $\mathrm{Ob}(\mathcal{C})$, and

[3] In this context the elements of R are called **scalars**.

(ii) for every pair X, Y of objects, a *set of morphisms from X to Y*, denoted by $\mathcal{C}(X, Y)$ or $[X, Y]$,[4] and

(iii) for every ordered triple of objects X, Y, Z an *application*

$$\circ : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$

called **composition**. One abbreviates intuitively $g \circ f := \circ(f, g)$.[5]

Objects, morphisms, and composition maps have to satisfy two axioms:

`(Associativity)` For all triples of composable morphisms it holds that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

`(Identities)` Every object X admits a morphism $\mathrm{id}_X : X \to X$ with

$$f \circ \mathrm{id}_X = f, \qquad \mathrm{id}_Y \circ f = f,$$

for every morphism $f : X \to Y$.

Example 2.2.13. The following are categories:

$\mathcal{Set}$ *Category of sets*: Objects are arbitrary sets, the set of morphisms $[X, Y]$ consists of all maps from X to Y, and $g \circ f$ has the usual meaning of composition of maps.

$\mathcal{AG}$ *Category of abelian groups*: Objects are arbitrary abelian groups, the set of morphisms $[X, Y] = \mathrm{Hom}(X, Y)$ consists of all homomorphisms from X to Y, and $g \circ f$ has the usual meaning.

$\mathcal{Top}$ *Category of topological spaces*: Objects are arbitrary topological spaces, the set of morphisms $[X, Y] = C^0(X, Y)$ consists of all (continuous) maps from X to Y, and $g \circ f$ has the usual meaning.

$\mathcal{Top}^{(2)}$ *Category of pairs of topological spaces*: Objects are arbitrary pairs of spaces, the set of morphisms $[(X, A), (Y, B)]$ consists of all maps of pairs from (X, A) to (Y, B), and $g \circ f$ has the usual meaning.

$\mathcal{C}^{\mathrm{op}}$ *Dual or opposite category*: Suppose $\mathcal{C}$ is a category. Set $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}^{\mathrm{op}}(X, Y) := \mathcal{C}(Y, X)$. Composition $*$ in $\mathcal{C}^{\mathrm{op}}$ is defined by

$$g * f := f \circ g, \qquad X \xrightarrow{f}_{\mathrm{op}} Y \xrightarrow{g}_{\mathrm{op}} Z.$$

R-$\mathcal{Mod}$ *Category of R-modules*: Objects are arbitrary R-modules, the set of morphisms $[M, N] = \mathrm{Hom}(M, N)$ consists of all homomorphisms from M to N, and $g \circ f$ has the usual meaning of composition of maps.

---

[4] A morphism f from X to Y is also denoted by $f : X \to Y$ or $X \xrightarrow{f} Y$.

[5] However, observe that the composition map is not in every category – although in many – given by composition of maps; see [Dol95, Ch. I Ex. 1.2 (iv)] where morphisms are not maps.

Definition 2.2.14 (Functor)**.** Given categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** T from $\mathcal{C}$ to $\mathcal{D}$, in symbols $T : \mathcal{C} \to \mathcal{D}$, consists of

(i) an *application* $T : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, and

(ii) *applications* $T = T_{XY} : \mathcal{C}(X, Y) \to \mathcal{D}(TX, TY)$, whenever $X, Y \in \mathrm{Ob}(\mathcal{C})$, which preserve composition and identities, namely

(iii) $T(g \circ f) = (Tg) \circ (Tf)$ for all morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$,

(iv) $T(\mathrm{id}_X) = \mathrm{id}_{TX}$ for every $X \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.2.15 (Cofunctor)**.** A **cofunctor** from $\mathcal{C}$ to $\mathcal{D}$ is, by definition, a functor from $\mathcal{C}$ to the dual category $\mathcal{D}^{\mathrm{op}}$ or, equivalently, a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.[6]

Direct and inverse limits

We follow the concise presentation in [BPS03, §4.6]; for further details see e.g. [Rot09, §5.2] or [GM03, §II.3.15].

Suppose $(I, \preceq)$ is a partially ordered set (**poset**). It is useful to view this poset as a category whose objects are the elements of I and where there is precisely one morphism from i to j whenever $i \preceq i$.

Suppose R is a commutative ring. A **partially ordered system** (**posyst**) **of *R*-modules over *I*** is a functor $(I, \preceq) \to R\text{-}\mathcal{Mod}$ written as a pair (M, ψ) where M assigns to each $i \in I$ an R-module M_i and ψ assigns to each related pair $i \preceq j$ of I a homomorphism $\psi_{ji} : M_i \to M_j$ such that

$$\psi_{kj}\psi_{ji} = \psi_{ki}, \qquad \psi_{\ell\ell} = \mathrm{id}_{M_\ell},$$

whenever $i \preceq j \preceq k$ and for every $\ell \in I$.

Definition 2.2.16 (Direct limit)**.** A poset $(I, \preceq)$ is **upward directed** if every pair of elements $i, j \in I$ admits a **common upper bound**, that is there is an $\ell \in I$ such that $i \preceq \ell$ and $j \preceq \ell$. In this case the posyst (M, ψ) above is called a **direct system of *R*-modules**. Its **direct limit** of such a direct system is defined as the quotient

$$\varinjlim M := \varinjlim_{i \in I} M_i := \{(i, x) \mid i \in I,\, x \in M_i\} \,/ \sim$$

where $(i, x) \sim (j, y)$ if and only if i and j admit a common upper bound ℓ such that $\psi_{\ell i} x = \psi_{\ell j} y$, that is x and y get mapped into a common R-module M_ℓ. Since I is upward directed this is an equivalence relation.

The direct limit is an R-module itself under the operations

$$[i, x] + [j, y] = [\ell, \psi_{\ell i} x + \psi_{\ell j} y], \qquad r[i, x] = [i, rx],$$

[6] In Definition 2.2.14 one only needs to replace (ii) by $T =: \mathcal{C}(X, Y) \to \mathcal{D}(TY, TX)$ and (iii) by $T(g \circ f) = (Tf) \circ (Tg)$.

where $\ell \in I$ is any common upper bound for i, j and $r \in R$ is any ring element. Furthermore, for each $i \in I$ there is an **insertion homomorphism**

$$\iota_i : M_i \to \varinjlim M, \quad x \mapsto [i, x],$$

which satisfies $\iota_i = \iota_j \psi_{ji}$ whenever $i \preceq j$. However, the homomorphism ι_i is not necessarily injective which is why it is called insertion and not injection.

Remark 2.2.17 (Universal property of direct limit)**.** Up to isomorphism the direct limit is characterized by the following universal property. Suppose N is any R-module and $(\tau_i : M_i \to N)_{i\in I}$ is a family of homomorphisms such that $\tau_i = \tau_j \psi_{ji}$ whenever $i \preceq j$. Then there is a unique homomorphism $\tau : \varinjlim M \to N$ such that $\tau_i = \tau \iota_i$ for every $i \in I$; indeed τ is given by $[i, x] \mapsto \tau_i(x)$

Definition 2.2.18 (Inverse limit)**.** A poset $(I, \preceq)$ is called **downward directed** if every pair of elements $i, j \in I$ admits a **common lower bound**, that is there is an $h \in I$ such that $h \preceq i$ and $h \preceq j$. In this case a posyst $(M, \psi) : (I, \preceq) \to R\text{-}\mathcal{Mod}$ is called an **inverse system of *R*-modules**. The **inverse limit** of such an inverse system is the submodule[7] of $\prod M_i$ defined by

$$\varprojlim M := \varprojlim_{i\in I} M_i := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} M_i \;\middle|\; i \preceq j \;\Rightarrow\; \psi_{ji} x_i = x_j \right\}.$$

For each $j \in I$ there is an obvious projection homomorphism

$$\pi_j : \varprojlim M \to M_j, \quad (x_i)_{i\in I} \mapsto x_j,$$

to the j^{th} component. These projections satisfy $\pi_j = \psi_{ji}\pi_i$ whenever $i \preceq j$, but it is not necessarily surjective.

Remark 2.2.19 (Universal property of inverse limit)**.** Up to isomorphism the inverse limit is characterized by the following universal property. Suppose N is any R-module and $(\tau_j : N \to M_j)_{j\in I}$ is a family of homomorphisms such that $\tau_j = \psi_{ji}\tau_i$ whenever $i \preceq j$. Then there is a unique homomorphism $\tau : N \to \varprojlim M$ such that $\tau_j = \pi_j \tau$ for every $j \in I$; indeed τ is given by $y \mapsto (\tau_i(y))_{i\in I}$.

Definition 2.2.20. A poset $(I, \preceq)$ is called **bidirected** if it is both up- and downward directed. In this case the posyst functor (M, ψ) is called a **bidirected system of *R*-modules**.

Remark 2.2.21 (Existence)**.**

Exhausting sequences

Computation of direct and inverse limits is greatly facilitated by the notion of exhausting sequences.

[7] An element $(x_i) \in \prod M_i$ such that $\psi_{ji} x_i = x_j$ whenever $i \preceq j$ is called a **thread**. Note that the set L of all threads is a submodule of $\prod M_i$.

Definition 2.2.22. Suppose (M, ψ) is a posyst of R-modules over $(I, \preceq)$. Denote $\mathbb{Z}^+ := \mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{Z}^- := \{\dots, -3, -2, -1\}$.
(i) A sequence $(i_\nu)_{\nu\in\mathbb{Z}^+}$ is called **upward exhausting** for (M, ψ) if and only if

- for every $\nu \in \mathbb{Z}^+$ one has $i_\nu \preceq i_{\nu+1}$ and $\psi_{i_{\nu+1} i_\nu} : M_{i_\nu} \to M_{i_{\nu+1}}$ is an isomorphism. Moreover, it is required that
- every $i \in I$ is dominated by some i_ν, that is $i \preceq i_\nu$ for some $\nu \in \mathbb{Z}^+$.

(ii) A sequence $(i_\nu)_{\nu\in\mathbb{Z}^-}$ is **downward exhausting** for (M, ψ) if and only if

- for every $\nu \in \mathbb{Z}^-$ one has $i_{\nu-1} \preceq i_\nu$ and $\psi_{i_\nu i_{\nu-1}} : M_{i_{\nu-1}} \to M_{i_\nu}$ is an isomorphism. Moreover, it is required that
- every $i \in I$ dominates some i_ν, that is $i_\nu \preceq i$ for some $\nu \in \mathbb{Z}^-$.

Lemma 2.2.23. *Suppose (M, ψ) is a posyst of R-modules over $(I, \preceq)$.*

(i) *If $(i_\nu)_{\nu\in\mathbb{Z}^+}$ is an upward exhausting sequence for (M, ψ), then $(I, \preceq)$ is upward directed and the insertion homomorphism $\iota_{i_\nu} : M_{i_\nu} \to \varinjlim M$ is an isomorphism for every $\nu \in \mathbb{Z}^+$.*

(ii) *If $(i_\nu)_{\nu\in\mathbb{Z}^-}$ is a downward exhausting sequence for (M, ψ), then $(I, \preceq)$ is downward directed and the projection homomorphism $\pi_{i_\nu} : \varprojlim M \to M_{i_\nu}$ is an isomorphism for every $\nu \in \mathbb{Z}^-$.*

Exercises

Exercise 2.2.24. Show that if A is a deformation retract of X, then $A \sim X$.

Exercise 2.2.25. Show that abelian groups and $\mathbb{Z}$-modules are the same.

Exercise 2.2.26. Show the assertion of Remark 2.2.3 (ii): For any abelian group G it holds that $\operatorname{rank} G := |\mathcal{B}| = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$ where $\mathcal{B} \subset G$ is any maximal linearly independent subset and the $\mathbb{Z}$-module tensor product $G \otimes_{\mathbb{Z}} \mathbb{Q}$ is viewed as a $\mathbb{Q}$-vector space with scalar multiplication $(r', g \otimes r) \mapsto g \otimes rr'$. [Hint: Show that $\{b \otimes 1 \mid b \in \mathcal{B}\}$ is a basis of the $\mathbb{Q}$-vector space.]

Exercise 2.2.27 (Direct limit is generalized union)**.**

Exercise 2.2.28 (Inverse limit is generalized intersection)**.**

Exercise 2.2.29 (Exhausting sequences)**.** Prove Lemma 2.2.23.

2.3 Analysis

Theorem 2.3.1 (Inverse Function Theorem)**.** *Suppose $f : E \supset U \to F$ is a C^r map between Banach spaces for some $r \geq 1$ and defined on an open set U. Let*

$x_0 \in U$ and assume that the derivative $f'(x_0) = df_{x_0} : E \to F$ is a **toplinear isomorphism**[8]*. Then f is a* **local diffeomorphism**[9] *at x_0 of class C^r.*

Proof. See e.g. [Lan93b, XIV §1]. □

2.4 Manifolds and transversality

Whenever the term **manifold** is used a smooth or at least C^r *differentiable* manifold[10] M of finite dimension, say n, is meant and the degree $r \geq 1$ of differentiability should be taken from the context. Excellent introductions to manifolds and transversality theory are the elementary texts [Jän01] and [GP74] and the slightly more advanced classics [Mil65b] and [Hir76].

Suppose $f : M \to N$ is a C^1 map between manifolds. A point $x \in M$ at which the linearization $df_x : T_xM \to T_{f(x)}N$ is not surjective is called a **critical point**, otherwise, it is a **regular point**. A point $y \in N$ is called a **critical value** of f if it is of the form $y = f(x)$ for some critical point x. Otherwise, in particular if $y \notin f(M)$, it is called a **regular value**. If $y \in f(M)$ is a regular value, then its pre-image $f^{-1}(y)$ is called a **regular level set**.

Theorem 2.4.1 (Regular Value Theorem)**.** *Suppose $f : M \to N$ is a C^r map with $r \geq 1$. If $y \in f(M)$ is a regular value, then $f^{-1}(y)$ is a C^r submanifold of M. Its dimension is the difference* $\dim M - \dim N$.

Proof. Inverse Function Theorem 2.3.1; see e.g. [Hir76, Ch. 1 Thm. 3.2]. □

For us the case $N = \mathbb{R}$, that is $f : M \to \mathbb{R}$, will be most important. In this case a regular level set is a submanifold of codimension 1, called a **hypersurface**. Since regular values give rise to submanifolds one would wish for a result asserting that the set of regular values is large or, equivalently, that its complement the set of critical values is small. For a proof of the following theorem in the smooth case we refer to [Hir76, Ch. 3 Thm. 1.3] where also the necessity of the differentiability requirement is discussed.

Theorem 2.4.2 (Sard's theorem)**.** *Suppose $f : M \to N$ is a C^r map with $r \geq 1$. If in addition $r > \dim M - \dim N$, then the set of critical values of f has measure zero in N. The set of regular values of f is dense in M.*

Substituting the regular value y in Theorem 2.4.1 by a whole family A one obtains a rather general and powerful tool called transversality.

Theorem 2.4.3 (Transversality)**.** *Suppose a C^r map $f : M \to N$ with $r \geq 1$ is* **transverse to a submanifold** *$A \subset N$ of class C^r (in symbols $f \pitchfork A$), that is*

$$df_x(T_xM) + T_{f(x)}A = T_{f(x)}N$$

[8] that is it admits an inverse (which is automatically linear) and this inverse is continuous.

[9] that is there are open neighborhoods U_0 of x_0 and V_0 of $f(x_0)$ such that f restricts to a bijection $f : U_0 \to V_0$ and the inverse of this bijection is of class C^r.

[10] that is a second countable Hausdorff space which is locally homeomorphic to euclidean space $\mathbb{R}^n$ such that all transition maps are C^r diffeomorphisms.

whenever $f(x) \in A$. Then the pre-image $f^{-1}(A)$ is a C^r submanifold of M. Its codimension is the codimension of A in N.

Proof. See e.g. [Hir76, Ch. 1 Thm. 3.3]. □

Remark 2.4.4. We say that two submanifolds $A, B \hookrightarrow N$ have **transverse intersection**, in symbols $A \pitchfork B$, if

$$T_y B + T_y A = T_y N, \qquad \forall y \in A \cap B,$$

that is together the two tangent subspaces span the whole ambient tangent space or, equivalenty, the inclusion $\iota : B \hookrightarrow N$ is transverse to A. Thus by Theorem 2.4.3 the intersection $A \cap B$ is a submanifold of N whose codimension is the sum of the codimensions of A and B. In this case one also says that the intersection $A \cap B$ is **cut out transversely**.

Manifolds-with-boundary

If in the above definition of *manifold* one generalizes to "...locally homeomorphic to euclidean space $\mathbb{R}^n$ or euclidean half-space $[0, \infty) \times \mathbb{R}^{n-1}$ such that..." one obtains what is called a (C^r differentiable) **manifold-with-boundary** M. By definition a point $x \in M$ belongs to the boundary ∂M of M if it is identified by some, hence every[11], coordinate chart with an element of the boundary $\{0\} \times \mathbb{R}^{n-1}$ of euclidean half-space. The boundary ∂M of a C^r manifold-with-boundary M of dimension n is itself a C^r manifold of dimension $n-1$.

Analogues of the Regular Value Theorem 2.4.1 and the Transversality Theorem 2.4.3 hold true for manifolds-with-boundary; see [Hir76, Ch. 1 §4].

Remark 2.4.5. Obviously a manifold M is a manifold-with-boundary whose boundary is the empty set. The latter is also called a **manifold-without-boundary**. In other words, *manifold* and *manifold-without-boundary* are equivalent notions. A compact manifold-without-boundary is called a **closed manifold**. On the other hand, to emphasize non-triviality of the boundary a manifold-with-boundary M with $\partial M \neq \emptyset$ is called a **∂-manifold**.

2.5 Group actions

We briefly introduce basic notions of group actions. For further redaing we recommend the textbooks [tD87, DK00].

Definition 2.5.1. A (left) **group action** of a group G on a set X is a map

$$\phi : G \times X \to X, \quad (g, x) \mapsto \phi(g, x) =: gx,$$

that satisfies the following two axioms:

[11] In other words, a coordinate change (also called a transition map) cannot map a boundary point of half-space to an interior point. If $r \geq 1$ this follows from the Inverse Function Theorem 2.3.1 and for $r = 0$ from a highly non-trivial result in topology called *invariance of domain*; see [Hir76, Ch. 1 §4].

`(identity)` The neutral element e of the group G acts as the identity on X, that is $\phi(e,\cdot) = \mathrm{id}_X$ or equivalently $ex = x$, $\forall x \in X$;

`(compatibility)` The action and the group operation are compatible in the sense that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

Note that the two axioms imply that for $g \in G$ the maps $\phi_g : X \to X$ are bijections, the inverse being $\phi_{g^{-1}}$, whose compositions satisfy $\phi_g \phi_h = \phi_{gh}$.

Definition 2.5.2. A group action of G on X is called

– **transitive** if X is non-empty and if for each pair x, y in X there is a group element g such that $gx = y$;

– **faithful**, or **effective**, if different group elements $g \neq h$ provide different bijections $\phi_g \neq \phi_g : X \to X$.

– **free**, or **fixed point free**, if none of the maps

$$\phi(g,\cdot) : X \to X, \quad g \in G \setminus \{e\},$$

admits fixed points. (Equivalently, if $gx = x$ has no solution unless $g = e$.)

Remark 2.5.3 (Orbits)**.** Consider a group action of a group G on a set X. Given a point $x \in X$, then all points $y \in X$ to which x can be moved by the elements of G is called the **orbit of x** and denoted by

$$Gx := \{gx \mid g \in G\}.$$

Orbits are the equivalence classes of the equivalence relation on X defined by $x \sim y$ if and only if $gx = y$ for some $g \in G$. Consequently X is partitioned by the orbits. Observe that the action is transitive if and only if it admits only one orbit, namely M. In this case M is called a **homogeneous space**. In general, the set of all orbits is called the **orbit space** and denoted X/G or X_G.

Definition 2.5.4 (G-manifold)**.** Consider a group action ϕ of G on M where G is a Lie group and M is a C^k manifold, $k \geq 1$, and such that map $(g,p) \mapsto \phi(g,p) =: gp$ is of class C^k. Such group action is called a **Lie group action** or a **smooth group action** of class C^k on the manifold M. A manifold equipped with a Lie group action is called a ***G*-manifold**. A function $f : M \to \mathbb{R}$ invariant under the action of G is called an **equivariant function**.

While in general (the quotient topology on) the orbit space M/G may not be Hausdorff, it is Hausdorff whenever G is compact. If G is compact and acts *freely*, then M/G is a manifold. This is a consequence of Koszul's tube theorem [Kos53] also called *slice theorem* or *equivariant tubular neighborhood theorem*; see also [Mos57, Pal61].

Theorem 2.5.5. *Suppose G is a compact Lie group which acts freely and of class C^k, $k \geq 1$, on a manifold M. Then the quotient space M/G has a unique structure of a C^k manifold of dimension $\dim M - \dim G$.*

Proof. [tD87, Prop. 5.2] or [DK00, Thm. 1.11.4] or [GGK02, App. B]. □

2.6 Local and global dynamics

For an overview, respectively the details, of the dynamics generated by vector fields on euclidean space and on finite dimensional manifolds we recommend the books by Hirsch [Hir76, Ch. 6 §2] and Zehnder [Zeh10, §IV.1], respectively Amann [Ama90, CH. 2] and Teschl [Tes12, Ch. 2 and §6.2]. The case of Banach spaces and Banach manifolds is treated in the book by Lang [Lan01, Ch. IV].

Flows

Following [Ama90, §10] suppose S is a metric space. For each $x \in S$ let $J_x = (T_x^-, T_x^+)$ be an open interval about $0 \in \mathbb{R}$ and consider the set

$$\Omega := \bigcup_{x \in S} J_x \times \{x\}.$$

Definition 2.6.1. A **(local) flow** or a **dynamical system on *S*** is a map

$$\varphi : \Omega \to S, \quad (t, x) \mapsto \varphi(t, x) =: \varphi_t(x) =: \varphi_t x,$$

with the following properties:

(i) the subset $\Omega \subset \mathbb{R} \times S$ is open;

(ii) the map $\varphi : \Omega \to S$ is continuous;

(iii) $\varphi_0 = \mathrm{id}_M$;

(iv) for all $x \in S$, $t \in J_x$, and $s \in J_{\varphi_t x}$ it holds that

$$s + t \in J_x, \qquad \varphi_s \varphi_t x = \varphi_{t+s} x.$$

If $\Omega = \mathbb{R} \times S$, that is $J_x = \mathbb{R}$ for every point x of S, then φ is called a **global flow** or a **global dynamical system**. For each $x \in S$, the curve

$$u_x : J_x = \left(T_x^-, T_x^+\right) \to S, \quad t \mapsto \varphi_t x,$$

is called the **flow line** or **trajectory through *x*** and the constants $T_x^- = T^-(x)$ and $T_x^+ = T^+(x)$ are called the **negative** and **positive life time of *x***, respectively. The images

$$\mathcal{O}^-(x) := \varphi_{(T_x^-, 0]} x, \qquad \mathcal{O}^+(x) := \varphi_{[0, T_x^+)} x, \qquad \mathcal{O}(x) := \varphi_{J_x} x$$

are called the **negative semi-orbit**, the **positive semi-orbit**, and the **orbit through *x***, respectively, and S is called the **phase space** of the flow.

Remark 2.6.2.

a) If S is a C^r-manifold where $r \geq 1$ and if (ii) above is replaced by

(ii') $\varphi \in C^r(\Omega, S)$

one obtains a $\boldsymbol{C^r}$**-flow**.

b) Each orbit $\mathcal{O}(x)$ comes with its **flow orientation**, that is the direction in which the flow line u_x runs through the orbit.

c) The orbits partition the phase space, that is $S = \cup_{x\in S}\mathcal{O}(x)$ and two orbits are either equal or disjoint.

Definition 2.6.3. Suppose Λ is a topological space and consider a function $f : \Lambda \to [-\infty, \infty] =: \bar{\mathbb{R}}$. Then f is called **lower semi-continuous at** $\boldsymbol{\lambda_*}$ if for every $a < f(\lambda_*)$ there is a neighborhood U_a of λ_* such that $f(\lambda) > a$ for every $\lambda \in U_a$. A function is **upper semi-continuous** if it is upper semi-continuous at every point. A function f is **upper semi-continuous** if $-f$ is lower semi-continuous.

Lemma 2.6.4. *Suppose φ is a flow on S. Then*

(i) the backward and forward life times $T^-, T^+ : S \to (0, \infty]$ are upper and lower semi-continuous, respectively;

(ii) for every $(t, x) \in \Omega$ it holds that $J_{\varphi_t x} = J_x - t$.

Proof. [Ama90, Lemma 10.5]. □

Flows generated by vector fields

Suppose $U \subset \mathbb{R}^n$ with $n \geq 1$ is an open subset and $F : U \to \mathbb{R}^n$ is a map of class C^r with $r \geq 1$.[12] Then the Cauchy or initial value problem

$$\dot{u} = F(u), \qquad u(0) = x, \tag{2.6.1}$$

admits a unique solution

$$u_x : J_x \to U, \qquad J_x = \left(T_x^-, T_x^+\right),$$

which is of class C^{r+1} and where $J_x \ni 0$ is the maximal interval of existence; see e.g. [Ama90, Thm 7.6]. Moreover, if the solution u_x remains in a compact subset Q of U, that is $u_x(t) \in Q$ for every $t \in J_x$, then $J_x = \mathbb{R}$. Now consider the map defined by

$$\varphi(t, x) := \varphi_t x := u_x(t)$$

on the set

$$\Omega^F := \{(t, x) \in \mathbb{R} \times U \mid t \in J_x, x \in U\}$$

which is in fact an open subset of $\mathbb{R} \times U$ by the following result.

Theorem 2.6.5. *If $F \in C^r(U, \mathbb{R}^n)$ with $r \geq 1$, then the solution*

$$\varphi = \varphi^F : \Omega^F \to U$$

of the Cauchy problem (2.6.1) is a C^r-flow on U, the **flow generated by** $\boldsymbol{F}$.

[12] (Local) Lipschitz continuity is sufficient to obtain a Lipschitz flow; see e.g. [Ama90, §10].

Proof. [Ama90, Thm 10.3]. □

These results remain valid if $\mathbb{R}^n$ is replaced by an arbitrary Banach space; see e.g. [Lan01, IV Thm. 1.16].

Manifolds and Banach manifolds

Using local coordinate charts one can lift the former results from $\mathbb{R}^n$ to differentiable manifolds of dimension n, see e.g. [Hir76, Ch. 6 §2], and from Banach spaces to Banach manifolds, see e.g. [Lan01, IV §2].

Hyperbolic gradient dynamics

Morse theory embeds naturally, upon choosing a Riemannian metric, into the well known field of hyperbolic dynamical systems, more precisely, into the subfield of hyperbolic gradient dynamics. For convenience of the reader and also to present and mirror some possibly less known results and techniques we shall provide all relevant results with proofs. To avoid conflict with the title of the present text this material is put in the appendix, given the large number of pages. We recommend consulting the overview given in the beginning of Appendix A and then to proceed with Chapter 3. The recent developments presented in Chapter 4 are based on large parts of Appendix A.5 though.

Part I

Morse theory

Part II

Conley theory

Part III

Appendices

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List of Symbols

$(\mathbb{D}^0, \mathbb{S}^{-1})$	$(\{0\}, \emptyset)$	37
$(\mathbb{D}^r, \mathbb{S}^{r-1})$	unit disk, unit sphere in $\mathbb{R}^r$	
$(M; \partial^- M, \partial^+ M)$	manifold triad	104
(N, L)	Conley pair	171,173
$(X, A) \times (Y, B)$	is defined by $(X \times Y, A \times Y \cup X \times B)$	10
$\lvert \mathrm{C} \rvert = \sum_i \dim \mathrm{C}_i$	total dimension of chain complex C	280
$\bar{\alpha}$	linear extension of function on basis	296
$\mathcal{B}_f^\# = \mathrm{Crit}^\# f = (\eta^x)_x$	formal dual basis of $\mathcal{B}_f = \mathrm{Crit} f$	150
$\big(\mathrm{H}^*(X) \otimes \mathrm{H}^*(Y), +, \bullet\big)$	tensor product algebra	318
$\dot{A} = \bar{A} \setminus \mathring{A}$	topological boundary of a set A	
$\cap$	cap product $\mathrm{H}^* \times \mathrm{H}_* \to \mathrm{H}_*$	311,314
$\mathrm{cat}(X)$	Lusternik-Schnirelmann category	62
χ	Euler characteristic	280
$\mathrm{cl}\, A = \bar{A}$	closure of a set A	
$\mathrm{Crit}_k f$	set of critical points of Morse index k	100
$\cup$	cup product on singular cochains	306
$\cup_\leq$	cup product on simplicial cochains	304
$\mathrm{cup}(X)$	cuplength of X	68
$\mathrm{cup}_{\mathbb{F}}(X)$	$\mathbb{F}$-cuplength of X	68
$\mathbb{D}^0 = \{\mathsf{pt}\}$, $\mathbb{S}^{-1} = \emptyset$	0-disk and (-1)-sphere	37
Δ	diagonal map $\Delta : X \to X \times X$, $x \mapsto (x, x)$	345
	diagonal set $\Delta = \{(x, x) \mid x \in X\}$	
$\delta^k : \mathrm{CM}^k \to \mathrm{CM}^{k+1}$	coboundary operator given by $\partial_{k+1}^\#$	149
Δ^m	standard euclidean m-simplex	295
$\eta^x : \mathrm{CM}_k \to \mathbb{Z}_2$	Dirac functional of $x \in \mathrm{Crit}_k f$	150
$\hat{\alpha}$	zero extension from $A \subset D$ to D	299
$\hat{u}(+\infty)$	asymptotic limit vector of orbit u	113
$\mathrm{HM}_*(M; \partial^- M, \partial^+ M)$	Morse homology of manifold triad	131
$\mathrm{I} = [0, 1]$	unit interval	
$\mathrm{int}\, A = \mathring{A}$	interior of a set A	

$\mathring{X}$	interior of topological space X	
$\mathcal{K}_{\mathrm{C}} = \left(\mathrm{C}, \mathrm{C}^{\#}, \langle\cdot,\cdot\rangle_{\mathrm{ev}}\right)$	canonical Kronecker triple	269
$\langle\alpha, c\rangle_{\mathrm{ev}}$	evaluation pairing $\alpha(c)$	269
$\langle\emptyset\rangle = \{0\}$	empty set generates trivial group	99,115,296
$\mathfrak{mb}_s = \mathfrak{mb}_s^{[a,b]}(f;\mathbb{F})$	Morse-Bott polynomial	52
$\mathfrak{mb}_s^G = \mathfrak{mb}_s^{G;[a,b]}(f;\mathbb{F})$	equivariant Morse-Bott polynomial	58
$\mathfrak{p}_s^G = \mathfrak{p}_s^G(M;\mathbb{F})$	equivariant Poincaré polynomial	57
$\mathfrak{p}_s(\mathrm{C};\mathbb{F})$	Poincaré polynomial of chain complex C	280
$\dot\bigcup_\lambda X_\lambda$	disjoint union: $\lambda \neq \beta \Rightarrow X_\lambda \cap X_\beta = \emptyset$	260,283
v/e/f	vertices/edges/faces	291
$\max\emptyset := -\infty$	$\min\emptyset := \infty$	
$\boldsymbol{\mathcal{K}}_{\mathbf{HC}} = (\mathrm{H}_*(\mathrm{C}), \mathrm{H}^*(\mathrm{C}), \langle\boldsymbol{\cdot},\boldsymbol{\cdot}\rangle)$	canonical Kronecker homology triple	270
$\mathcal{MS}(M) \ni h = (f,g)$	positive exhaustive Morse-Smale pairs	141
$\mathcal{MS}(M,A)$	relative Morse-Smale pairs	156
$\nu_Q = \nu_Q^M$	normal bundle of Q in M	351
$\mathrm{O}(2)$	orthogonal group $\mathrm{SO}(2) \rtimes \mathbb{Z}_2$	59
$\partial^{\#}$	dual operator or transpose	268
$\partial_k = \partial_k^{\mathrm{M}}$	Morse-Witten boundary operator	103
$\mathrm{PD}: \mathrm{H}_{n-\ell} \to \mathrm{H}^{\ell}$	Poincaré duality operators	347
${}^p\sigma$	simplicial/singular front face	304,306
$\Psi_*^{\beta\alpha} := [\psi_*^{\beta\alpha}(h_{\alpha\beta})]$	continuation isomorphism	117
$\mathbb{R}^+ = (0,\infty)$		
$\mathrm{SB}(X,A) := \sum_k b_k = \dim \mathrm{H}_*(X,A;\mathbb{Q})$	sum of Betti numbers	40
$\sigma^{\#}$	gluing map for orientations	111
σ^q	simplicial/singular back face	304,306
$\mathrm{SP}_m^m(K)$	simplicial cochain group $2_{\mathrm{fin}}^{\mathcal{S}_m(K)}$	285
$\mathrm{S}X$	singular chain complex	297
$\mathrm{S}^{\#}X$	singular cochain complex	297
$\mathrm{S}_m^m(X)$	singular cochain group $2_{\mathrm{fin}}^{\mathcal{S}_m(X)}$	295
$\mathcal{S}_m(X)$	canonical basis of singular group $\mathrm{S}_m(X)$	296
$\sum'$	formal sum	151
$\theta_{\mathrm{rel}}^{\boldsymbol{t}}: \mathrm{H}^*(F,F') \to \mathrm{H}^*(E,E')$	CEF: cohomology extension of the fiber associated to Thom class $\boldsymbol{t} \in \mathrm{H}^r(E,E')$	268
$\mathrm{Thom}: \mathrm{H}^m(B) \to \mathrm{H}^{m+r}(E,E')$	Thom isomorphism $\boldsymbol{\beta} \mapsto \boldsymbol{p}^*\boldsymbol{\beta} \cup_{\boldsymbol{\cdot}} \boldsymbol{t}$	337
$\times$	(co)homology cross product	321
$\times_\theta$	bundle cross product	329
$\times_{\theta_{\mathrm{rel}}}$	relative bundle cross product	336
$\underline{\mathrm{Thom}}: \mathrm{H}_m(E,E') \to \mathrm{H}_{m-r}(B)$	homology Thom isom. $\boldsymbol{c} \mapsto \boldsymbol{p}_*(\boldsymbol{t}_{\boldsymbol{\cdot}}\cap \boldsymbol{c})$	337
$\underline{\times}$	homology cross product	322
$\underline{\times}_\theta$	homology bundle cross product	329
$\varphi^{(f,g)} = \varphi = \{\varphi_t\}$	downward gradient flow (of $-\nabla^g f$)	100
$A \otimes_R B$	tensor product of R-modules	316
A^{C}	complement $X \setminus A$ of subset A	297
$A_* \otimes_R B_*$	tensor product of graded R-modules	318
e^k	k-cell	36