Proposition 1: A convergent sequence is a Cauchy sequence.
Proof: This is Ross, Lemma 10.9.

Definition: The series \( \sum_1^\infty a_k \) converges means that the sequence \((s_n = \sum_1^n a_k)\) of partial sums is a convergent sequence. (Ross, section 14.2).

Proposition 2: (Cauchy criterion) the series \( \sum_1^\infty a_k \) converges if and only if for every \( \epsilon > 0 \) there exists an \( N \) such that \( m,n > N, m > n \) implies \( \sum_{n+1}^m a_k < \epsilon \).
Proof: This is Ross, Theorem 14.4.

Proposition 3: If the series \( \sum_1^\infty a_k \) converges then \( \lim a_k = 0 \).
Proof: Need to show for any \( \epsilon > 0 \) there exists an index \( N \) such that if \( n > N \) then \( |a_n| < \epsilon \). If the series converges, it satisfies the Cauchy criterion: there exists an \( N' \) such that if \( m,n > N' \) (and \( m \geq n \)) then \( |\sum_{n+1}^m a_k| < \epsilon \). Take \( N = N' + 1 \). If \( n > N \) then \( n - 1 > N' \) and \( |\sum_{n-1+1}^m a_k| < \epsilon \), i.e. \( |\sum_{n}^m a_k| < \epsilon \). In particular, take \( m = n \). Then \( |a_n| = |\sum_{n}^m a_k| < \epsilon \), as required. [This is better than the argument I gave in class, which required proving first that \( \lim a_k \) exists.]

Proposition 4 (Comparison Test): Suppose \( \sum a_k \) is a convergent series with positive terms (every \( a_k \geq 0 \)). Then if the terms of a series \( \sum b_k \) satisfy \( |b_k| \leq a_k \) for every \( k \), the series \( \sum b_k \) converges.

Proof. We use the Cauchy criterion. For every \( \epsilon > 0 \) there exists an index \( N \) such that if \( m,n > N \) implies \( \sum_{n+1}^m a_k < \epsilon \). Suppose then \( m,n > N \); then \( |\sum_{n+1}^m b_k| \leq \sum_{n+1}^m |b_k| \leq \sum_{n+1}^m a_k < \epsilon \) (\( \ast \)), so \( \sum b_k \) satisfies the Cauchy criterion and therefore converges. Triangle inequality used in (\( \ast \)).

Definition: A series \( \sum a_k \) converges absolutely means that \( \sum |a_k| \) converges.

Proposition 5: If a series converges absolutely, it converges.
Proof: Since \( a_k \leq |a_k| \) this follows from the Comparison Test.

Proposition 6 (Ratio Test): Suppose a series \( \sum a_k \) of non-zero terms satisfies \( \lim \left| \frac{a_{n+1}}{a_n} \right| = R \). Then if \( R < 1 \) the series \( \sum a_k \) converges absolutely, and if \( R > 1 \) it diverges.

Proof. First suppose \( R < 1 \). Let \( \epsilon = \frac{1}{2}(1 - R) \). Note that \( \frac{1}{2}(1 - R) > 0 \) so there exists an index \( N \) such that if \( n > N \) then
\[
\left| \frac{a_{n+1}}{a_n} \right| - R < \frac{1}{2}(1 - R),
\]
which means
\[
R - \frac{1}{2}(1 - R) < \left| \frac{a_{n+1}}{a_n} \right| < R + \frac{1}{2}(1 - R).
\]
Set \( \rho = R + \frac{1}{2}(1 - R) = \frac{1}{2}(1 + R) \) and note that \( \rho < 1 \). In particular,

\[
\frac{|a_{N+2}|}{a_{N+1}} < \rho \quad \text{so} \quad |a_{N+2}| < \rho |a_{N+1}|
\]

\[
\frac{|a_{N+3}|}{a_{N+2}} < \rho \quad \text{so} \quad |a_{N+3}| < \rho |a_{N+2}| < \rho^2 |a_{N+1}|
\]

\[
\frac{|a_{N+i}|}{a_{N+i-1}} < \rho \quad \text{so} \quad |a_{N+i}| < \rho |a_{N+i-1}| < \ldots < \rho^{i-1} |a_{N+1}|
\]

\[
\ldots
\]

With the \( N \) we have obtained, let us write

\[
\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{\infty} |a_k|.
\]

The first sum is finite. We can apply the Comparison Test to the second sum, which equals \( |a_{N+1}| + |a_{N+2}| + \cdots \). The series \( \sum_{k=N+1}^{\infty} |a_k| \) is equal to \( \sum_{i=1}^{\infty} |a_{N+i}| \) (just rewriting the indices). Since \( |a_{N+i}| \leq \rho^{i-1} |a_{N+1}| \) (note that this holds for \( i = 1 \) also), and \( \sum_{i=1}^{\infty} \rho^{i-1} |a_{N+1}| \) is a geometric series converging to

\[
|a_{N+1}| \sum_{i=1}^{\infty} \rho^{i-1} = |a_{N+1}| \sum_{i=0}^{\infty} \rho^i = \frac{|a_{N+1}|}{1 - \rho}
\]

the Comparison Test tells us that \( \sum_{k=N+1}^{\infty} |a_k| \) converges. Throwing in the finite sum \( \sum_{k=1}^{N} |a_k| \) exhibits \( \sum_{k=1}^{\infty} |a_k| \) as a convergent sequence, as was to be shown.

Now suppose \( R > 1 \), and take \( \epsilon = \frac{1}{2}(R - 1) \). Arguing as before, we can find an \( N \) such that if \( n > N \) then

\[
R - \frac{1}{2}(R - 1) < \left| \frac{a_{n+1}}{a_n} \right| < R + \frac{1}{2}(R - 1).
\]

Set \( \rho = R - \frac{1}{2}(R - 1) = \frac{1}{2}(R + 1) \) and note that \( \rho > 1 \). In particular,

\[
\frac{|a_{N+2}|}{a_{N+1}} > \rho \quad \text{so} \quad |a_{N+2}| > \rho |a_{N+1}|
\]

\[
\frac{|a_{N+i}|}{a_{N+i-1}} > \rho \quad \text{so} \quad |a_{N+i}| > \rho |a_{N+i-1}| > \ldots > \rho^{i-1} |a_{N+1}|
\]

\[
\ldots
\]

If \( \sum a_k \) converges, then (Proposition 3) \( \lim_{n \to \infty} a_n = 0 \). But here \( \lim_{i \to \infty} |a_{N+i}| > \lim_{i \to \infty} \rho^{i-1} |a_{N+1}| = \infty \) since \( \rho > 1 \). Since the terms indexed beyond \( N + 1 \) are going to \( \infty \) in absolute value, they have no chance of going to zero, so the sum does not converge.