Definition: \((s_n)\) is a Cauchy sequence if for every \(\epsilon > 0\) there exists a natural number \(N\) such that if \(m, n > N\) then \(|s_m - s_n| < \epsilon\).

Example: Using the Fibonacci sequence \(F_1 = 1, F_2 = 1\) and \(F_{n+1} = F_n + F_{n-1}\) for \(n \geq 2\), define \(s_n = F_{n+1}/F_n\), so \(s_1 = 1, s_2 = 2, s_3 = 3/2\) etc.

Proposition: In this example, \((s_n)\) is a Cauchy sequence.

Proof: First note that
\[
|s_{n+1} - s_n| = \left| \frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n} \right| = \left| \frac{F_{n+2}F_n - F_{n+1}^2}{F_{n+1}F_n} \right| = \left| \frac{1}{F_{n+1}F_n} \right| \tag{*}
\]
using a Fibonacci identity proved earlier.

Also note that supposing (as we may w.l.o.g) that \(m > n\),
\[
|s_m - s_n| = |(s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \cdots + (s_{n+2} - s_{n+1}) + (s_{n+1} - s_n)|
\]
which we may rewrite as
\[
|s_m - s_n| = |(s_{n+1} - s_n) + (s_{n+2} - s_{n+1}) + \cdots + (s_{m-1} - s_{m-2}) + (s_m - s_{m-1})|.
\]
By the triangle inequality,
\[
|s_m - s_n| \leq |s_{n+1} - s_n| + |s_{n+2} - s_{n+1}| + \cdots + |s_{m-1} - s_{m-2}| + |s_m - s_{m-1}|.
\]
Using \((*)\), this becomes (we can leave out the \(||\) because the terms are positive)
\[
|s_m - s_n| \leq \frac{1}{F_{n+1}F_n} + \frac{1}{F_{n+2}F_{n+1}} + \cdots + \frac{1}{F_{m-1}F_{m-2}} + \frac{1}{F_mF_{m-1}}
\]
which we can write in compressed form as
\[
|s_m - s_n| \leq \sum_{k=n}^{m-1} \frac{1}{F_{k+1}F_k} \leq \sum_{k=n}^{\infty} \frac{1}{F_{k+1}F_k}
\]
since we can add extra positive terms to the right-hand side. [This is now different from the attempt in class]. The terms \(\frac{1}{F_{k+1}F_k}\) go to zero very rapidly, but for our purposes it is enough to remark that \(F_k > k\) as soon as \(k \geq 5\) so
\[
|s_m - s_n| < \sum_{k=n}^{\infty} \frac{1}{k(k+1)} < \sum_{k=n}^{\infty} \frac{1}{k^2} < \int_{n-1}^{\infty} \frac{1}{x^2} \, dx = \frac{1}{n-1}
\]
using the “integral test.” Now we know that \(|s_m - s_n| < \frac{1}{n-1}\) where \(n\) is the smaller of the two indices. So if \(N \geq 1/\epsilon + 1\) then \(m, n > N\) implies
\[
|s_m - s_n| < \frac{1}{n-1} < \frac{1}{N-1} \leq \epsilon,
\]
as required.