Math 125 Solutions to Midterm 2 (Inigo Montoya)

1. For each of the functions $f(x)$ given below, find $f'(x)$.

(a) $f(x) = \frac{1 + 2x^2}{1 + x^5}$

**Solution:** This is a straightforward quotient rule problem:

$$f'(x) = \frac{(4x)(1 + x^5) - (1 + 2x^2)(5x^4)}{(1 + x^5)^2} = \frac{4x - 5x^4 - 6x^6}{(1 + x^5)^2}$$

The simplification is not required.

(b) $f(x) = \sin(3x) \tan(x)$

**Solution:** Apply the product rule, with a chain rule for the $\sin(3x)$ term to get

$$f'(x) = 3 \cos(3x) \tan(x) + \sin(3x) \sec^2(x).$$

(c) $f(x) = \arctan \left( \sqrt{1 + 4x} \right)$

**Solution:** Applying the chain rule, we get

$$\frac{1}{1 + (\sqrt{1 + 4x})^2} \cdot \frac{1}{2} (1 + 4x)^{-1/2} \cdot (4) = \frac{2}{(2 + 4x)\sqrt{1 + 4x}}$$

2. Compute each of the following derivatives as indicated:

(a) $\frac{d}{du} \left[ \frac{u^3}{2} + \frac{2}{u^3} \right]$

**Solution:** Write this as $\frac{1}{2}u^3 + 2u^{-3}$ and apply the power rule to get

$$\frac{3}{2}u^2 - 6u^{-4}.$$

(b) $\frac{d}{dx} \left[ e^x - \pi^4 \right]$

**Solution:** Remember that $\pi^4$ is a constant and so its derivative is zero. Thus, we have $\frac{d}{dx} \left[ e^x - \pi^4 \right] = e^x.$
(c) \( \frac{d}{dw} \left[ \sqrt{1 + \sqrt{1 + w}} \right] \)

**Solution:** View this as \( \frac{d}{dw} \left[ \left(1 + (1 + w)^{1/2}\right)^{1/2} \right] \) and apply the chain rule:

\[
\frac{1}{2} \left(1 + (1 + w)^{1/2}\right)^{-\frac{3}{2}} \cdot \frac{1}{2} (1 + w)^{-\frac{1}{2}} = \frac{1}{4\sqrt{1 + w} \sqrt{1 + \sqrt{1 + w}}}
\]

3. The set of points \((x, y)\) which satisfy the relationship

\[
y^2(y^2 - 4) = x^2(x^2 - 5)
\]

lie on what is known as a “devil’s curve”, shown at right.

Write the equation of the line tangent to the given devil’s curve at the point \((\sqrt{5}, 2)\).

**Solution:**

First, we use implicit differentiation to determine the slope of the tangent line. This will be slightly easier if we rewrite the equation as \(y^4 - 4y^2 = x^4 - 5x^2\) first. Differentiating with respect to \(x\) gives

\[
4y^3 y' - 4 \cdot 2y \cdot y' = 4x^3 - 5 \cdot 2x \quad \text{and so} \quad y' = \frac{x(2x^2 - 5)}{y(2y^2 - 4)}.
\]

At our desired point, \(x = \sqrt{5}\) and \(y = 2\), and so the slope is

\[
y' = \frac{\sqrt{5} \cdot 5}{2 \cdot 4} = \frac{5\sqrt{5}}{8}.
\]

This means the desired line is

\[
y - 2 = \frac{5\sqrt{5}}{8}(x - \sqrt{5}).
\]
4. Let \( f(x) = x \ln(3x) \)

(a) Calculate \( f'(x) \)

**Solution:** Applying the product rule (and the chain rule) gives
\[
f'(x) = \ln(3x) + x \cdot \frac{1}{3x} \cdot 3 = \ln(3x) + 1.
\]

(b) Calculate \( f''(x) \)

**Solution:** Taking the derivative of the above, we get \( f''(x) = \frac{1}{x} \).

(c) For what values of \( x \) is \( f(x) \) increasing?

**Solution:** As we all know, \( f(x) \) is increasing when \( f'(x) > 0 \). Thus, using our answer from part (a) tells us that we need to know when
\[
\ln(3x) + 1 > 0 \quad \text{or, equivalently,} \quad \ln(3x) > -1.
\]
Exponentiating both sides gives \( 3x > e^{-1} \), so we know that
\[
f(x) \text{ is increasing for } x > \frac{1}{3e}.
\]

(d) For what values of \( x \) is \( f(x) \) concave down?

**Solution:** We need to determine when \( f''(x) < 0 \). From part (b), this means
\[
\frac{1}{x} < 0 \quad \text{that is,} \quad x < 0.
\]
However, remember that \( \ln(\ln(4x)) \) is only defined for \( x > 0 \). Thus \( f(x) \) is concave up for all values of \( x \) in its domain. There are no values of \( x \) where \( f(x) \) is concave down.

5. Give the \( x \) and \( y \) coordinates of the (absolute) maximum and minimum values of the function
\[
y = x^4 - 8x^2 - 2 \quad \text{where} \quad -1 \leq x \leq 3.
\]

**Solution:** First, we locate the critical points. Since the function is a polynomial, \( f'(x) \) is defined everywhere, so we only need concern ourselves with the \( x \) for which \( f'(x) = 0 \).
Since \( f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2) \), we have the critical points

\[
x = 0 \quad x = 2 \quad x = -2
\]

However, since we are concerned only with \(-1 \leq x \leq 3\), we discard \( x = -2 \).

Now we evaluate \( f \) at each of the critical points, and the endpoints:

- \( f(0) = -2 \).
- \( f(2) = 16 - 32 - 2 = -18 \).
- \( f(-1) = 1 - 8 - 2 = -9 \).
- \( f(3) = 81 - 72 - 2 = 7 \).

The largest value of the above occurs at \( x = 3, y = 7 \). This is our absolute maximum. The smallest occurs when \( x = 2 \) and \( y = -18 \), which is our absolute minimum.

6. Calvin’s family is visiting a winery in Cutchogue, and he wanders off into the fermenting room and dives into one of the large cylindrical\(^\dagger\) wine vats. The vat has a diameter of 6 feet and is 8 feet tall. The vinter hears the splash and quickly opens the taps to drain the vat, which drains at a rate of 5 cubic feet per minute. How quickly is the height of wine in the tank dropping when the wine is 4 feet deep?

\(^\dagger\)The volume of a cylinder of height \( h \) and radius \( r \) is \( \pi r^2 h \) and its surface area (excluding top and bottom) is \( 2\pi rh \). The density of the wine is about .98 kg/L or 61 pounds per cubic foot. 5 cubic feet is about 38 gallons or 142 liters. The wine is a rather sweet Riesling, but is probably less sweet after Calvin has been in it.

**Solution:** We have the formula for the volume of a cylinder \( V = \pi r^2 h \). In our case, \( r = 3 \) since the diameter is 6, so we have \( V = 9\pi h \). We want to know \( dh/dt \).

Since the vat is draining at a rate of 5 cubic feet per minute, we have \( dV/dt = 5 \).

Using implicit differentiation, we get \( dV/dt = 9\pi (dh/dt) \). So, we see that

\[
\frac{5}{9\pi} = \frac{dh}{dt}
\]
7. For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box. If the graph does not occur, use the letter X.