1. Let $f(x) = x^2 + 3x$ with domain all real numbers. Let $A = (1, f(1))$ and $B = (2, f(2))$. There is also the point $C = (x, f(x))$ with $x$ close to 1.

(a) Calculate the slope of the line through $A$ and $B$.

**Solution.** The line through two points $(x_1, y_1)$ and $(x_2, y_2)$ has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$  

In this case take $(x_1, x_2) = A = (1, f(1)) = (1, 4)$ and $(x_2, y_2) = B = (2, f(2)) = (2, 10)$.

This gives

$$m = \frac{10 - 4}{2 - 1} = 6.$$  

(b) Give an equation for the line through $A$ and $B$.

**Solution.** An equation for the line with slope $m$ which contains a point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1).$$

By part (a) we know that the slope is $m = 6$. Taking $(x_1, y_1) = A = (1, 4)$ gives the equation

$$y - 4 = 6(x - 1)$$

which can be simplified to

$$y - 6x + 2 = 0.$$  

(c) Explain that the slope of the line through $A$ and $C$ is given by

$$\text{slope} = \frac{x^2 + 3x - 4}{x - 1}.$$  

**Solution.** By the same reasoning used in part (a), the slope of the line through $A = (1, 4)$ and $C = (x, f(x))$ is

$$\text{slope} = \frac{f(x) - 4}{x - 1} = x^2 + 3x - 4x - 1.$$
(d) Calculate the slope of the tangent line to the graph of \( f \) at \( A \).

**Solution.** The slope of the tangent line to the graph of \( f \) at \( A \) is the limit as \( C \) approaches \( A \) of the slope of the line through \( A \) and \( C \). As \( C \) approaches \( A \), \( x \) approaches 1. Using the result of (c), we can write the slope of the tangent line to the graph of \( f \) at \( A \) as

\[
\text{slope} = \lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1}.
\]

To calculate this limit we use the factorization

\[
x^2 + 3x - 4 = (x + 4)(x - 1).
\]

Now we can calculate the limit:

\[
\text{slope} = \lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1} = \lim_{x \to 1} \frac{(x + 4)(x - 1)}{x - 1} = \lim_{x \to 1} (x + 4) = 1 + 4 = 5.
\]

2.

(a) Calculate the limit

\[
\lim_{x \to 2} \frac{3x^2 - 15x + 18}{x - 2}.
\]

**Solution.** Observe that we can factor the numerator as

\[
3x^2 - 15x + 18 = 3(x^2 - 5x + 6) = 3(x - 2)(x - 3).
\]

This allows us to calculate the limit:

\[
\lim_{x \to 2} \frac{3x^2 - 15x + 18}{x - 2} = \lim_{x \to 2} \frac{3(x - 2)(x - 3)}{x - 2} = \lim_{x \to 2} 3(x - 3) = 3(2 - 3) = -1.
\]

(b) Calculate the limit

\[
\lim_{x \to 2} \frac{3x^2 - 15x + 19}{x - 2}.
\]
Solution. This limit does not exist (even as an infinite limit). First note that
\[ \lim_{x \to 2^-} \frac{1}{x - 2} = -\infty, \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{x - 2} = +\infty. \]
Since \( \lim_{x \to 2^-} (3x^2 - 15x + 19) = 1 \), the limit laws (which are valid for
infinite limits) tell us that
\[
\lim_{x \to 2^-} \frac{3x^2 - 15x + 19}{x - 2} = \left( \lim_{x \to 2^-} (3x^2 - 15x + 19) \right) \left( \lim_{x \to 2^-} \frac{1}{x - 2} \right) = -\infty.
\]
The analogous calculation shows that
\[
\lim_{x \to 2^+} \frac{3x^2 - 15x + 19}{x - 2} = +\infty.
\]
Since the left limit is not equal to the right limit, we conclude that the
limit does not exist.

3. Explain whether the function
\[ f(x) = \begin{cases} 
\frac{x^2 - 3x}{x^2 - 9} & x \neq 3 \\
21 & x = 3 
\end{cases} \]
is continuous at \( x = 3 \) or not.

Solution. The function is continuous at \( x = 3 \) if and only if \( \lim_{x \to 3} f(x) = f(3) \). But
\[
\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9} = \lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(x + 3)} = \lim_{x \to 3} \frac{x}{x + 3} = \frac{3}{6} = \frac{1}{2}.
\]
Therefore the value of the limit is different from \( f(3) = 21 \), so the function is
not continuous at \( x = 3 \).

4. Given the function
\[ f(x) = \left[ \frac{1}{1-x} + \frac{1}{x-3} \right] + \cos(\pi x), \]
with domain the numbers between 1 and 3, \( 1 < x < 3 \).
(a) Calculate \(f(2)\).

Solution. Since \(\cos(2\pi) = 1\),

\[
f(2) = \left[\frac{1}{2 - 1} + \frac{1}{2 - 3}\right] + \cos(2\pi) = [1 - 1] + 1 = 0 + 1 = 1.
\]

(b) Is there a solution, a number \(x\) between 1 and 3, of \(f(x) = 0\)?

Solution. Yes. First note that

\[
f\left(\frac{5}{2}\right) = \left[\frac{1}{\frac{5}{2} - 1} + \frac{1}{\frac{5}{2} - 3}\right] + \cos\left(\frac{5\pi}{2}\right) = \left[\frac{1}{\frac{3}{2} - \frac{1}{2}}\right] + 0 = \frac{2}{3} - 2 = -\frac{4}{3}.
\]

The function \(f\) is continuous on the closed interval \([2, \frac{5}{2}]\) and satisfies \(f(2) > 0, f\left(\frac{5}{2}\right) < 0\). By the intermediate value theorem there exists a number \(x \in (2, \frac{5}{2})\) with \(f(x) = 0\).

5. Calculate

\[
\lim_{x \to \infty} \frac{3x^2 + 21}{7x^4 + 31x}.
\]

Solution. First write

\[
\frac{3x^2 + 21}{7x^4 + 31x} = \frac{3x^2 + 21}{7x^4 + 31x} \cdot \frac{1/x^4}{1/x^4} = \frac{3/x^2 + 21/x^4}{7 + 31/x^3}.
\]

Using the limit laws and the fact that

\[
\lim_{x \to \infty} \frac{1}{x^n} = 0
\]

for any positive integer \(n\), we get

\[
\lim_{x \to \infty} \frac{3x^2 + 21}{7x^4 + 31x} = \lim_{x \to \infty} \frac{3/x^2 + 21/x^4}{7 + 31/x^3} = \frac{3 \lim_{x \to \infty} (1/x) + 21 \lim_{x \to \infty} (1/x^4)}{7 + 31 \lim_{x \to \infty} (1/x^3)} = \frac{3 \cdot 0 + 21 \cdot 0}{7 + 31 \cdot 0} = 0.
\]

6.
(a) Calculate
\[ \lim_{x \to 0^+} e^{-1/x}. \]

*Solution.* If \( x > 0 \) then \(-1/x < 0\), and
\[ \lim_{x \to 0^+} (-1/x) = -\infty. \]

By the law for limits of compositions,
\[ \lim_{x \to 0^+} e^{-1/x} = \lim_{y \to -\infty} e^y = 0. \]

(b) Calculate
\[ \lim_{x \to 0^-} e^{-1/x}. \]

*Solution.* If \( x < 0 \) then \(-1/x > 0\) and
\[ \lim_{x \to 0^-} (-1/x) = +\infty. \]

By the law for limits of compositions,
\[ \lim_{x \to 0^-} e^{-1/x} = \lim_{y \to +\infty} e^y = +\infty. \]

7. Explain in words
\[ \lim_{x \to \infty} f(x) = L. \]

*Solution.* This means that the values of the function \( f(x) \) can be made arbitrarily close to \( L \) by taking \( x \) sufficiently large.
8. Sketch the graph of an example of a function $f$ which satisfies all of the following conditions.

- $f(0) = 0$
- $f(7) = 11$
- $\lim_{x \to 7^-} f(x) = 3$
- $\lim_{x \to 7^+} f(x) = -3$
- $\lim_{x \to \infty} f(x) = 0$
- $\lim_{x \to 2^+} f(x) = \infty$
- $\lim_{x \to 2^-} f(x) = -\infty$
- $f(1) = 3$
- $f(2) = 3$

Solution: One such graph is shown below. Other choices are possible, some are right, some are wrong.