Math 125 Solutions to Second Midterm, Vers. 3

1. For each of the functions $f(x)$ given below, find $f'(x)$.

(a) 4 points $f(x) = x^5 + 5x^4 + 4x^2 + 9$

Solution:

$$f'(x) = 5x^4 + 20x^3 + 8x$$

(b) 4 points $f(x) = x^8 e^x$

Solution: This requires the product rule. Recall that the derivative of $e^x$ is $e^x$.

$$f'(x) = 8x^7 e^x + x^8 e^x$$

(c) 4 points $f(x) = \frac{3x^2 + 9}{x^3 + 2 \tan x}$

Solution: Using the quotient rule,

$$\frac{6x(x^3 + 2 \tan x) - (3x^2 + 9)(x^2 + 2 \sec^2 x)}{(x^3 + 2 \tan x)^2}$$

There is little point in trying to simplify this.

2. Compute each of the following derivatives as indicated:

(a) 4 points $\frac{d}{d\theta} \cos \left( \frac{\pi}{180} \theta \right)$

Solution: This is just the derivative of the $\cos \theta$, when $\theta$ is in degrees. Using the chain rule, we get

$$-\frac{\pi}{180} \sin \left( \frac{\pi}{180} \theta \right)$$

(b) 4 points $\frac{d}{du} [\sin(3u) \sin(5u)]$

Solution: Use the product rule to get

$$\left( \frac{d}{du} \sin(3u) \right) \sin(5u) + \sin(3u) \left( \frac{d}{du} \sin(5u) \right)$$

and then use the chain rule to get the answer, which is

$$3 \cos(3u) \sin(5u) + 5 \sin(3u) \cos(5u).$$
(c) **4 points** \[ \frac{d}{dt} \left[ \frac{t - 5}{t} \right] \]

**Solution:** If you rewrite this as \( \frac{1}{5}t - 5t^{-1} \), it is clear the derivative is \( \frac{1}{5} + 5t^{-2} \)

3. **8 points** Write a limit that represents the slope of the graph

\[
y = \begin{cases} 
2 + x \ln |x| & x \neq 0 \\
2 & x = 0 
\end{cases}
\]

at \( x = 0 \). You **do not need to evaluate the limit**.

**Solution:** To do this, we need to remember the definition of the derivative, which is \( f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \). In the current case, \( a = 0 \), and notice that \( f(0) = 2 \), so we have

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{(2 + h \ln |h|) - 2}{h}
\]

This simplifies to

\[
\lim_{h \to 0} \frac{h \ln |h|}{h} = \lim_{h \to 0} \ln |h| = -\infty,
\]

although it wasn’t required for you to do this.

4. At right is the graph of **the derivative** \( f' \) of a function.

(a) **4 points** List all values of \( x \) with \(-3 \leq x \leq 4\) where \( f(x) \) has a local maximum.

**Solution:** A local maximum for \( f(x) \) will occur where \( f'(x) \) changes from positive to negative. This happens at \( x = 0 \).

(b) **4 points** At \( x = -1 \), is \( f(x) \) concave up, concave down, or neither?

**Solution:** We know that a function is concave up when its second derivative is positive, and concave down when \( f'' \) is negative. The graph shows \( f'(x) \), which is decreasing near \( x = -1 \). That means the derivative of \( f'(x) \) is negative near \( x = -1 \), so \( f''(-1) < 0 \). Hence \( f(x) \) is concave down at \( x = -1 \).
5. **16 points** For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

- **H**: 
  - [Graph]
  - A: 
    - [Graph]
  - B: 
    - [Graph]

- **F**: 
  - [Graph]
  - C: 
    - [Graph]
  - D: 
    - [Graph]

- **E**: 
  - [Graph]
  - E: 
    - [Graph]
  - F: 
    - [Graph]

- **D**: 
  - [Graph]
  - G: 
    - [Graph]
  - H: 
    - [Graph]
6. Let \( f(x) = xe^{-6x} \).

(a) 3 points Calculate \( f'(x) \)

**Solution:** We use the product rule and the chain rule:

\[
f'(x) = e^{-6x} - 6xe^{-6x}
\]

(b) 3 points Calculate \( f''(x) \)?

**Solution:** Taking the derivative of the above gives

\[
f''(x) = -6e^{-6x} - 6e^{-6x} + 36xe^{-6x}
\]

which simplifies to

\[
36xe^{-6x} - 12e^{-6x}
\]

(c) 4 points For what values of \( x \) is \( f(x) \) increasing?

**Solution:** To answer this, we need to know when \( f'(x) > 0 \), that is, where

\[
e^{-6x} - 6xe^{-6x} > 0
\]

Factoring out the exponential term gives \( e^{-6x}(1 - 6x) > 0 \), and since \( e^{-6x} \) is always positive, we only need ask where \( 1 - 6 > 0 \). This happens for

\[
x < \frac{1}{6}.
\]

(d) 4 points For what values of \( x \) is \( f(x) \) concave down?

**Solution:** We need to know when \( f''(x) < 0 \), so factor \( f''(x) \) as

\[
12e^{-6x}(3x - 1).
\]

As before, we can ignore the exponential term, since it is always positive, and we see that \( f''(x) < 0 \) when \( x < 1/3 \).
7. **10 points** Write the equation of the line tangent to the curve
\[ y = 3x^4 - x + \sqrt{x} \quad \text{at} \quad x = 1 \]

**Solution:** To write the equation of a line, we need a point and a slope. Since the line is tangent to the curve at \( x = 1 \), it contains the point \( (1, f(1)) = (1, 3) \).

To get the slope, we calculate \( f'(1) \). Taking the derivative gives
\[ f'(x) = 12x^3 - 1 + \frac{1}{2}x^{-1/2}, \]
so \( f'(1) = 12 - 1 + \frac{1}{2} = \frac{23}{2} \). Hence the line is
\[ y - 3 = \frac{23}{2}(x - 1), \quad \text{or, equivalently,} \quad y = \frac{23}{2}x - \frac{17}{2} \]

8. **10 points** A ladder 12 feet long rests against a vertical wall. Let \( \theta \) be the angle between the top of the ladder and the wall, and let \( \ell \) be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does \( \ell \) change with respect to \( \theta \) when \( \theta = \frac{\pi}{6} \)?

**Solution:** Since the ladder forms a right triangle with the wall, we have \( \ell = 12 \sin \theta \). The rate of change of \( \ell \) with respect to \( \theta \) is \( \frac{d\ell}{d\theta} \), which is \( 12 \cos \theta \). We want its value when \( \theta = \frac{\pi}{6} \), so that is
\[ 12 \cos \left( \frac{\pi}{6} \right) = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3} \]