INSTRUCTIONS – PLEASE READ

Please turn off your cell phone and put it away.

Please write your name and your section number right now.

This is a closed book exam. You are NOT allowed to use a calculator or any other electronic device or aid.

The midterm has 6 problems worth a total of 100 points. Look over your test packet as soon as the exam begins. If you find any missing pages or problems please ask a proctor for another test booklet.

Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.

Academic integrity is expected of all Stony Brook University students at all times, whether in the presence or absence of members of the faculty.

<table>
<thead>
<tr>
<th>LEC 01</th>
<th>MWF 10:00-10:53am</th>
<th>Joseph Adams</th>
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<tbody>
<tr>
<td>R01</td>
<td>F 1:00-1:53pm</td>
<td>Jaroslaw Jaracz</td>
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<td>R02</td>
<td>Tu 4:00-4:53pm</td>
<td>Charles Cifarelli</td>
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<td>R03</td>
<td>Tu 1:00-1:53pm</td>
<td>Jaroslaw Jaracz</td>
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<td>R04</td>
<td>Th 8:30-9:23am</td>
<td>Alaa Abd-El-Hafez</td>
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<td>R05</td>
<td>M 1:00-1:53pm</td>
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<td>R06</td>
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<td>Zhuang Tao</td>
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<td>R07</td>
<td>W 11:00-11:53am</td>
<td>Dyi-Shing On</td>
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<td>LEC 02</td>
<td>TuTh 2:30-3:50pm</td>
<td>Raluca Tanase*</td>
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<td>R08</td>
<td>Tu 4:00-4:53pm</td>
<td>Gaurish Telang</td>
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<td>R09</td>
<td>Tu 1:00-1:53pm</td>
<td>Yuan Gao</td>
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<td>R10</td>
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<td>Alaa Abd-El-Hafez</td>
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<td>Ruijie Yang</td>
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<td>R12</td>
<td>W 12:00-12:53pm</td>
<td>Christopher Ianzano</td>
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<td>R13</td>
<td>M 10:00-10:53am</td>
<td>Zhuang Tao</td>
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<td>R14</td>
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<td>LEC 03</td>
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<td>David Kahn</td>
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<td>Mariangela Ferraro</td>
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<td>R32</td>
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<td>R33</td>
<td>Tu 1:00-1:53pm</td>
<td>Yu Zeng</td>
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Problem 1. (17 points)

a) Find the function $f(y)$ such that $f'(y) = e^y + \cos(y)$ and $f(0) = 10.$

$$f(y) = \int f'(y) \, dy = \int (e^y + \cos(y)) \, dy = e^y + \sin(y) + c$$

$$f(0) = 10 \Rightarrow e^0 + \sin(0) + c = 10 \Rightarrow c = 9$$

$$f(y) = e^y + \sin(y) + 9$$

b) Let $h(y) = \int_{2}^{y} \frac{1}{\sqrt{t^3 + 1}} \, dt.$ Compute $h(2)$ and $h'(2)$.

$$h(2) = \int_{2}^{2} \frac{1}{\sqrt{t^3 + 1}} \, dt = 0$$

$$h'(y) = \frac{d}{dy} \left( \int_{2}^{y} \frac{1}{\sqrt{t^3 + 1}} \, dt \right) = \frac{1}{\sqrt{y^3 + 1}}$$

$$h'(2) = \frac{1}{\sqrt{2^3 + 1}} = 3$$

c) Suppose that $g$ is an integrable functions on $[0, 3]$, such that $\int_{1}^{3} g(x) \, dx = 6$, $\int_{1}^{0} g(x) \, dx = -5$ and $\int_{2}^{3} g(x) \, dx = 1$. Compute $\int_{0}^{3} (-2g(x) + 5) \, dx$.

$$\int_{0}^{2} g(x) \, dx = \int_{0}^{1} g(x) \, dx + \int_{1}^{2} g(x) \, dx = -\int_{1}^{3} g(x) \, dx + \int_{1}^{2} g(x) \, dx - \int_{2}^{3} g(x) \, dx$$

$$= -(-5) + 6 - 1 = 10$$

$$\int_{0}^{2} (-2g(x) + 5) \, dx = -2 \int_{0}^{2} g(x) \, dx + \int_{0}^{2} 5 \, dx = -20 + 10 = -10$$
Problem 2. (20 points) Consider the function \( f(x) = x^2 + 1 \), defined on the interval \([-1, 2]\).

![Graph of \( f(x) = x^2 + 1 \)]

a) Approximate the area between the graph of \( f \) and the \( x \)-axis using 3 right hand rectangles with equal widths.

\[
\Delta x = \frac{2 - (-1)}{3} = \frac{3}{3} = 1
\]

\[
A \approx \left( f(0) + f(1) + f(2) \right) \Delta x = (1 + 2 + 5) \cdot 1 = 8
\]

b) Write a formula for a Riemann Sum with \( n \) right hand rectangles.

\[
R_n = \sum_{k=1}^{n} f(-1 + k \Delta x) \Delta x \quad \text{where} \quad \Delta x = \frac{3}{n}
\]

\[
R_n = \sum_{k=1}^{n} \left( (-1 + 3 \frac{k}{n})^2 + 1 \right) \cdot \frac{3}{n}
\]

c) Evaluate the limit of the Riemann Sum from part (b) as \( n \to \infty \), either using integrals, or by direct computation, using the formula \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \).

\[
\lim_{n \to \infty} R_n = \int_{-1}^{2} f(x) \, dx = \int_{-1}^{2} (x^2 + 1) \, dx = \left[ \frac{x^3}{3} + x \right]_{-1}^{2}
\]

\[
= \left( \frac{8}{3} + 2 \right) - \left( \frac{-1}{3} - 1 \right) = \frac{9}{3} + 3 = 6
\]
\[ R_n = \sum_{k=1}^{n} \frac{\kappa}{\eta} + \sum_{k=1}^{n} \frac{2\pi k^2}{m^3} - \sum_{k=1}^{n} \frac{18k}{n^2} \]

\[ R_n = 6 \cdot \frac{\pi}{k \kappa} + \frac{2\pi}{m^3} \sum_{k=1}^{n} k^2 - \frac{18}{n^2} \sum_{k=1}^{n} k \]

\[ = 6 + \frac{2\pi}{m^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2} \cdot \frac{n(n+1)}{2} \]

\[ = 6 + \frac{9(n+1)(2n+1)}{2n^2} - \frac{9n+1}{n} \]

\[ = 6 + 9 \left( \frac{2n^2 + 3n + 1}{2n^2} \right) - 9 \left( 1 + \frac{1}{n} \right) \]

\[ = 6 + 9 \left( 1 + \frac{3}{2n} + \frac{1}{2n^2} \right) - 9 \left( 1 + \frac{1}{n} \right) = 6 + 9 \left( \frac{1}{2n} + \frac{1}{2n^2} \right) \]

\[ \lim_{n \to \infty} R_n = \lim_{n \to \infty} 6 + 9 \left( \frac{1}{2n} + \frac{1}{2n^2} \right) = 6. \]
Problem 3. (21 points) Evaluate the following expressions:

a) \( \int_2^5 \left( 6x^2 + 4x - 1 \right) dx \) 
\[
= \left. \frac{6x^3}{3} + \frac{4x^2}{2} - x \right|_2^5 \\
= \left( 250 + 50 - 5 \right) - \left( 36 + 8 - 2 \right) \\
= 295 - 22 = 273
\]

b) \( \int \frac{\sin^2(x)}{\sec(x) - \sec(x) \cos^2(x)} dx \) 
\[
= \int \frac{\sin^2(x)}{\sec(x) \left( 1 - \cos^2(x) \right)} dx \\
= \int \frac{\sin^2(x)}{\sec(x) \cdot \sin^2(x)} dx \\
= \int \frac{1}{\sec(x)} dx = \int \cos x dx \\
= \sin x + C
\]

c) \( \frac{d}{dx} \left( \int_2^5 \tan(\ln(t^2)) dt \right) \) 
\[
= \tan(\ln(x^5)) \cdot (x^5)' - \tan(\ln(2x^2)) \cdot (2x)' \\
= \tan(\ln(x^{10})) \cdot 5x^4 - \tan(\ln(4x^2)) \cdot 2
\]
Problem 4. (14 points) Calculate the following integrals, using the appropriate substitution:

\( \int \frac{1}{x(\ln x)^2} \, dx = \int \frac{1}{u^2} \, du = \int u^{-2} \, du = \int \frac{d}{u^2} \frac{du}{u} = \int \frac{du}{u^2} + c = -\frac{1}{u} + c \),

\( u = \ln x \)

\( du = \frac{1}{x} \, dx \)

\( = -\frac{1}{\ln x} + c \)

\( \int_0^{\frac{\pi}{3}} x^2 \sin(x^3) \, dx = \int_0^{\frac{\pi}{3}} x^2 \sin(u) \, du = -\frac{\cos(u)}{3} \bigg|_0^{\frac{\pi}{3}} \)

\( u = x^3 \)

\( du = 3x^2 \, dx \)

\( x = 0 \Rightarrow u = 0 \)

\( x = \frac{\sqrt{\pi}}{3} \Rightarrow u = \frac{\pi}{9} \)

\( = \left( -\frac{\cos(\frac{\pi}{9})} {3} - (-\cos 0) \right) \)

\( = \frac{-(-1) - (-1)}{3} = \frac{2}{3} \)
Problem 5. (20 points) Consider the function $f(x)$ graphed below:

Now define a new function $F(x) = \int_{-3}^{x} f(t) \, dt$ on the interval $[-3, 4]$.

a) Compute $F(-3)$, $F(1)$ and $F(4)$.

\[
F(-3) = \int_{-3}^{-3} f(t) \, dt = 0 \\
F(1) = \int_{-3}^{1} f(t) \, dt = 2 \\
F(4) = \int_{-3}^{4} f(t) \, dt = -\frac{5}{2}
\]

b) Where is $F$ increasing? Where is $F$ decreasing?

$F'(x) = \frac{d}{dx} \left( \int_{-3}^{x} f(t) \, dt \right) = f(x)$, by FTC

$F$ is increasing when $f$ is positive, that is on $[-2, 1]$.

$F$ is decreasing when $f(x) < 0$, that is on $[-3, -2] \cup [1, 4]$.

c) Where is $F$ concave-up? Where is $F$ concave-down?

$F''(x) = f'(x)$

$F$ is concave-up when $F''(x) > 0$, that is when $f'(x) > 0$, or equivalently when $f$ is increasing (its slope is positive).

So $F$ is concave-up on $[-3, -1]$ and concave-down on $[-1, 4]$. 


Problem 6. (8 points) Determine whether the following statements are true or false. Circle your response and give a brief explanation (a reason why it’s true or an example where it fails).

a) **True**

Suppose that \( f \) and \( g \) are two integrable functions on \([0,1]\). Then

\[
\int_0^1 f(x)g(x)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx.
\]

Take for example \( f(x) = x \) and \( g(x) = x \)

\[
\int_0^1 f(x)g(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}
\]

\[
\int_0^1 f(x)\,dx \int_0^1 g(x)\,dx = \left(\int_0^1 x\,dx\right)^2 = \left(\frac{x^2}{2}\bigg|_0^1\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}
\]

\[
\frac{1}{3} \neq \frac{1}{4}
\]

b) **False**

Let \( f \) be an integrable function on \([a,b]\). The definite integral \( \int_a^b f(x)\,dx \) represents the area of the region enclosed between the graph of the function and the \( x \)-axis.

Assume \( f \) is negative. Take for example \( f(x) = x \) on \([-1,0]\)

\[
\int_{-1}^0 f(x)\,dx = \int_{-1}^0 x\,dx = \frac{x^2}{2}\bigg|_{-1}^0 = -\frac{1}{2}
\]

However, the area of the region between the graph of \( f \) and the \( x \)-axis is positive

\[
A = \frac{1 \cdot 1}{2} = \frac{1}{2}
\]

So in this case

\[
\int_{-1}^0 f(x)\,dx = -A
\]