MAT 141 FALL 2002
MIDTERM I

!!! WRITE YOUR NAME, SSN AND SECTION BELOW !!!

NAME :

SSN :

SECTION :

THERE ARE 5 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE.
SHOW YOUR WORK!!!

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1
1. [30 points] Let $f(x) = < x >$ be the decimal part function.

For example: if $x = 2.65$, then $< x > = .65$; if $x = 3.567$, then $< x > = .567$.

Determine the following limit.

\[
\lim_{x \to \infty} \frac{< x >}{x}.
\]

Solution:

$0 \leq < x > < 1$

$0 \leq \frac{< x >}{x} < \frac{1}{x}$

By the Sandwich Theorem, $\lim_{x \to \infty} 0 \leq \lim_{x \to \infty} \frac{< x >}{x} \leq \lim_{x \to \infty} \frac{1}{x}$

$0 \leq \lim_{x \to \infty} \frac{< x >}{x} \leq 0$

Answer: $\lim_{x \to \infty} \frac{< x >}{x} = 0$
2. [50 points] Let

\[ f(x) = x^3 \left( \frac{1}{x-1} - \frac{1}{x+1} \right) \]

Find the asymptotes of \( f(x) \) and draw them on the \( xy \)-plane together with a qualitative sketch of the graph of the curve near the asymptotes.

Solution)

Vertical Asymptotes: \( x - 1 = 0 \) and \( x + 1 = 0 \).

\[ f(x) = \frac{2x^3}{x^2-1} = \frac{2x^3 - 2x^2 + 2x}{x^2 - 1} = 2x + \frac{2x}{x^2 - 1} \]

Oblique Asymptotes: \( y = 2x \)

Horizontal Asymptotes: None, since

\[ \lim_{x \to \infty} \frac{x^3}{x^2 - 1} = \infty \]

and

\[ \lim_{x \to -\infty} \frac{x^3}{x^2 - 1} = -\infty \]
3. [40 points] Let $f(x)$ be a function defined for every value of $x$.
   Assume that
   a) $f(x)$ is differentiable for every value of $x$;
   b) for $x \neq 0$, $f(x) = \frac{\sin(x^3 + x^2)}{\sqrt{x^3 + x^2}}$.

   Find $f(0)$. Justify your answer.

   Solution)
   Since $f(x)$ is differentiable, it is continuous. So $\lim_{x \to 0} f(x) = f(0)$ holds.
   Letting $t = x^3 + x^2$,
   \[
   \lim_{x \to 0} \frac{\sin(x^3 + x^2)}{\sqrt{x^3 + x^2}} = \lim_{t \to 0^+} \frac{\sin t}{\sqrt{t}} = \lim_{t \to 0^+} \sqrt{t} \frac{\sin t}{t} = 0 \cdot 1 = 0
   \]
   Hence, $f(0) = 0$. 
Let \( u \) and \( v \) be differentiable functions.
Assume that
\[
\begin{align*}
  u(1) &= 1, & u'(1) &= -1, & v(1) &= 2, & v'(1) &= -3.
\end{align*}
\]
Compute:

a) \((\frac{u}{v})'(1)\)

\[
\left(\frac{u}{v}\right)'(1) = \frac{u'v - uv'}{v^2}(1) = \frac{u'(1)v(1) - u(1)v'(1)}{v(1)^2}
= \frac{-1 \cdot 2 - (-1) \cdot (-3)}{2^2} = \frac{-2 + 3}{4} = .25
\]

b) \((uv^2)'(1)\)

\[
(uv^2)'(1) = \{u' \cdot v^2 + u \cdot (v^2)\}'(1) = \{u' \cdot v^2 + u \cdot (v'v + vv')\}(1)
= u'(1)v(1)^2 + u(1) \cdot (v'(1)v(1) + v(1)v'(1)) = -1 \cdot 2^2 + 1 \cdot (-3 \cdot 2 + 2 \cdot (-3)) = -4 - 12 = -16
\]
5. [40 points] A body moves along the s-axis with velocity
\[ v = t^2 - 4t + 3. \]

a) Find the body’s acceleration each time the velocity is zero.

\[ v = 0 \text{ when } t^2 - 4t + 3 = 0. \]
\[ (t - 3)(t - 1) = 0, \text{ so } t = 1 \text{ or } t = 3. \]
\[ a = v' = 2t - 4 \]
So when \( t = 1 \) acceleration is \( a = -2 \) and when \( t = 3, \ a = 2. \)

b) When is the body moving forward?

Body moves forward when \( v > 0. \)
\[ (t - 1)(t - 3) > 0. \text{ So } t < 1 \text{ or } t > 3. \]

c) When is the body’s velocity increasing?

\[ v' > 0. \text{ So } 2t - 4 > 0. \]
\[ t > 2. \text{ After } t=2 \text{ velocity increases.} \]