Let 
\[ y = f(x) = Ax^2 + Bx + C \]
be the equation of a parabola.
Let \( a \) be a number and \( h \) be a positive number.
Assume 
\[ f(a - h) = y_0, \quad f(a) = y_1, \quad f(a + h) = y_2. \]
Find an expression for 
\[ I := \int_{a-h}^{a+h} f(x) \, dx \]
involving only \( y_0, y_1 \) and \( y_2 \) and \( h \).
(Hint: it makes no difference if you assume \( a = 0 \) (why?))

**Solution.** Let us assume that \( a = 0 \). The answer will not depend on this assumption, since since we can shift the graph to the left (if \( a > 0 \)) or to the right (if \( a < 0 \)) without changing the values of \( h, y_0, y_1 \) and \( y_2 \). This will change \( A, B \) and \( C \), but not the answer. So we can also use the same letters \( A, B, \) and \( C \). We have 
\[ I := \int_{-h}^{h} (Ax^2 + Bx + C) \, dx = A/3x^3 + B/2x^2 + Cx \bigg|_{-h}^{h} = 2/3Ah^3 + 2Ch. \]
Since \( f(0) = y_1 \), by plugging in \( f(x) \), we get \( C = y_1 \) and \( I = 2/3Ah^3 + 2y_1h \).
We have 
\[ f(-h) = Ah^2 - Bh + y_1 = y_0 \quad f(h) = Ah^2 + Bh + y_1 = y_2 \]
and adding the second to the first we have 
\[ 2Ah^2 = y_0 + y_2 - 2y_1. \]
So 
\[ I = 1/3(2Ah^2)h + 2y_1h = 1/3(y_0 + 4y_1 + y_2)h. \]