Remarks on Piecewise Monotone Maps

John Milnor

Stony Brook University

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PM-maps $f : (\mathcal{I}, \partial \mathcal{I}) \to (\mathcal{I}, \partial \mathcal{I})$ where $\mathcal{I} = [0, 1]$.

Maximal intervals of monotonicity: $\mathcal{I}_j = [c_{j-1}, c_j]$ where $0 = c_0 < c_1 < \cdots < c_{d-1} < c_d = 1$.

The vector $v = (v_0, v_1, \ldots, v_d) \in \mathcal{I}^{d+1}$ where $v_j = f(c_j)$ will be called the critical value vector.

Caution: In this talk the word “critical” will be used to mean local maximum or minimum point. Inflection points are not “critical”.
The Polynomial Case.

**Theorem.** Given a PM-map $f(x)$ with critical value vector $(v_0, v_1, \ldots, v_d)$, there is one and only one polynomial PM-map $g(x)$ of degree $d$ with the same critical value vector.

Proofs by deMelo and vanStrien, 1993; by Milnor and Tresser (and also by Douady and Sentenac in appendix), 2000.

**PROBLEMS:**
1. The proofs are non-constructive.
2. I don’t know any efficient algorithm for finding $g$ if $d \geq 5$.
Construction of polynomial map from critical values.
Construction of polynomial map from critical values.
Construction of polynomial map from critical values.
Construction of polynomial map from critical values.
An Easy Lemma

Suppose that we are given two different PM-maps $f$ and $g$ with the same critical value vector.

**LEMMA.** There is one and only one “connecting homeomorphism”

$h = h_{f,g}$ from $(\mathcal{I}, \partial \mathcal{I})$ to itself which maps each interval of monotonicity $\mathcal{I}_j(f)$ to the corresponding interval $\mathcal{I}_j(g)$ and which satisfies $g \circ h = f$.

Graph of $f$,  

$\mathcal{I}_j(f)$  

$\mathcal{I}_j(g)$  

Graph of $g$,  

$h(x)$
Suppose that we start with any PM-map $f_0$. 

\[ \mathcal{I} \xrightarrow{f_0} \mathcal{I}. \]
Suppose that we start with any PM-map $f_0$. Then there is a unique polynomial map $g_0$ of minimal degree with the same critical value vector.
Suppose that we start with any PM-map $f_0$.

Then there is a unique polynomial map $g_0$ of minimal degree with the same critical value vector.

By the Lemma, there is a connecting homeomorphism

$$h_0 = h_{f_0,g_0} \quad \text{with} \quad g_0 \circ h_0 = f_0.$$
The Thurston Tower Construction

Suppose that we start with any PM-map \( f_0 \).

Then there is a unique polynomial map \( g_0 \) of minimal degree with the same critical value vector.

By the Lemma, there is a connecting homeomorphism

\[ h_0 = h_{f_0,g_0} \quad \text{with} \quad g_0 \circ h_0 = f_0. \]

Now define \( f_1 \) to be \( h_0 \circ g_0 \).
Continue Inductively.
Continue Inductively.
Continue Inductively.
Continue Inductively.
Iteration.

This construction defines a continuous correspondence

\[ f_0 \mapsto f_1 \]

such that \( f_1 \) is topologically conjugate to \( f_0 \).

Iterating, we obtain an infinite sequence of topologically conjugate maps

\[ f_0 \mapsto f_1 \mapsto f_2 \mapsto \cdots. \]

Problem: For which \( f_0 \) does this sequence converge uniformly to a polynomial map?

According to Thurston:

If \( f_0 \) is critically finite, with at least 4 postcritical points, and with no Thurston obstruction, then the auxiliary sequence \{\( g_n \)\} converges uniformly.

But what can one say about the \( f_n \) and the \( h_n \)?
An Example with $d = 3$. 
An Example with $d = 3$. 
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An Example with $d = 3$. 

(movie 1)
Here $f_0$ has critical orbit:

$$(2) \leftrightarrow (5) \leftrightarrow (1) \leftrightarrow (4) \leftrightarrow (3).$$
Critically preperiodic example (continued):
Critically preperiodic example (continued):
Critically preperiodic example (continued):
Critically preperiodic example (continued):
Jumping from $f_0$ to $f_n$.

We can skip the intermediate steps and look at the topological conjugacy

\[ I \xrightarrow{f_n} I \]

\[ I \xrightarrow{f_0} I \]

where

\[ H_n = h_{n-1} \circ h_{n-2} \circ \cdots \circ h_1 \circ h_0. \]

Thus $f_n = H_n \circ f_0 \circ H_n^{-1}$. 

(movie 3)
A Critically Periodic Example

Critical orbit: \( (2) \leftrightarrow (3) \leftrightarrow (1) \leftrightarrow (2) \). (movie 4)
Empirical conclusions.

“Good Convergence”: For “many” choices of $f_0$ the sequences $\{f_n\}$ and $\{g_n\}$ seem to converge uniformly to a common polynomial limit $f_\infty$, and the sequence $\{h_n\}$ seems to converge to the identity.

But this limit map $f_\infty$ may not be topologically conjugate to $f_0$.

And the sequence of compositions $H_n = h_{n-1} \circ \cdots \circ h_1 \circ h_0$ may not converge to a homeomorphism.

The graph of $H_n$ does seem to converge to a limit in the Hausdorff topology.

The set of all limits of graphs of homeomorphisms forms a compact metric space.

Every such limit is a geodesic in the Manhattan metric.

$|dx| + |dy|$.
Numerical Problem:

We have

\[ f_n = H_n \circ f_0 \circ H_n^{-1}, \]

where the maps \( H_n \) and \( H_n^{-1} \) may have points with derivative tending to infinity as \( n \to \infty \).

Therefore computation of \( f_n \) could become very unstable as \( n \to \infty \).

This seems to be particularly a problem for maps with topological entropy zero.
A More General Construction.

Given \( d \geq 2 \) and \( v_0 \in \{0, 1\} \):

Let \( \mathcal{F} = \mathcal{F}(d, v_0) \) be the metric space consisting of all PM-maps \( f \) with \( d \) intervals of monotonicity and with \( f(0) = v_0 \), where

\[
\text{dist}(f, g) = \max_x (|f(x) - g(x)|).
\]

Definition. A subset \( \mathcal{G} \subset \mathcal{F} \) is parametrized by critical values if, for any \( f \in \mathcal{F} \) there is one and only one \( g = g_f \in \mathcal{G} \) with the same critical value vector \( \mathbf{v} \).

For each such \( \mathcal{G} \) there is an associated Thurston tower construction

\[
\Theta_{\mathcal{G}} : f \mapsto h_{f,g} \circ g \quad \text{where} \quad g = g_f
\]

which maps each \( f \in \mathcal{F} \) to a topologically conjugate map \( \Theta_{\mathcal{G}}(f) \in \mathcal{F} \).
Examples of sets $\mathcal{G}$ parametrized by critical values.

1. Polynomials.

   The space $\mathcal{G}_{\text{poly}}$ of polynomial maps of $(I, \partial I)$ with all critical points real and distinct, and in the interior of $I$.

2. A trivial example. Take evenly spaced critical points $c_j = j/d$, and suppose that $g$ is linear on each $I_j = [c_{j-1}, c_j]$.

3. Constant Slope. By definition, a map $f$ of the interval has constant slope $s \geq 0$ if $f$ is piecewise linear with derivative satisfying $|f'(x)| = s$ almost everywhere.

   **Lemma.** The set $\mathcal{G}_{\text{CS}} \subset \mathcal{F}$ consisting of all PM-maps with constant slope is parametrized by critical values.

   **Proof Outline:** Suppose that $g_f \in \mathcal{G}_{\text{CS}}$ has the same critical value vector as $f$. Then the slope $s$ of $g_f$ must be equal to the total variation of $f$ (or of $g_f$):

   $$s = \sum_{j=1}^{d} |v_j - v_{j-1}| > 0.$$ 

   Now compute the critical points of $g_f$ inductively $\ldots$. □
Topological Entropy

Theorem of Misiurewicz and Slenk:

If \( g : \mathcal{I} \rightarrow \mathcal{I} \) has constant slope \( s \geq 0 \), then its topological entropy is given by

\[
    h_{\text{top}}(g) = \log^+(s) \geq 0.
\]

Thus if iteration of \( \Theta_{\mathcal{G}_{cs}} \) converges to a map of constant slope, then we can easily compute the topological entropy of the limit map \( f_\infty \).

**Question.** For which \( f_0 \in \mathcal{F} \), does the sequence

\[
    \Theta_{\mathcal{G}_{cs}} : f_0 \mapsto f_1 \mapsto f_2 \mapsto \cdots
\]

converge uniformly to a map of constant slope, **with the same entropy**?
A Degree Four Example

In this example, \( f_n \) converges to the standard tent map, and \( s \) converges to 2. Therefore

\[
\mathbf{h}_{\text{top}}(f_0) = \log(2)
\]

**Conjecture.** For any (reasonable?) \( f_0 \), the associated sequence of constant slope maps \( g_n \) converges, and yields the correct topological entropy \( \mathbf{h}_{\text{top}}(f_0) = \log^+(s(g_\infty)) \).

(However, the sequence of topologically conjugate maps \( f_n \) does not always converge to a constant slope map; and the sequence of \( h_n \) does not always converge to the identity map.)
Example: \( f_0(x) = 3.8 \, x(1 - x) \).
Anomalous Convergence: \( f_0(x) = 2.8 \, x(1 - x) \).

There seems to be uniform convergence:

\[
f_n \to f_\infty, \quad g_n \to g_\infty, \quad h_n \to h_\infty \quad \text{as} \quad n \to \infty; \]

but \( f_\infty \neq g_\infty \) , and the homeomorphism \( h_\infty \) is not the identity map.
A Conditional Result.

**Theorem.** If the sequence \( \{f_n\} \) converges uniformly, then the sequences \( \{g_n\} \) and \( \{h_n\} \) also converge uniformly; and the limit maps \( f_\infty, g_\infty, \) and \( h_\infty \) commute with each other. Furthermore

\[
h_{\text{top}}(f_\infty) = h_{\text{top}}(g_\infty) = \log^+ (s(g_\infty)) ;
\]

and if \( h_{\text{top}} > 0 \) we have “good convergence”:

\[ f_\infty = g_\infty, \quad \text{and} \quad h_\infty \text{ is the identity map.} \]

**Lemma.** If \( g = g_\infty \) has constant slope \( s > 1 \), then no non-trivial orientation preserving homeomorphism \( h = h_\infty \) can commute with \( g \).

**Proof:**

**Step 1.** Precritical points of \( g \) are everywhere dense,

**Step 2.** Any precritical point of \( g \) must be fixed by \( h \).

But does \( h_{\text{top}}(f_\infty) = \lim_{n \to \infty} h_{\text{top}}(f_n) \) ?
Appendix: The Balmforth-Spiegel-Tresser Algorithm
(Phys. Rev. Let. 72, 1994; or arXiv, 1993)

Given a PM-map \( f \) with critical points \( c_j \), let \( P_m \subset I \) be the finite set consisting of all \( f^0h(c_j) \) with \( 0 \leq h < m \).
This subdivides \( I \) into finitely many intervals \( J_1, \ldots, J_N \).

Construct an \( N \times N \) matrix \( M = [a_{ik}] \) with

\[
a_{ik} = \begin{cases} 
1 & \text{if } f(J_i) \supset J_k, \\
0 & \text{if } f(J_i) \text{ is disjoint from the interior of } J_k, \\
.5 & \text{if } f(J_i) \text{ covers part of } J_k.
\end{cases}
\]

If we replace each \(.5\) by a zero, we get a matrix \( M_0 \) whose leading eigenvalue is a lower bound for \( s = \exp(h_{\text{top}}(f)) \).
Similarly, if we replace each \(.5\) by a one, we get a matrix \( M_1 \) whose leading eigenvalue is an upper bound for \( s \).

**Theorem (BST).** As \( m \to \infty \), these upper and lower bounds both converge to \( \exp(h_{\text{top}}(f)) \).
It began with a classic Mechoui

And with Misha and Carsten and Cui

But time’s running out

So let’s get up and shout

Three cheers for John Hamal Hubbard,
and for Dynamical Holomorphie!