Math 539 Possible topics for Presentations

February 26, 2004

Here are some suggestions. I would like you to have formed into four groups of about
three each and settled on topics by the end of next week (March 5) at the very latest. Each
person in the group should talk for about 20–25 mins, so you will need to divide up the
topic. For these presentations to be successful they will have to be very well planned and
focussed. You should pick one or two main results to concentrate on. You won’t be able
to go into all details, but you should try to explain all concepts and definitions and state
theorems clearly, then do one or two proofs or do some examples.

I will be around through March 10 to help you plan. Tony Phillips and Jack Milnor have
agreed to act as consultants and one of them will come to the lectures.

1. **Homological algebra** Explain the basic concepts: chain complex and its homology,
maps between chain complexes, what is the equivalent of a homotopy of complexes? the
five lemma; short exact sequence of chain complexes gives rise to long exact sequence in
homology (and cohomology.) (Thm 7.4.10 in Aguilar, Gitler and Prieto). For reference look
at almost any book on algebraic topology except AGP.

2. **Framed cobordism and the homotopy groups of spheres** You could try to explain
Pontriagin’s proof that \( \pi_3(\mathbb{S}^2) = \mathbb{Z} \) while \( \pi_4(\mathbb{S}^3) = \mathbb{Z}/2\mathbb{Z} \). Pontriagin showed that if \( M \) is a
smooth manifold of dimension \( k \) without boundary then the set of homotopy classes of maps
\( M \to \mathbb{S}^p \) are in 1-1 correspondence with the set of framed codimension \( k – p \) submanifolds
of \( M \). (Proofs are given in §7 of Milnor: *Topology from a differentiable viewpoint*. Also see
problems 16,17 (p.54) for how to make framed cobordisms classes into a group. This kind of
geometric interpretation for homotopy classes is important now, for example in the recent
proof of the Mumford conjecture about the stable homology of the mapping class groups.)
To calculate \( \pi_{p+1}(\mathbb{S}^p) \) Pontriagin then classified framed 1-dimensional submanifolds of \( \mathbb{S}^{p+1} \).
These are circles, but you have to understand the framing. It would be hard to give a
complete proof here, but at least you could explain why the standard framed circle has
infinite order in \( \mathbb{S}^3 \) but order 2 in \( \mathbb{S}^4 \).

You could well find other possible subjects in Milnor’s book, such as Brouwer degree and
the Poincaré–Hopf theorem on the index of vector fields. These may be too close to topics
covered by the last semester’s Differential Topology course. Oh the other hand, you might
be interested in looking again at those topics and interpreting them in a more topological
way. eg you could put the Poincaré–Hopf theorem in the context of the Euler class of the
tangent bundle. (See also Topic 4 below.)

**Topic 3: Hopf invariant** You could do a lecture on different ways of understanding the
Hopf invariant for a map \( f : \mathbb{S}^{2p-1} \to \mathbb{S}^p \) (in particular, for the Hopf map \( \pi : \mathbb{S}^3 \to \mathbb{S}^2 \).)
Problems 14 and 15 in Milnor outline its construction as a linking number. Ch 9.3 of
Spanier (Algebraic Topology) describes it in terms of CW complexes and long exact homo-
topy sequences. (It is in a chapter called applications of the homology spectral sequence,
but I don’t think he uses either homology or spectral sequences.) There is a discussion in
AGP Ch 10.6, but it looks rather too advanced for the present. We might do it at the end of the semester.

**Topic 4: Vector bundles and Characteristic classes**

(a) **First Chern class** You can define the first Chern class of a complex line bundle over a Riemann surface by counting the zeros of a generic section. Using this you can define the first Chern for any complex vector bundle. Relate this to the Euler number of a vector field as mentioned in 2. (This approach is outlined for example in McDuff–Salamon: *Introduction to Symplectic Topology* Chapter 2 (see Thm 2.69, Remark 2.70) but in the context of symplectic vector bundles.)

(b) **Classifying spaces** You could discuss other approaches to characteristic classes, or talk about the classifying space for (complex) vector (or line) bundles. This is a space, usually called $BU(n)$ that carries a universal rank $n$ vector bundle $E \to BU(n)$ and has the property that the set of isomorphism classes of vector bundle over a paracompact space $X$ is in bijective correspondence with the homotopy classes of maps $X \to BU(n)$, i.e.

$$\text{Vect}(X) = [X, BU(n)].$$

Reference: Milnor and Stasheff: *Characteristic classes* or AGP Ch 8.