MAT 324, Fall 2015  
PROBLEM SET 7, Due Tuesday, December 1  
Product Measures

(1) Suppose $E \subset \mathbb{R}^2$ is measurable and that $E_y = \{x : (x, y) \in E\} \subset \mathbb{R}^1$ has measure zero for almost every $y$. Show that $E$ has measure zero in $\mathbb{R}^2$.

(2) If $f$ and $g$ are measurable on $\mathbb{R}$, show that $h(x, y) = f(x)g(y)$ is measurable on $\mathbb{R}^2$.

(3) Show
\[
\int_{\mathbb{R}^n} e^{-|x|^2} \, dm = \pi^{n/2}.
\]
(Hint: For $n = 1$ use,
\[
(\int e^{-x^2} \, dx)^2 = \int \int e^{-x^2-y^2} \, dxdy,
\]
and polar coordinates. For $n > 1$, use $|x|^2 = x_1^2 + \cdots + x_n^2$ and Fubini’s theorem to reduce to the $n = 1$ case.)

(4) The convolution $f * g$ of two functions $f, g \in L^1(\mathbb{R})$ is defined as the function
\[
f * g(y) = \int_{\mathbb{R}} f(y-x)g(x) \, dx.
\]
Show that $f * g$ is in $L^1(\mathbb{R})$ and
\[
\int f * g(y) \, dy = (\int_{\mathbb{R}} f(x) \, dx)(\int_{\mathbb{R}} g(x) \, dx).
\]

(5) If $E \subset \mathbb{R}$ is closed, let $\delta(y) = \text{dist}(y, E) = \inf\{|x-y| : x \in E\}$. Show that
\[
M(x) = \int_{0}^{1} \frac{\delta^{\alpha}(y) \, dy}{|x-y|^{1+\alpha}} < \infty,
\]
for almost every $x \in E$. (Hint: integrate $M$ over $E$.)