I’m going to write down the second principle once again. Let $X^a := \{x \in X \mid f(x) \leq a\}$ and suppose that $f^{-1}([a,b])$ is compact. Suppose that there is exactly one critical point $p$ of $f$ in $f^{-1}([a,b])$ and $a < f(p) < b$. Then

$$X^b \simeq X^a \cup_\sigma \mathbb{B}^\lambda$$

are homotopy equivalent, where $\mathbb{B}^\lambda$ is the unit closed ball in $\mathbb{R}^\lambda$ and $\sigma: \partial \mathbb{B}^\lambda \to \partial X^a$ is the attaching map. Let’s draw a picture:

Let me give you the standard example of the two dimensional torus.

**Remark 1.** We’ll come back to this example later but suppose $X$ is compact and $f: X \to \mathbb{R}$ has exactly 2 critical points which are non-degenerate. So $f(x) = x_1^2 + \cdots + x_n^2$ and $f(y) = -y_1^2 - \cdots - y_n^2$ at the critical points. The flow of $\nabla f$ is where the knotting takes place, i.e., it gives you a way to identify the boundary of the two Morse cells which are $n$-disks.

As another example, let’s look at:

The level sets are as follows:
We see that in this example as well as the torus we have 4 critical points although the resulting spaces are different. The reason is that in the torus the Morse cells add to the topology as we pass through critical points while it kills some homology at the intermediate level for the bumpy sphere. This illustrates the important fact: Attaching a $\lambda$-cell either increases $\dim H_\lambda$ by one or decreases $\dim H_{\lambda-1}$ by one.

**Convention** All homology has field coefficients (for e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_p$).

To see the fact mentioned above, let’s apply Mayer-Vietoris to $X^a$ and $e^\lambda$:

$$\cdots \to H_*(S^{(\lambda-1)}) \to H_*(X^a) \oplus H_*(e^\lambda) \to H_*(X^a \cup e^\lambda) \to H_{*(1)}(S^{(\lambda-1)}) \to \cdots$$

Note that if $* \neq \lambda, \lambda + 1$ then $H_*(X^a) \cong H_*(X^a \cup e^\lambda)$. If $* = \lambda$ then

$$0 \to H_\lambda(X^a) \to H_\lambda(X^a \cup e^\lambda) \to k \xrightarrow{\varphi} H_{\lambda-1}(X^a) \to H_{\lambda-1}(X^a \cup e^\lambda) \to 0.$$ 

Since $k$ is a field, either $\varphi$ is injective or the zero map. This leads to the two possibilities:

- $\varphi$ is injective: $H_{\lambda-1}(X^a \cup e^\lambda) \cong k \cong H_{\lambda-1}(X^a)$
- $\varphi$ is trivial: $H_\lambda(X^a \cup e^\lambda) \cong H_\lambda(X^a) \oplus k$.

Let $b_k = b_k(X^a; k) := \dim_k H_k(X^a; k)$ be the $k$th Betti number with $k$-coefficients.

**Proposition 2.** When attaching a $\lambda$-cell as above, either $b_\lambda$ increases by one or $b_{\lambda-1}$ decreases by one. All other Betti numbers are unchanged.

**Definition 3.** A critical point of index $\lambda$ is *completable* if $b_\lambda$ increases. If every critical point of $f$ is completable then $f$ is called *perfect*.

The height functions on the torus and the round 2-sphere are both perfect while the height function on the bumpy sphere is not (there are *handle cancellations* going on).

One can make two immediate deductions:

1. **The Lacunary Principle**
   Let $f : X \to \mathbb{R}$ have non-degenerate critical points whose indices take values $\lambda_1 < \lambda_2 < \cdots < \lambda_N$. Suppose that $\lambda_k - \lambda_{k-1} \leq 2$ for all $k$. Then $f$ is perfect and $H_\lambda(X; k) \cong k^{n_\lambda}$ where $n_\lambda$ is the number of critical points of index $\lambda$.

   **Proof** At any stage $X^a$ has no critical points of index less than $\lambda$, i.e., $H_{\lambda-1}(X^a; k) = 0$ by induction. So $b_{\lambda-1}$ cannot decrease and hence $b_\lambda$ increases.

   **Example 4.** Use homogeneous coordinates on $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ and use

   $$f([z]) = \frac{1}{\|z\|^2} \left( |z_0|^2 + 2|z_1|^2 + \cdots + (n+1)|z_n|^2 \right).$$

   All indices are even.

2. **The Morse Inequalities**
   The classical *Poincaré polynomial* is

   $$P(X, t) := \sum_{\lambda \geq 0} \dim H_\lambda(X; k)t^\lambda = \sum_{\lambda \geq 0} b_\lambda t^\lambda.$$ 

   We have the *Morse polynomial* of $f$ given by

   $$M(X, f, t) := \sum_{\lambda \geq 0} n_\lambda(f)t^\lambda,$$

   where $n_\lambda(f)$ is the no of critical points of index $\lambda$ for a Morse function $f$ which has non-degenerate critical points.
Theorem 5. (Morse Inequality)

\[ M(X, f, t) - P(X, t) = Q(t)(t + 1) \]

where \( Q(t) = \sum_{\lambda \geq 0} m_{\lambda} t^{\lambda} \) and \( m_{\lambda} \geq 0 \).

This is how Bott used to describe Morse inequalities. However, if you search in books or even google it, this is not how it’s presented. The corollary of this that we’ll write down is the often seen formal version.

**Proof**  Let \( M(t), P(t) \) be the respective polynomials for \( X^a \) and \( \tilde{M}(t), \tilde{P}(t) \) be the analogous ones for \( \tilde{X}^a = X^a \cup e^\lambda \). Then

\[
\begin{align*}
\tilde{M}(t) &= M(t) + t^\lambda \\
\tilde{P}(t) &= \begin{cases} 
    P(t) + t^\lambda & \text{if } b_{\lambda} \text{ increases by } 1 \\
    P(t) - t^{\lambda-1} & \text{if } b_{\lambda-1} \text{ decreases by } 1.
\end{cases}
\end{align*}
\]

Therefore,

\[
\begin{align*}
\tilde{M}(t) - \tilde{P}(t) &= \begin{cases} 
    M(t) - P(t) & \text{if } b_{\lambda} \text{ increases by } 1 \\
    M(t) - P(t) + t^\lambda + t^{\lambda-1} & \text{if } b_{\lambda-1} \text{ decreases by } 1
\end{cases} \\
&= \begin{cases} 
    Q(t)(t + 1) & \text{if } b_{\lambda} \text{ increases by } 1 \\
    (Q(t) + t^{\lambda-1})(t + 1) & \text{if } b_{\lambda-1} \text{ decreases by } 1.
\end{cases}
\end{align*}
\]

**Corollary 6.** For all \( k \)

\[ n_k - n_{k-1} + \cdots + (-1)^k n_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0. \]

Moreover, \( n_k \geq b_k \) for all \( k \).

Observe that the last statement above simple states that the the rank of the homology is at most the rank of the chains.

**Proof**  Write \( Q(t) \) by inverting \((t + 1)\) and conclude that

\[
\sum_{\lambda = 0}^{k} (n_{\lambda} - b_{\lambda})(-1)^{k-\lambda} = m_k \geq 0.
\]

**Note**  (1) These inequalities apply to \( X^a \) for any non-critical \( a \).

(2) Suppose that \( X \) is not compact but \( f^{-1}([a,b]) \) is compact and all critical points in \( f^{-1}([a,b]) \) non-degenerate and in the interior. Then everything holds if we replace \( H_*(X^c; k) \) by \( H_*(X^c, X^a; k) \).

\[
\begin{array}{ccc}
 f = c & f = a & f = b \\
\end{array}
\]