We will work over a compact Riemannian manifold and consider Hermitian bundles $E, F \to X$. We had the notion of pseudodifferential operators $\Psi DO(E, F)$ which maps smooth sections to smooth sections. Let $\{U_{\beta}, \psi_{\beta}\}$ be good pairs of $E, F$. Let $P_{\beta}$ be the representative of $P$ on $U_{\beta}$ which is elliptic. Let $Q_{\beta}$ be a parametrix for $P_{\beta}$. We have

$$Q_{\beta}P_{\beta} = 1 - S_{\beta}, \quad P_{\beta}Q_{\beta} = 1 - S'_{\beta}.$$ 

Assume that $\text{supp}(\psi_{\beta}) \subset B(1)$ and $Q_{\beta}S_{\beta}, S'_{\beta}$ are 1-local and $\text{supp}(\psi_{\beta}Q_{\beta}) \subset B(2)$. This gives $\Gamma(E) \to \Gamma(F)$.

**Definition 1.**

$$Q := \sum \psi_{\beta}Q_{\beta}, \quad Q' := \sum Q_{\beta}\psi_{\beta}, \quad S := \sum \psi_{\beta}S_{\beta}, \quad S' = \sum S'_{\beta}\psi_{\beta}.$$ 

This implies that

$$PQ'u = \sum P\psi_{\beta}u = \sum P_{\beta}Q_{\beta}\psi_{\beta}u = \left(\sum \psi_{\beta}\right)u - S'u = u - S'u.$$ 

Similarly we have 

$$QP'u = u - Su, \quad Q \sim Q(PQ') = (QP)Q' \sim Q'.$$

**Note** A pseudodifferential operator $P \in \Psi DO_m(E, F)$ is **elliptic** if its principal symbol

$$\text{Sym}(P) \in \Gamma(T^*X, \text{Hom}(\pi^*E, \pi^*F))$$

is invertible outside a compact set and satisfies

$$|P^{-1}(\eta)| \leq C(1 + |\eta|)^{-m}.$$ 

**Theorem 2.** Let $P \in \Psi DO_m(E, F)$ be an elliptic pseudodifferential operator of order $m$. Then there exists an operator $Q \in \Psi DO_{-m}(E, F)$, unique up to equivalence, such that

$$PQ = \text{Id} - S', \quad QP = \text{Id} - S,$$

where $S, S'$ are infinitely smoothing operators.

Now I want to discuss the basic results.

**Definition 3.** A bounded linear map $L : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces is **Fredholm** if

(i) $\dim (\ker L) < \infty$

(ii) $\dim (\coker L) \equiv \dim (\mathcal{H}_2/\text{Im} L) = \dim (\ker L^*) < \infty$, and

(iii) $\text{Im} L$ is closed.

Equivalently,

(iii) $L(\mathcal{H}_1) = \overline{L(\mathcal{H}_1)}$

(i) $\dim (\ker L) < \infty$

(ii) $\dim (\ker L^*) < \infty$.

The **index** is defined to be

$$\text{(1) \quad \text{index} (L) := \dim (\ker L) - \dim (\coker L).}$$
**Definition 4.** A bounded linear map $L : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces is *compact* if for every bounded sequence $\{v_j\}_{j=1}^{\infty} \subset \mathcal{H}_1$ ($\|v_j\| \leq C \forall j$ and some $C$) there exists a subsequence such that $L(v_{j_n})$ converges in $\mathcal{H}_2$.

Recall that the inclusion $\iota : L^2_{s'} \to L^2_{s}$ for $s > s'$ is compact. In particular, if $S : \Gamma(E) \to \Gamma(F)$ is infinitely smoothing then every extension $S : L^2_s(E) \to L^2_{s-m}(F)$ is compact since it factors

$$L^2_s(E) \xrightarrow{S} L^2_{s-m}(F) \xrightarrow{S} L^2_{s'}(F)$$

for $s' > s - m$.

**Proposition 5.** Suppose $P : \mathcal{H}_1 \to \mathcal{H}_2, Q : \mathcal{H}_2 \to \mathcal{H}_1$ are bounded linear maps such that

$$PQ = \text{Id} - S', \quad QP = \text{Id} - S,$$

where $S, S'$ are compact operators. Then $P$ and $Q$ are Fredholm.

**Proof** We have

$$QP|_{\ker P} = (\text{Id} - S)|_{\ker P} = 0$$

whence $\text{Id}|_{\ker P}$ is compact and $\dim (\ker P) < \infty$. Take an orthonormal basis $\{e_n\}$ such that $\|e_n - e_m\| = \sqrt{2}$ if $n \neq m$. Then

$$(PQ)^* = Q^*P^* = \text{Id} - (S')^*$$

and therefore $\dim (\ker P^*) < \infty$. We need to show that $P$ has closed range. Assume without loss of generality that $P$ is injective since otherwise we may restrict $P$ to $(\ker P)^\perp$. Let $v_j = P(u_j), j = 1, 2, \ldots$ be a sequence such that $v_j \to v \in \mathcal{H}_2$. We have to show that there exists $u \in \mathcal{H}_1$ such that $P(u) = v$.

Assume there is a $C$ such that $\|u_j\| \leq C$ for all $j$. Then

$$Q(v_j) =QP(u_j) = u_j - S(u_j) \to Q(v).$$

Passing to a subsequence $S(u_j) \to u_\infty \in \mathcal{H}_1$ and

$$u_j = Q(v_j) + S(u_j) \to Q(v) + u_\infty.$$

Therefore,

$$v_j = P(u_j) \to P(Q(v) + u_\infty) \in \text{Im } P.$$

To prove the assumption, assume the contrary. There is a subsequence such that $\|u_j\| \to \infty$. Then

$$P \left( \frac{u_j}{\|u_j\|} \right) = \frac{1}{\|u_j\|}P(u_j) = \frac{v_j}{\|u_j\|} \to 0.$$

Since $QP = \text{Id} - S$ we have

$$\frac{u_j}{\|u_j\|} - S \left( \frac{u_j}{\|u_j\|} \right) \to 0.$$

Since $S$ is compact, passing to a subsequence

$$\lim_{j \to \infty} S \left( \frac{u_j}{\|u_j\|} \right) = w = \lim_{j \to \infty} \frac{u_j}{\|u_j\|},$$

whence $\|w\| = 1$. But we also have $P(w) = 0$, a contradiction to the fact that $P$ is injective. \qed

This brings us to the main theorem.
Theorem 6. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic pseudodifferential operator of order $m$ on a compact
manifold $X$. Then

1. For all $U^\text{open} \subset X$ and $u \in L^2_s(E)$,
   \[ Pu|_U \text{ is } C^\infty \Rightarrow u|_U \text{ is } C^\infty. \]

2. $P$ extends to a Fredholm map $P : L^2_s(E) \to L^2_{s-m}(F)$ for all $s \in \mathbb{R}$ and $\text{ind } P$ is independent of $s$.

3. For all $s$ there is a constant $C_s$ such that
   \[ \|u\|_s \leq C_s \left(\|u\|_{s-m} + \|Pu\|_{s-m} \right), \forall u \in \Gamma(E). \]
   Equivalently,
   \[ \|\cdot\|_s \sim \|\cdot\|_{s-m} + \|P\|_s \]
on $\Gamma(E)$.

Proof

1. Same as in $\mathbb{R}^n$.

2. We know that $P$ has an extension
   \[ P_s : L^2_s(E) \to L^2_{s-m}(F) \]
   for all $s$. The existence of a parametrix $Q$ and Proposition 5 imply that $P_s$ is Fredholm. (1) implies that
   \[ \text{ker } P_s \text{ consists of } C^\infty \text{ sections and} \]
   \[ \text{ker } P_s = \text{ker } P_s \cap \Gamma(E) = \text{ker } P|_{\Gamma(E)} \]
is independent of $s$. So $\dim (\text{ker } P_s)$ is independent of $s$ and similarly, $\dim (\text{coker } P_s)$ is also independent of $s$.
   This finishes the proof of (2).

3. There is a parametrix $Q \in \Psi DO_{-m}(E,F)$ and $u = QPu + Su$. Then
   \[ \|u\|_s \leq \|QPu\|_s + \|Su\|_s \leq C \|Pu\|_{s-m} + C\|u\|_{s-m} \]
   where the last term in the last inequality follows from $S$ being infinitely smoothing. $\square$

Definition 7. For any elliptic $P \in \Psi DO_m(E,F)$, index $(P)$ is well defined and equals $\text{ind } (P_s)$ for any $s$.

Theorem 8. (Local fundamental elliptic estimate)

Let $D = \sum_{|\alpha| \leq m} A_\alpha(x)D^\alpha$ be an elliptic differential operator on a domain $\Omega \subset \mathbb{R}^n$. Given $K^\text{cpt} \subset \Omega$ and
$k \geq 0$ an integer, there exists $C_{K,k} > 0$ such that

2. \[ \|u\|_{K,C^k} \leq C_{K,k}\|u\|_{\Omega,L^2} \]
whenever $Du = 0$.

The point is that on any compact set you can estimate all the derivatives (up to any order you like) by the
$L^2$ norm. So, if you get something converge to $L^2$ then it gives an extremely powerful estimate.

Proof

Choose $\phi \in C^\infty_0(\Omega)$ such that $\phi \equiv 1$ on $K$. Then
   \[ D(\phi u) = \phi Du + \sum_{|\beta| \leq m-1} A_\beta(x)D^\beta u = \sum_{|\beta| \leq m-1} A_\beta(x)D^\beta u =: \tilde{P}u \]
has order $m - 1$. Apply (3) of Theorem 6 on $\mathbb{R}^n \subset S^n$:

\[ \|u\|_{K,L^2_x} \leq \|\phi u\|_{\Omega,L^2_x} \]
\[ \leq C \left(\|\phi u\|_{\Omega,L^2_{x-m}} + \|P(\phi u)\|_{\Omega,L^2_{x-m}} \right) \]
\[ \leq C' \left(\|u\|_{\Omega,L^2_{x-1}} + \|u\|_{\Omega,L^2_{x-1}} \right) \]
\[ \leq C'' \|u\|_{\Omega,L^2_{x-1}}. \]
Now apply this to domains

\[ K \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_N \subset \Omega \]

to get

\[ \|u\|_{K,L^2} \leq C\|u\|_{\Omega_N,L^2_{-N}}. \]

Choose \( s - N > \frac{n}{2} + k \) and apply Sobolev embedding. There exists \( C_{K,k} \) such that the required condition holds.

Let me just write down what we're going to do next time. If \( D : \Gamma(E) \to \Gamma(F) \) is an elliptic differential operator of order \( m \) and \( D^* : \Gamma(F^*) \to \Gamma(E^*) \) then

\[ \Gamma(E) = \ker D \oplus \text{Im} D^*, \]

where this is an \( L^2 \)-orthogonal decomposition of smooth sections and \( \text{Im} D^* = D^*(\Gamma(F)) \). In particular, if \( D \) is self-adjoint and \( E = F \) then

\[ \Gamma(E) = \ker D \oplus \text{Im} D. \]

When we apply this to \( E = \Lambda^*(T^*M) \) and \( D = d + d^* \) we get the Hodge decomposition.