Cohomology of Lie groups and Lie algebras

1 Introduction

The aim of this expository essay is to illustrate one example of a local-to-global phenomenon. What we mean by that is better illustrated by explaining the topic at hand. We aim to understand the de Rham cohomology groups of a Lie group. But instead of doing it using actual differential forms, we shall use the properties of a Lie group (especially the fact that it is a group) to reduce calculations on the de Rham complex to calculations involving the Lie algebra and its tensor powers. On the one hand, this reduces the problem to a local one while on the other, makes it easier to solve by virtue of being a linear problem. In what follows, we explain the passage from global to local in §2 and then exhibit a few computations of the cohomology groups in §3.

2 Cohomology of Lie groups

Let $G$ be a connected compact Lie group of dimension $n$ with a normalized bi-invariant measure $\mu$ on it. Let $L_g : G \to G$ denote the left multiplication by $g \in G$ and $m : G \times G \to G$ the multiplication. We will be working with real coefficients throughout this section unless specified otherwise.

Definition 2.1. Let $(C(G), d)$ denote the cochain complex of de Rham differential forms on $G$. An element $\alpha \in C^i(G)$ is called a left-invariant form if $L_g^* \alpha = \alpha$ for any $g \in G$. The space of all left-invariant forms will be denoted by $C^L(G)$.

Observe that $C^1_L(G) = g^*$ is the dual to the left-invariant vector fields and $C_L(G)$ is the exterior algebra over $g^*$. We have an averaging map $\rho : C(G) \to C_L(G)$ defined by

$$\alpha \mapsto \int_G L_g^* \alpha \, d\mu.$$ 

This is a map of cochain complexes which is identity on $C_L(G)$ and $\rho \circ \iota$ is the identity on $C_L(G)$. This implies that $\iota_*$ is injective. We claim that

Proposition 2.2. The map $\iota_* : H^*_L(G; \mathbb{R}) \to H^*(G; \mathbb{R})$ is an isomorphism.

Proof Suppose we have constructed a chain map $h : C^i(G) \to C^{i-1}(G)$ of degree $-1$ such that

$$\iota \circ \rho - \text{Id} = dh + hd$$
on $C(G)$. Since $\iota_* \circ \rho_* = \text{Id}$, $\iota_*$ is surjective. It is injective from the previous discussion, whence it is an isomorphism. We construct $h$ as a composition of $h_G \circ m^*$ where $h_G : C(G \times G) \to C(G)$ is homogeneous of degree $-1$.

Let $\pi_1 : G \times G \to G$ denote the projection of the trivial $G$-bundle to $G$. We have a map $f^G : C(G \times G) \to C(G)$ called the fibre integral and is defined at $g \in G$ by integrating it over the fibre at $g$. It is a homogeneous map of degree $-n$ and commutes with $d$. We now define a degree 0 map

$$I_\Omega : C(G \times G) \to C(G)$$

by setting

$$I_\Omega(\omega)(g) := \int^G \omega \wedge \pi_1^* \Omega,$$

where $\Omega$ is the normalized left-invariant volume form on $G$. For any $\alpha \in C(G)$

$$[(I_\Omega \circ m^*) \alpha](g) = \int^G m^* \alpha \wedge \pi_1^* \Omega = \int^G (L_g^* \alpha)(g) d\mu.$$  

This proves that $I_\Omega \circ m^* = \rho$. Let $i : G \to G \times G$ denote the map sending $g$ to $(g, 1)$. Then $m \circ i = \text{Id}$ and consequently $i^* \circ m^* = \text{Id}$. If we construct

$$h_G : C(G \times G) \to C(G)$$

such that $I_\Omega - i^* = dh_G + h_G d$ then it follows that

$$\iota \circ \rho - \text{Id} = I_\Omega \circ m^* - i^* \circ m^* = (dh_G + h_G d) \circ m^* \circ m^* = d(h_G m^*) + (h_G m^*) d,$$

where the last equality holds since $L^*$ is a cochain map.

If we change the the volume form $\Omega$ to another $n$-form $\Psi$ supported in a contractible local chart $U \ni 1$ of $G$ such that $\int_G \Psi = 1$, then $\Omega - \Psi = d\eta$ for some $(n-1)$-form $\eta$. Then the maps $I_\Omega$ and $I_\Psi$ are chain homotopic. The homotopy is given by

$$h_\eta(\alpha) = (-1)^i \int^G \alpha \wedge \pi_1^* \eta, \ \alpha \in C^i(G \times G).$$

Choosing $\Psi$ has the advantage that $I_\Psi : C(U \times G) \to C(G)$ and clearly $U \times G$ deformation retracts to $G$. Thus, $I_\Psi$ and $i^*$ are chain homotopic, whence $I_\Omega$ and $i^*$ are also chain homotopic. \qed

**Remark 2.3.** *The same proof, with slight modifications, works well for a $G$-action on a manifold $M$ by a compact, connected Lie group. We can prove that the inclusion of the subcomplex of $G$-invariant forms on $M$ into the complex of all forms on $M$ is an isomorphism in cohomology.*

We shift our focus to *invariant forms*, i.e., forms invariant under the left and right actions $L_g$ and $R_g$ respectively. In particular, these are invariant under the adjoint action $\text{Ad}_g = L_g \circ R_{g^{-1}}$. These forms are invariant under $d$. If we define the action $I$, of $G \times G$ on $G$, by

$$I_{g_1, g_2}(g) = g_1 g_2^{-1}$$

then the algebra of differential forms that are invariant under this action is precisely the space of invariant forms, denoted $C_I(G)$. 

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Lemma 2.4. $C_I(G)$ consists of closed forms.

Proof. First observe that if $\tau: G \to G$ denotes the inverse map, then

$$d\tau_g = -(R_{g^{-1}})_* \circ (L_{g^{-1}})_*, \ g \in G$$

and $\tau^* \alpha = (-1)^p \alpha$ for $\alpha \in C^p_I(G)$. Since $d\alpha \in C^{p+1}_I(G)$,

$$(-1)^{p+1}d\alpha = \tau^* d\alpha = d\tau^* \alpha = (-1)^p d\alpha,$$

whence $d\alpha = 0$. \framebox{}

Since $C_I(G)$ is closed, $H^*_I(G) = C_I(G)$ and by the remark, it is isomorphic to $H^*(G)$. We have isomorphisms

(2.1)  

$$C_I(G) \cong H^*_I(G) \cong H^*(G).$$

If $G$ is semisimple, this isomorphism is just a manifestation of the Hodge theorem. More precisely, it is known that for any semisimple group one can find a bi-invariant Riemannian metric on $G$. Hodge had proved that the harmonic forms with respect to such a metric are exactly $C_I(G)$.

We have the multiplication $m: G \times G \to G$ and $m^*: C(G) \to C(G \times G)$ which induces a map

$$\Delta: H^*(G) \to H^*(G \times G) \cong H^*(G) \otimes H^*(G).$$

of degree 0. Let $i_1, i_2: G \to G \times G$ be the inclusion maps opposite 1. If $\gamma \in H^*(G \times G)$ then

$$\gamma = i_1^* \gamma \otimes 1 + \beta + 1 \otimes i_2^* \gamma,$$

where $\beta \in H^+(G)^{\otimes 2}$. Since $m \circ i_1 = m \circ i_2 = Id$,

$$\Delta(\alpha) = \alpha \otimes 1 + \beta + 1 \otimes \alpha, \ \alpha \in H^*(G), \beta \in H^+(G)^{\otimes 2}.$$

Definition 2.5. An element $\alpha \in H^+(G)$ is called primitive if

(2.2)  

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha.$$

Remark 2.6. It is classically known that any compact connected Lie group is rationally homotopy equivalent to a product of odd spheres. The volume forms of these spheres generate the primitive elements of $H^*(G)$.

The primitive elements form a graded subspace, $P_G$, of $H^*(G)$. Notice that there are no even primitives because if $\alpha$ was one such then $1 \otimes \alpha$ and $\alpha \otimes 1$ would commute, both being even. Now let $k$ be the least positive number such that $\alpha^k = 0$. Then

$$0 = \Delta(\alpha^k) = (\alpha \otimes 1 + 1 \otimes \alpha)^k = \sum_{i=1}^{k-1} \alpha^i \otimes \alpha^{k-i}.$$

In particular, $\alpha \otimes \alpha^{k-1} = 0$, whence $\alpha = 0$. Since every homogeneous element of $P_G$ is odd, it’s square is zero. Thus, the inclusion $P_G \hookrightarrow H^*(G)$ extends to a homomorphism

(2.3)  

$$\lambda_G : \Lambda P_G \to H^*(G)$$

of graded algebras. It can be shown using properties of power maps and its eigenspaces that dim $P_G = \text{rank} G$ and $\lambda_G$ is an isomorphism. Thus, $H^*(G)$ is of dimension $2^{\text{rank} G}$. 

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3 Cohomology of Lie algebras

Let \( \mathfrak{g} \) be a finite dimensional Lie algebra. By Lie’s theorem, it corresponds to a simply connected Lie group \( G \). To each \( \mathfrak{g} \)-module \( M \) we can associate a cochain complex \( C^k(\mathfrak{g}; M) \), whose cohomology is defined to be the *Lie algebra cohomology of \( \mathfrak{g} \) with values in \( M \). We define

\[
C^k(\mathfrak{g}; M) := \text{Hom}(\Lambda^k \mathfrak{g}, M), \quad k = 0, 1, \ldots, \text{dim} \, \mathfrak{g},
\]

the vector space of real valued multilinear, skew maps with values in \( M \). The coboundary operator \( \delta : C^k(\mathfrak{g}; M) \to C^{k+1}(\mathfrak{g}; M) \) is defined by

\[
(\delta \omega)(x_0, \ldots, x_k) := \sum_{i=0}^{k} (-1)^i x_i \cdot \omega(\ldots, \hat{x}_i, \ldots)
+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([x_i, x_j], \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots).
\]

It is easily verified, using Jacobi and the properties of the \( \mathfrak{g} \)-action on \( M \), that \( \delta \circ \delta = 0 \).

Since our main object of interest is cohomology with values in \( \mathbb{R} \), we set \( M = \mathbb{R} \) with the trivial \( \mathfrak{g} \)-action. We will also abbreviate notation and denote \( C^k(\mathfrak{g}; \mathbb{R}) \) by \( C^k(\mathfrak{g}) \) and the corresponding cohomology groups \( H^k(\mathfrak{g}; \mathbb{R}) \) by \( H^k(\mathfrak{g}) \). Observe that the cohomology groups so obtained are just the of left-invariant forms on \( G \) and \( \delta \) is exactly \( d \). By definition, \( C^0(\mathfrak{g}) = \mathbb{R} \) and \( C^1(\mathfrak{g}) = \mathfrak{g}^* \cong \mathfrak{g} \). The first three coboundary maps are :

\[
\begin{align*}
(\delta \alpha)(x) &= 0, \\
(\delta \beta)(x, y) &= -\beta([x, y]), \\
(\delta \gamma)(x, y, z) &= -\gamma([x, y], z) - \gamma([y, z], x) - \gamma([z, x], y).
\end{align*}
\]

where \( x, y, z \in \mathfrak{g} \) and \( \alpha, \beta, \gamma \) are 0, 1 and 2-cochains.

For small values of \( k \), the cohomology groups have certain interesting interpretations. The first equation (3.3) implies that

\[
H^0(\mathfrak{g}) = \mathbb{R}.
\]

Using (3.4) we see that \( H^1(\mathfrak{g}) \) is exactly the kernel of \( \delta : C^1(\mathfrak{g}) \to C^2(\mathfrak{g}) \) since the map \( \delta : C^0(\mathfrak{g}) \to C^1(\mathfrak{g}) \) is zero. Elements \( \alpha \) in the kernel are precisely the ones that vanish on commutators, i.e., \( \alpha([x, y]) = 0 \) for any \( x, y \in \mathfrak{g} \). Alternatively, these can be viewed as maps from \( \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \) to \( \mathbb{R} \), whence

\[
H^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].
\]

In particular, the first cohomology vanishes for a semisimple Lie algebra.

To interpret \( H^2(\mathfrak{g}) \) we need to understand the kernel of (3.5), i.e., 2-cochains \( \omega \) such that

\[
\omega(([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0.
\]

The restraint above is called the *cocycle condition* and is equivalent to \( \omega \) being closed. Any such \( \omega \) defines a central extension

\[
0 \to \mathbb{R} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0
\]
with the Lie bracket on \( \tilde{g} \) given by

\[
[(x, s), (y, t)] := ([x, y], \omega(x, y)).
\]

The bracket satisfies the Jacobi identity due to (3.8) and is skew since \( \omega \) is. Conversely, given a central extension, the bracket on \( \tilde{g} \) is defined as in (3.9) and \( \omega \) must satisfy (3.8). Thus, the central extensions of \( g \) by \( \mathbb{R} \) are in bijective correspondence with the 2-cocycles.

We try to see what relations are forced on the 2-cocycles \( \omega, \omega' \) if the corresponding central extensions \( \tilde{g}, \tilde{g}' \) are equivalent. Recall that two extensions \( \tilde{g} \) and \( \tilde{g}' \) are equivalent if there exists a map \( \varphi : \tilde{g} \to \tilde{g}' \) of Lie algebras such that the following commutes:

\[
\begin{array}{cccc}
0 & \to & \mathbb{R} & \to & \tilde{g} & \to & g & \to & 0 \\
\downarrow{id} & & \downarrow{\varphi} & & \downarrow{id} & & \downarrow{id} & & \downarrow{0} \\
0 & \to & \mathbb{R} & \to & \tilde{g}' & \to & g & \to & 0.
\end{array}
\]

Both the extensions are \( g \oplus \mathbb{R} \) as vector spaces and \( \varphi : g \oplus \mathbb{R} \to g \oplus \mathbb{R} \) is the identity when restricted to \( \mathbb{R} \). Moreover, \( \varphi \) is an isomorphism (by the five-lemma) and \( \varphi(x, 0) = x + \alpha(x) \) where \( \alpha \in C^1(g) \).

We have

\[
[\varphi(x, 0), \varphi(y, 0)] = [[x, \alpha(x)], (y, \alpha(y))] = ([x, y], \omega'(x, y))
\]

and we also have

\[
\varphi([[(x, 0), (y, 0)]]) = \varphi(([[x, y], \omega(x, y)])) = ([x, y], \alpha([x, y]) + \omega(x, y)).
\]

Thus, the 2-cocycles are cohomologous via \( \alpha \).

**Proposition 3.1.** Equivalence classes of central extensions of \( g \) by \( \mathbb{R} \) are in bijective correspondence with elements of \( H^2(g) \).

It can be deduced that if \( g \) is semisimple then there are no non-trivial central extensions.

**Remark 3.2.** If \( G \) is simply connected then \( H^2(G; \mathbb{Z}) \hookrightarrow H^2(G; \mathbb{R}) \) is an injection. Recall that isomorphism classes of circle bundles over \( G \) correspond to \( H^2(G; \mathbb{Z}) \) and the total space of any such bundle can be made into a group, i.e., there is a short exact sequence of groups

\[
1 \to S^1 \to \tilde{G} \to G \to 1
\]

realizing such a bundle. The map of Lie algebras then give us the integral central extensions.

To discuss \( H^3(g) \), we shall restrict ourselves to algebras such that \( H^1(g) = 0 = H^2(g) \). The Lie algebras of any connected compact semisimple Lie group \( G \) satisfies this property. It follows from (3.4) that the negative of the dual of \( \delta \) is a map

\[
(3.10) \quad \delta^* : \Lambda^2 g^* \to g, \ x \wedge y \mapsto [x, y].
\]

Since \( \delta : \Lambda^2 g \to \Lambda g \) satisfies \( \delta^2 = 0 \), the map \( \delta^* \) extends to \( \Lambda g \) and satisfies \( \delta^* \circ \delta^* = 0 \). The resulting homology groups will be called the homology groups of \( g \) and denoted by \( H_i(g) \). By our assumption that the first two cohomology groups vanish, it follows from the duality of \( \delta \) and \( \delta^* \) that \( H_1(g) = 0 = H_2(g) \). In fact, the explicit formula of \( \delta^* \) is

\[
(3.11) \quad x_0 \wedge \cdots \wedge x_p \xrightarrow{\delta^*} \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_0 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_p.
\]
Notice that $\delta^*$ may not be a derivation.

Since $g \cong g^*$ as $g$-modules, the space of (symmetric) invariant bilinear forms on $g$, $\text{Bil}(g) = (S^2g)^g$, is isomorphic to $(S^2g^*)^g$. With this identification, define a map

$$
\varphi : (S^2g^*)^g \to (L^3g^*)^g
$$

(3.12)

$$
B \mapsto \varphi(B) : (x \wedge y \wedge z) \to B([x,y],[z]) = B(\delta^*(x \wedge y), z).
$$

The 3-form $\varphi(B)$ is anti-symmetric since $B$ is invariant and symmetric and $[,]$ is skew. The invariance follows from the Jacobi identity and the invariance of $B$, viz,

$$
\varphi(B)([w,x] \wedge y \wedge z) = \varphi(B)(x \wedge [w,y] \wedge y) + \varphi(B)(x \wedge y \wedge [w,z])
$$

$$
= B([[w,x],y],z) + B([y,w],[x],z) + B([x,y],[w,z])
$$

$$
= -B([[[x,y],w],z] + B([x,y],[w,z])
$$

$$
= 0.
$$

Let $\omega \in (L^3g^*)^g$. Since $\omega$ is closed, we have

$$
0 = \omega([x_0,x_1] \wedge x_2 \wedge x_3) - \omega([x_0,x_2] \wedge x_1 \wedge x_3) + \omega([x_0,x_3] \wedge x_1 \wedge x_2)
$$

$$
= \omega([x_0,x_1] \wedge x_2 \wedge x_3) - \omega([x_0,x_2] \wedge x_1 \wedge x_3) + \omega([x_0,x_3] \wedge x_1 \wedge x_2)
$$

$$
= \omega([x_0,x_1] \wedge x_2 \wedge x_3) - \omega([x_0,x_2] \wedge x_1 \wedge x_3) + \omega([x_0,x_3] \wedge x_1 \wedge x_2)
$$

$$
= \omega([x_0,x_1] \wedge x_2 \wedge x_3) - \omega([x_0,x_2] \wedge x_1 \wedge x_3)
$$

$$
= \omega([x_0,x_2] \wedge x_1 \wedge x_3) - \omega([x_0,x_1] \wedge x_2 \wedge x_3)
$$

This implies

$$
\omega(u \wedge \delta^*v) = \omega(\delta^*u \wedge v)
$$

(3.13)

$$
\omega(\delta^*w \wedge y) = 0
$$

(3.14)

for $u,v \in (L^2g)^g, w \in (L^3g)^g$. We are now prepared to prove the following proposition which provides the connection between $\text{Bil}(g)$ and $H^3(G) \cong (L^3g^*)^g$.

**Proposition 3.3.** The map $\varphi : (S^2g^*)^g \to (L^3g^*)^g$ is an isomorphism for any semisimple Lie algebra $g$.

**Proof** Injectivity of $\varphi$ follows from $H^1(g) = 0$ (equivalently $g = [g,g]$). To prove surjectivity, let $\omega \in (L^3g^*)^g$. Define $B \in (S^2g^*)^g$ by

$$
B(x,y) = \omega(u \wedge y), \text{ where } \delta^*u = x.
$$

This is well defined since if $\delta^*v = x$ then $\delta^*(u-v) = 0$. Since $H_2(g;\mathbb{R}) = 0$, there exists $w \in (L^3g)^g$ such that $\delta^*w = u - v$. Then $\omega(\delta^*w \wedge y) = 0$ by (3.14). Using (3.13) and the surjectivity of $\delta^* : L^2g \to g$,

$$
B(\delta^*u, \delta^*v) = \omega(u \wedge \delta^*v) = \omega(v \wedge \delta^*v) = B(\delta^*v, \delta^*u),
$$

$$
B(\delta^*u, \delta^*v) = \omega(u \wedge \delta^*v) = \omega(v \wedge \delta^*u) = B(\delta^*v, \delta^*u),
$$

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the symmetry of $B$ follows. By definition $\varphi(B) = \omega$. Since

$$B([x, w], y) = \omega(x \wedge w \wedge y) = \omega(w \wedge y \wedge x) = B(x, [w, y]),$$

$B$ is invariant.

In view of this result and the discussion preceding it, we conclude that $\text{Bil}(\mathfrak{g})$ is isomorphic to $H^3(G; \mathbb{R})$. If $G$ is simple, then it is 1-dimensional since any such bilinear form is a multiple of the Killing form on $\mathfrak{g}$. □