Result: $O$ - category "generally doesn't work"

\[
O = \bigoplus_{\lambda \in P^+} O_{\lambda}, \quad O_{\lambda} = \{ M \mid (z, x, (\varepsilon)) \text{ is nilpotent on } M \}, \quad z \in \mathbb{Z}(U_q) \text{.}
\]

\[\text{has simple obj. } L(w, \lambda) \text{, we } \lambda \in \mathfrak{h} \\text{.}
\]

also contains $M(w, \lambda), \text{ we } \lambda \in \mathfrak{h} \\text{.}$

\[
[M(w, \lambda)] = [L(w, \lambda)] + \sum_{\lambda' \prec \lambda} [L(w', \lambda')]
\]

(if $\lambda \in P_+$).

Main goal today: prove

Then for any $\lambda, \mu \in P_+$ (abelian)

we have equivalence of cat.

\[O_{\lambda} \simeq O_{\mu}.\]

In part, $\geq O_0$ - "the regular block".

Pf: Use "Translation functors".

Before that, Preliminaries:

(i) Consider $L$ - f.d. $g$-module.

Then $M \rightarrow L \otimes M$ is functor $0 \rightarrow 0$.

Moreover, it is exact (just see on level of v.s.)

Properties:

1. $\text{Hom}(L \otimes A, B) \simeq \text{Hom}(A, L \otimes B)$

2. $L \otimes M(N)$ has free comp. series

of factors $M(\lambda + \nu)$, $\nu$ a word of $L$.

appearing dimL\nu times.

Note: $0$ is not tensorial.

\[\text{& violates for generation.}\]

\[\text{look at characteristic.}\]

If $\lambda \neq \mu$, $L = C^\mu$, right, $\nu$.

\[C^\lambda \otimes M(\mu) \cong M(\lambda - \mu) \cong 0\]

M(min).

(\& generic, will have $\otimes$.)

(\& if $\lambda \neq \text{integ}$.)

\[\text{[exercise]}.\]
Pf: (1) trivial

(2) \[ M(\lambda) = U^g \otimes \mathcal{C}_\lambda \]

\[ \mathcal{C}_\lambda = \text{Hom}_L(\lambda, \mathfrak{h}) \]

\[ \mathfrak{h} = \text{span} \{ h \in \mathfrak{g} \mid h \lambda = \lambda \} \]

\[ h \otimes v - \lambda(h) \cdot v, \quad h \in \mathfrak{h} \]

\[ \forall v \in \lambda, \quad v \in \mathcal{P}_\lambda \]

\[ \Rightarrow \mathcal{C}_\lambda \otimes L = U^g(\mathcal{C}_\lambda \otimes L) \]

\[ = U^g(\mathcal{C}_\lambda \otimes L) \]

Thus, suffices to check that

\[ [\mathcal{C}_\lambda \otimes L] = \sum \text{dim } L_\nu \cdot [\mathcal{C}_\nu] \]

in the weight mult.

Recall \( \lambda \) is admissible, i.e., \( \mathcal{C}_\lambda \otimes L \) has comp. series as \( \mathfrak{h} \)-mod.

In this case \( \lambda = \mu, \mu \in \mathcal{P}_\lambda \).

\[ \text{Def. } \mu = \lambda, \mu \in \mathfrak{h}^*, \quad \nu = \mu - \lambda \in \mathcal{P}_\lambda \]

\[ \text{Def. } L = \text{L}(\nu^*) \quad \text{where } \nu^* = \text{W}(\nu) \cap \mathfrak{h}^* \]

Define the "map of extremal weight" for \( \mu \)

\[ T^\mu : \Omega_\lambda \rightarrow \Omega_\mu \]

\[ M \rightarrow \pi_\mu (L \otimes M) \quad \pi_\mu : \Omega \rightarrow \Omega_\mu \text{ is projection of } \]

\[ \text{T}^\mu : \text{Bla translate functors} \quad \text{T}^\mu \text{ will provide the desired equivalence.} \]

Proposition:
Properties of $T^\lambda_n$:

1. $T^\lambda_n$ is exact. (since $0 \to \mathbb{P}_n$, $\pi_\lambda$ is exact too).

2. Adjointness property: $\text{Hom}(T^\lambda_n A, B) \cong \text{Hom}(A, T^{\lambda^\circ}_n B)$, $A \in \mathbb{P}_n$.

$\text{Hom}(L \otimes A, B) \cong \text{Hom}(A, L^\circ \otimes B)\tag{\alpha \Rightarrow \lambda \mu \text{ is extremal right}}$

Lemma: Let $\lambda \mu \in \mathbb{P}_n$, $\Lambda = \mathbb{P}_n$, $L = L(\lambda \mu)$. Then $\lambda \mu + L \subseteq \mathbb{P}_n$, $\mu \in L(\lambda \mu)$.

Proof: Exercise

Corollary: as an corollary of Lemma, $T^\lambda_n M(\mu) = M(\lambda \mu)$

Proof: $L \otimes M(\mu)$ decomposes as factors $M(\lambda \mu)$

$T^\lambda_n M(\lambda \mu) = \pi_\lambda (L \otimes M(\mu)) = M(\lambda \mu)$, by Lemma.

$T^\lambda_n M(\lambda \mu, \lambda \mu) = M(\lambda \mu, \lambda \mu)$, apply the duality of Lemmas.

$T^\lambda_n T^\lambda_n M(\lambda \mu, \lambda \mu) = M(\lambda \mu, \lambda \mu)$, if $w(\lambda \mu, \lambda \mu)$ instead of $P_\lambda$.

As Upshot... just study $U_0$.

Next time: use geometry to get $S_c$.

\vspace{0.5cm}

\vspace{0.5cm}

$M = M_0 \to M_1 \to \ldots \to M_n = 0$

$X_i = M_i / M_{i-1}$, $\pi$ exact

$\pi(\lambda \mu) \to \pi(\mu) \to \ldots \to \pi(M_n) = 0$

$\pi(M_i) / \pi(M_{i+1}) = \pi(X_i)$, exact.

If $\pi$ kills all factors $X_i$, but 1, then $\pi(M) = \pi X_i$.

(If it doesn't kill 2, then something still has some semisimplicity.)