The Siegel Modular Form Defining $\Omega\mathcal{M}_3^{odd}(4)$

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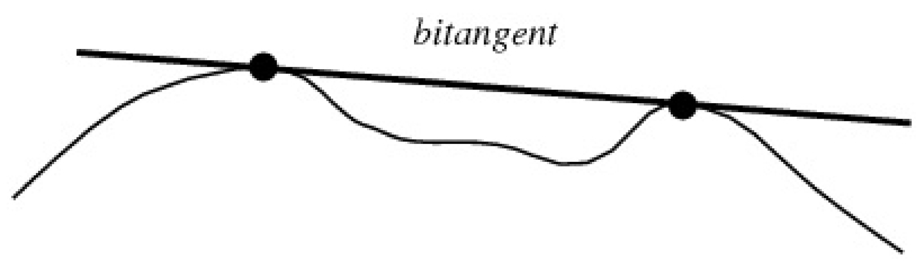
Background

A general plane quartic has 28 bitangents. When the two bitangent points coincide, we call the four-fold point a *hyperflex*. Admitting a hyperflex is a closed condition that cuts out a Cartier divisor (denoted by $\mathcal{HF}$) in $\mathcal{M}_3$.



A plane quartic with a chosen bitangent corresponds to a point in $\Omega\mathcal{M}_3^{odd}(2,2)$, while the locus $\mathcal{HF}$ is the image of the stratum $\Omega\mathcal{M}_3^{odd}(4)$ in $\mathcal{M}_3$ by forgetting the differential.

Outline

- We determine an explicit modular form defining the locus $\mathcal{HF}$;
- recompute the class of $\mathcal{HF}$ as in [Cuk89];
- investigate the deeper boundary strata of $\mathcal{HF}$.

Theta Functions in Genus Three

- **Riemann theta function with characteristics** $(\epsilon,\delta)\in(\mathbb{Z}/2\mathbb{Z})^g\times(\mathbb{Z}/2\mathbb{Z})^g$:
$$\theta[^{\epsilon}_{\delta}](\tau,z) = \Sigma_{k\in\mathbb{Z}^g}\exp\big(\pi i(k+\frac{\epsilon}{2})^t\tau(k+\frac{\epsilon}{2}) + 2\pi i(k+\frac{\epsilon}{2})^t(z+\frac{\delta}{2})\big).$$
- The parity of $\theta[^{\epsilon}_{\delta}](\tau,z)$ in $z \longleftrightarrow$ The parity of the characteristics $e(\epsilon,\delta):=\epsilon\cdot\delta$.
- $\operatorname{grad}\theta[^{\epsilon}_{\delta}](\tau,0)$ with odd characteristics $\longleftrightarrow$ 28 bitangent lines of the canonical image of the quartic on $\mathbb{P}^2\simeq\mathbb{P}H^0(C,K_C)$.
- Theta constants $\theta[^{\epsilon}_{\delta}](\tau,0)$ are **modular forms** of weight $\frac{1}{2}$; $\operatorname{grad}\theta[^{\epsilon}_{\delta}](\tau,0)$ are vector-valued modular forms.

Main Ideas

- To simplify notation: $(\epsilon,\delta)\leftrightarrow(i,j)$ $i=4\epsilon_1+2\epsilon_2+\epsilon_3$, $j=4\delta_1+2\delta_2+\delta_3$, e.g. $([1,1,0],[0,1,1])=(6,3)$.
- In [DPFSM14], **the equation of a plane quartic in terms of its bitangents globally over** $\mathcal{M}_3$ was derived (see lemma).
- By carefully investigating the condition for the hyperflex to exist, using the lemma, we get the modular form expression.

Lemma [DPFSM14, Cor. 6.3]

The equation of the canonical image of C is the determinant of the symmetric matrix:
$$Q(\tau,z):=\begin{pmatrix} 0 & \frac{D(31,13,26)}{D(77,31,26)}b_{77} & \frac{D(22,13,35)}{D(77,31,26)}b_{64} & \frac{D(77,64,46)}{D(77,31,26)}b_{51} \\ * & 0 & \frac{D(22,13,35)}{D(77,46,51)}b_{13} & \frac{D(77,13,31)}{D(77,31,26)}b_{26} \\ * & * & 0 & \frac{D(64,13,22)}{D(77,31,26)}b_{35} \\ * & * & * & 0 \end{pmatrix},$$
where τ denotes the period matrix of C, $b_{ij}:=\operatorname{grad}_z\theta_{ij}(\tau,z)|_{z=0}$, and $D(n_1,n_2,n_3):=b_{n_1}\wedge b_{n_2}\wedge b_{n_3}$.

Main Theorem

On $\mathcal{A}_3(2)$, the modular form $\Omega_{77}(\tau)$ defined by:
$$\Omega_{77}(\tau):=[\theta_{01}\theta_{10}\theta_{37}\theta_{43}\theta_{52}\theta_{75}\theta_{42}\theta_{06}\theta_{30}\theta_{21}\theta_{55}+\theta_{02}\theta_{25}\theta_{34}\theta_{40}\theta_{67}\theta_{76}\theta_{33}\theta_{05}\theta_{14}\theta_{60}\theta_{42}]^2 - 4\theta_{01}\theta_{02}\theta_{10}\theta_{25}\theta_{34}\theta_{37}\theta_{40}\theta_{43}\theta_{52}\theta_{67}\theta_{75}\theta_{76}\theta_{00}\theta_{04}\theta_{57}\theta_{70}\theta_{61}\theta_{73}\theta_{20}\theta_{07}\theta_{00}\theta_{16}.$$
where $\theta_{ij}:=\theta_{ij}(\tau,0)$, vanishes at the period matrix τ of a smooth plane quartic iff the bitangent line corresponding to $(i,j)=(7,7)$ is a hyperflex.

Divisor Class of the Locus

- **Weight of the modular form = The coefficient of** λ (by definition).
- **Period Matrixes:** By [Fay73] and [Yam80], for a one parameter plumbing family C_s whose limit curve lies
  - in Δ_0: $\tau_s=\begin{bmatrix}\ln s & b\\ b^t & \tau'\end{bmatrix}+O(s)$;
  - in Δ_1: $\tau_s=\begin{bmatrix}\tau_1 & 0\\ 0 & \tau_2\end{bmatrix}-s\begin{bmatrix}0 & R\\ R^T & 0\end{bmatrix}+O(s)$.
- **Fourier-Jacobi expansion**: compute
$$\operatorname{ord}\theta_m(\tau,0) \qquad \operatorname{ord}\operatorname{grad}_z\theta_m(\tau,z)|_{z=0}$$
on the pre-image of Δ_0 and Δ_1 on $\overline{\mathcal{A}_3}(2)$.

Corollary 1 [Cuk89]

The divisor class of the closure of the hyperflex locus in $\operatorname{Pic}_{\mathbb{Q}}\overline{\mathcal{M}}_3$ is
$$[\mathcal{HF}]=308\cdot\lambda-32\cdot\delta_0-76\cdot\delta_1.$$

Deeper Boundary Strata

One can use the generalized Fourier-Jacobi expansion of a modular form near any boundary stratum to determine the intersection of the closure of $\Omega\mathcal{M}_g$ with it.
Example: The locus T of "banana curves": two genus one curves intersecting at two nodes.

Corollary 2 [Che15]

The boundary locus $T\subset\overline{\mathcal{HF}}$.

Main tool [Tan89]: *For a plumbing family C_s, the function*
$$f_{h,k}(s):=\exp(2\pi i\tau_{h,k}(s))\cdot\Pi_{i=1}^n s_i^{-N_{i,h}\cdot N_{i,k}}$$
is holomorphic for small enough s, where $N_{i,j}:=S_i\times B_j$, and $\tau_{h,k}(s)$ is the period matrix for C_s.

Future Research

- Use the modular form to study the $SL_2\mathbb{R}$-orbit closure in $\Omega\mathcal{M}_3^{odd}(4)$, especially rank 1.
- Similar ideas can be applied to study the hyperelliptic locus using the theta-null modular form.

References

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