RESEARCH STATEMENT

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My interests are the theory of Riemann surfaces, Teichmüller theory, and algebraic geometry. To be more specific, I use algebraic and analytic tools to study the algebro-geometric properties of the moduli spaces of algebraic curves, the moduli spaces of principally polarized abelian varieties (ppav), and the moduli spaces of abelian differentials, as well as their compactifications.

1. Overview

In this section I will give an overview of my thesis work and anticipated future projects, with the finer details being addressed in section $2 \sim 5$.

The main objects I focus on are the *moduli spaces of abelian differentials* (or *strata* of abelian differentials). Let $\mu = (m_1, \ldots, m_n)$ be an integral partition of $2g-2$, then set-theoretically a stratum is defined as

$$\Omega\mathcal{M}_{g,n}(\mu) := \left\{ (X; p_1, \ldots, p_n; \omega) : \begin{array}{c} X \text{ smooth genus } g \text{ curve with } n \text{ marked points at } p_i, \\ \omega \in H^{1,0}(X, \mathbb{C}), \qquad \operatorname{div}(\omega) = \sum_{i=1}^{g} m_i p_i \end{array} \right\},$$

These spaces are intensively studied in Teichmüller dynamics. Kontsevich and Zorich [KZ03] classified the connected components of $\Omega\mathcal{M}_{g,n}(\mu)$ for all g and μ. The compactification of strata was worked out by Bainbridge, Chen, Grushevsky, Gendron, and Möller [BCGGM16] (see also Farkas-Pandharipande [FP16]).

There is an $\mathrm{SL}_2(\mathbb{R})$-action on $\Omega\mathcal{M}_{g,n}(\mu)$ naturally arising from the study of dynamics in the game of billiards. The guiding problem in Teichmüller dynamics is to classify all the orbit closures of this action. Given that the action itself is nowhere near algebraic, it is striking that its orbit closures are indeed algebraic varieties. This result is due to the renowned recent works of Eskin-Mirzakhani-Mohammadi [EM13] [EMM15] and Filip [Fil16], and can be seen as an analog - for the Teichmüller flow - of Ratner's theorems on the unipotent flows on homogeneous spaces.

The closed orbits of the $\mathrm{SL}_2(\mathbb{R})$-action are known to be totally geodesic with respect to the Teichmüller metric (these are called *Teichmüller curves*). The Teichmüller curves are extremely interesting objects due to their analytic and dynamical nature and their algebro-geometric and number-theoretic properties: they parametrize curves whose Jacobians admit real multiplication. The classification of Teichmüller curves in genus 2 was done by McMullen [McM05a] [McM05b] [McM06]. Partial work in genus 3 and 4 is due to Bainbridge, Möller, Habegger, and Zagier [BM12] [BM14] [BHM16] [MZ16]. However, given any stratum in genus ≥ 3, a complete classification of Teichmüller curves within the stratum still remains an open problem.

With the tools introduced in my thesis work [Hu17] [HN17], I believe that it is possible to classify the Teichmüller curves in the minimal stratum $\Omega\mathcal{M}_3^{odd}(4)$ in genus 3. Our approach in [HN17] also allows us to study the degeneration of *period coordinates* on any stratum, which we hope will lead to an analytic alternative proof of the algebraicity of the $\mathrm{SL}_2(\mathbb{R})$-orbit closure.

1.1. A modular form for $\Omega\mathcal{M}_3^{odd}(4)$.

In genus 3, the stratum of lowest dimension is $\Omega\mathcal{M}_3(4)$. Its generic points correspond to plane quartics with a hyperflex point. We call the component containing a generic point the *hyperflex* locus $\Omega\mathcal{M}_3^{odd}(4)$. In [Hu17], I study the image $\mathcal{M}_3^{odd}(4)$ of the hyperflex locus in $\mathcal{M}_3$ (forgetting the differential), and prove the following theorem:

Theorem 1.1 (=Theorem 3.1)**.** *The modular form defined by:*

$$\begin{aligned}\Omega_{77}(\tau) :&= [\theta_{01}\theta_{10}\theta_{37}\theta_{43}\theta_{52}\theta_{75}\theta_{42}\theta_{06}\theta_{30}\theta_{21}\theta_{55} + \theta_{02}\theta_{25}\theta_{34}\theta_{40}\theta_{67}\theta_{76}\theta_{33}\theta_{05}\theta_{14}\theta_{60}\theta_{42}]^2 \\ &\quad - 4\theta_{01}\theta_{02}\theta_{10}\theta_{25}\theta_{34}\theta_{37}\theta_{40}\theta_{43}\theta_{52}\theta_{67}\theta_{75}\theta_{76}\theta_{00}\theta_{04}\theta_{57}\theta_{70}\theta_{61}\theta_{73}\theta_{20}\theta_{07}\theta_{00}\theta_{16}.\end{aligned}$$

cuts out the Torelli image of the locus of plane quartics with a hyperflex $\mathcal{M}_3^{odd}(4)$ *on a level cover of* $\mathcal{A}_3$*, where* $\mathcal{A}_3$ *is the moduli space of ppav of dimension* 3*.*

The θ_{ij} in the theorem are the *Riemann theta constants with even characteristics*, which we will properly define in section 3. Using the modular form and the degeneration of theta constants, we also compute the class of the closure of the hyperflex locus in the Deligne-Mumford compactification $\overline{\mathcal{M}}_3$:

$$[\overline{\mathcal{M}_3^{odd}(4)}] = 308\lambda - 32\delta_0 - 76\delta_1,$$

where λ is the Hodge class on $\overline{\mathcal{M}_3}$, and δ_0, δ_1 are the classes of the boundary divisors. This class is first computed in [Cuk89] using a different approach.

1.2. **Degeneration of abelian differentials and period matrices.** The variational formula of the period matrices in *plumbing coordinates* is first considered by Yamada [Yam80], Fay [Fay73], and Taniguchi [Tan89]. In [HN17], we improve their results in full generality. We give variational formulas for arbitrarily degenerate abelian differentials in plumbing coordinates:

Theorem 1.2 (=Theorem 4.1)**.** *Given any stable differential* $(X, \Omega) \in \partial\Omega\overline{\mathcal{M}}_g$ *where* X *is a stable curve with* n *nodes, there is an explicit construction of a family of smooth differentials* $(X_{\underline{s}}, \Omega_{\underline{s}})$ *where* $\underline{s} = (s_1, \dots, s_n)$ *are the plumbing coordinates. Moreover, the* $\underline{s}$*-expansion of* $\Omega_{\underline{s}}$ *can be written down explicitly to an arbitrary degree of precision.*

We then explicitly compute the variational formula for a general period of a given differential (see Theorem 4.2), and as a corollary, we give the variational formula of the period matrices (see Corollary 4.3). The special case $n = 1$ in Theorem 1.2 gives the main result by Yamada and Fay. Corollary 4.3 gives the same logarithmic term as in [Tan89]. We moreover explicitly compute the constant and linear terms.

1.3. **Incidence variety compactification of strata.** In [BCGGM16], the authors define the *incidence variety compactification* (IVC) of strata $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{inc}(\mu)$ by taking the closure of the strata in the projectivized compactification of the Hodge bundle $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$. They further prove that the existence of a *twisted differential* Ξ and a level function l on the vertices of the dual graph of X - with certain compatibility conditions - is necessary and sufficient for a stable differential (X, Ω) to lie in the boundary of the IVC. In [HN17], we give an alternative proof of such conditions, which gives more information about the neighborhood of the boundary.

1.4. **Future work.** I will now briefly describe some natural questions that I plan to investigate in the near future. Please find a more detailed discussion of these projects in section 5.

Project 1. Using the expansion of a general period of any abelian differential given in our paper [HN17], one can understand the degeneration of the *period coordinates* on the strata. Furthermore, one can study the boundary of *affine invariant submanifolds* in the strata, since by definition they are locally $\mathbb{R}$-linear in the period coordinates. From this approach, I look forward to giving an alternative proof of Filip's celebrated result on the algebraicity of the affine invariant submanifolds [Fil16].

Project 2. I want to study the Teichmüller curves in the minimal stratum in genus 3 using the modular form given in [Hu17]. It is known that the Teichmüller curves only intersect the deepest boundary strata where points correspond to the totally degenerate curves. The degeneration of the period matrix near such boundary strata is given in [HN17]. The understanding of such degenerations and the real-multiplication condition on the Jacobians gives strong constraints on the whereabouts of the Teichmüller curves in the minimal stratum.

Other projects. Our degeneration technique in [HN17] can also be applied in a more general setting to study the degeneration of any global section of a vector bundle as the curve degenerates, for instance, the Higgs field. Moreover, I am also interested in algebro-geometric questions about the strata, in particular their cohomology. It seems promising to give a bound on the dimension of the complete subvarieties in the strata by imitating the proof of Diaz's theorem given in Grushevsky-Krichever [GK09].

2. Background

2.1. Period coordinates. The strata $\Omega\mathcal{M}_{g,n}(\mu)$ are equipped with local coordinates. For a pointed abelian differential $(X; p_1, \ldots, p_n; \omega)$, a basis of the integral relative homology group $H_1(X, \Sigma; \mathbb{Z})$ is given by a symplectic basis $(A_1, \ldots, A_g; B_1, \ldots, B_g)$ of the absolute homology $H_1(X; \mathbb{Z})$, together with paths $(\gamma_1, \ldots, \gamma_{n-1})$ where γ_k connects p_1 and p_k. The *period coordinates* on strata are locally given by the integrals of ω over $\{A_i, B_i\}$ (these are called *absolute* coordinates), and over $\{\gamma_k\}$ (these are called *relative* coordinates).

2.2. Action of $\mathrm{SL}_2(\mathbb{R})$ and Teichmüller curves. A *translation structure* on a Riemann surface X is an atlas of complex charts $\{(U_\alpha, f_\alpha)\}_{\alpha \in I}$, where all transition functions are locally translations. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, the group $\mathrm{SL}_2(\mathbb{R})$ acts on the set of translation structures on X by post-composing the chart maps f_α with the linear map.

For an abelian differential (X, ω) it is known that $X \backslash Z_{red}(\omega)$ has a translation structure. Therefore the group $\mathrm{SL}_2(\mathbb{R})$ acts on the Hodge bundle $\Omega\mathcal{M}_g$. This action preserves the number and multiplicities of the zeroes of the 1-form, i.e., the stratum $\Omega\mathcal{M}_{g,n}(\mu)$ is $\mathrm{SL}_2(\mathbb{R})$-invariant. One can also interpret this action using the flat model of an abelian differential (see [Zor06], [Wri15]). The breakthrough work of Eskin-Mirzakhani-Mohammadi [EM13] [EMM15] gives:

Theorem 2.1. *Any $\mathrm{SL}_2(\mathbb{R})$-orbit closure is locally cut out by linear equations of real coefficients in period coordinates.*

The above-stated locally $\mathbb{R}$-linear objects are called *affine invariant submanifolds* of the strata. In this context, Filip [Fil16] further showed the following:

Theorem 2.2. *All affine invariant submaniolds are quasi-projective varieties.*

The closure of an $\mathrm{SL}_2(\mathbb{R})$-orbit can have dimension between 1 and the dimension of the stratum. When the orbit is closed, its image under the projection from $\Omega\mathcal{M}_g$ to $\mathcal{M}_g$ is totally geodesic with respect to the Teichmüller metric [Vee95] [SW04], and the converse is also true [McM03]. Such geodesics are called *Teichmüller curves.*

2.3. Moduli spaces of ppav and Hilbert modular varieties. The arithmetic perspective has proven to be effective in tackling the problem of classifying Teichmüller curves. One important arithmetic property of the Teichmüller curve is that for any point $[X]$ on the curve, $\mathrm{Jac}(X)$ admits *real multiplication* (see [McM03] for genus two and [Möl06] for general genus).

Let F be a *totally real* number field with degree g, i.e., F is a degree g extension of $\mathbb{Q}$ with the property that all its g embeddings into the complex numbers factor through $\mathbb{R}$. *Real multiplication* by F on a principally polarized abelian variety (ppav) A is an embedding $\rho : F \to \mathrm{End}_{\mathbb{Q}}(A)$. Such embeddings depend on the integral structure of F: ρ is determined by the underlying map from an order $\mathfrak{o}$ of F to $\mathrm{End}_{\mathbb{Z}}(A)$.

Let $\mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$ be the moduli space of ppav of dimension g, where $\mathbb{H}_g$ is the *Siegel upper half plane.* We have the *Torelli map* from $\mathcal{M}_g$ to $\mathcal{A}_g$ sending X to $\mathrm{Jac}(X)$. By the Riemann bilinear relations, the local coordinates on $\mathcal{A}_g$ are given by the *period matrices.* Given a symplectic basis $(A_1, \ldots, A_g; B_1, \ldots, B_g)$ of $H_1(X, \mathbb{Z})$. Let $\{v_1, \ldots, v_g\}$ be a normalized basis of $H^{1,0}(X, \mathbb{C})$ dual to the A-cycles. The period matrix of X is then defined as the $g \times g$ matrix $(\int_{B_i} v_j)_{i,j=1,\ldots,g}$.

Hilbert modular varieties $\mathcal{H}_{\mathfrak{o}} := \mathbb{H}^g / SL(\mathfrak{o} \otimes \mathfrak{o}^\vee)$ parametrize abelian varieties with real multiplication, and map to $\mathcal{A}_g$ by descending from the map $\phi : \mathbb{H}^g \to \mathbb{H}_g$:

$$\phi : (\tau_1, \ldots, \tau_g) \mapsto M \cdot \mathrm{diag}(\tau_1, \ldots, \tau_g) \cdot M^T,$$

where M is a $g \times g$ matrix determined by the order $\mathfrak{o}$ and the g distinct embeddings of F.

When $g = 2$, $\mathcal{H}_{\mathfrak{o}}$ is called the Hilbert modular surface. It is used in classifying the Teichmüller curves in genus two by McMullen [McM03]. A further study of the relationship between Hilbert modular varieties and Teichmüller curves in $g = 2, 3, 4$ is done by Bainbridge-Möller [BM12] [BM14], Bainbridge-Habegger-Möller [BHM16], and Möller-Zagier [MZ16].

3. Minimal stratum in genus three

3.1. The Riemann theta function and the theta-null modular form. The stratum $\Omega\mathcal{M}_3^{odd}(4)$ is called the *minimal stratum* in genus three. The locus $\mathcal{M}_3^{odd}(4) \subset \mathcal{M}_3$ obtained by forgetting the differential can be seen as the locus where one of the 28 bitangent lines of a plane quartic X is in fact a hyperflex line, i.e. its two tangent points comes together. Equivalently, it is the locus where X has a Weierstrass point p of weight 4, i.e. $4p \equiv K_X$. The classification of the affine invariant submanifolds of dimension ≥ 2 in $\Omega\mathcal{M}_3^{odd}(4)$ was done by Aulicino-Nguyen-Wright [ANW16]. The classification of the Teichmüller curves in this stratum is still an open problem.

It is known that for a given curve X, the *Riemann theta function with characteristics* is a section of the line bundle $\frac{1}{2}K$ twisted by a two-torsion line bundle. One can identify the set of two-torsion points in $\mathrm{Jac}(X)$ with $(\mathbb{Z}_2)^3 \times (\mathbb{Z}_2)^3$, sending $m = (\tau\epsilon + \delta)/2$ to the characteristics (ϵ, δ), where τ is the period matrix of X. Technically our discussion depends on the choice of such an identification, and hence should be on a level cover of $\mathcal{A}_3$. In order to simplify the discussion, we choose to neglect these issues here.

The Riemann theta constants with characteristics $\theta[{\epsilon \atop \delta}](\tau, 0)$ are known to be *modular forms* of weight $\frac{1}{2}$. The theta-null modular form is defined as $\Theta_{null}(\tau) := \prod_{(\epsilon,\delta)\text{even}} \theta[{\epsilon \atop \delta}](\tau, 0)$, where the parity on the characteristics is given by the Weil pairing. It is known that the theta-null modular form cuts out the hyperelliptic locus in genus 3.

3.2. A modular form for the stratum $\Omega\mathcal{M}_3^{odd}(4)$. In [Hu17], I give a modular form that cuts out the locus $\mathcal{M}_3^{odd}(4)$. For simplicity, let us denote the characteristics (ϵ, δ) by (i, j), where $i = 4\epsilon_1 + 2\epsilon_2 + \epsilon_3, j = 4\delta_1 + 2\delta_2 + \delta_3$. For instance, $([1,1,0],[0,1,1])$ is denoted by $(6,3)$.

Theorem 3.1 ([Hu17, Theorem 2.5])**.** *On $\mathcal{A}_3$, the modular form $\Omega_{77}(\tau)$ defined by*

$$\begin{aligned}\Omega_{77}(\tau) := {} & [\theta_{01}\theta_{10}\theta_{37}\theta_{43}\theta_{52}\theta_{75}\theta_{42}\theta_{06}\theta_{30}\theta_{21}\theta_{55} + \theta_{02}\theta_{25}\theta_{34}\theta_{40}\theta_{67}\theta_{76}\theta_{33}\theta_{05}\theta_{14}\theta_{60}\theta_{42}]^2 \\ & - 4\theta_{01}\theta_{02}\theta_{10}\theta_{25}\theta_{34}\theta_{37}\theta_{40}\theta_{43}\theta_{52}\theta_{67}\theta_{75}\theta_{76}\theta_{00}\theta_{04}\theta_{57}\theta_{70}\theta_{61}\theta_{73}\theta_{20}\theta_{07}\theta_{00}\theta_{16}\end{aligned}$$

vanishes at the period matrix τ of a smooth plane quartic X if and only if X has a Weierstrass point P of weight 4 such that the 2-torsion point $[\frac{1}{2}K_X - 2P]$ on A_τ corresponds to the characteristic $(i,j) = (7,7)$. Here $\theta_{ij} := \theta_{ij}(\tau, 0)$ is the Riemann theta constant with characteristics (i,j).

The requirement that the 2-torsion point correspond to the characteristic $(i,j) = (7,7)$ is merely a technical condition to fix the choice of the identification $A_\tau[2] \simeq (\mathbb{Z}/2\mathbb{Z})^6$. In other words, the modular form Ω_{77} cuts out a locus on the level two cover of $\mathcal{A}_3$ that maps one-to-one onto the image of $\mathcal{M}_3^{odd}(4)$ in $\mathcal{A}_3$.

3.3. Computation of the class. Let $\overline{\mathcal{M}}_3$ be the Deligne-Mumford compactification of $\mathcal{M}_3$. I use the modular form Ω_{77} to compute the divisor class of the closure of the locus $\mathcal{M}_3^{odd}(4)$ in $\overline{\mathcal{M}}_3$:

$$[\overline{\mathcal{M}_3^{odd}(4)}] = 308\lambda - 32\delta_0 - 76\delta_1,$$

where λ is the Hodge class on $\overline{\mathcal{M}}_3$, and δ_0, δ_1 are the classes of the boundary divisors. This class is first computed in [Cuk89] using a different method.

Generally speaking, the weight of a modular form F gives the multiplicity of the Hodge class in the class of the locus cut out by F. In order to compute the analogous coefficients of the boundary divisor classes, one computes the vanishing orders of F at the boundary components. We therefore study the degeneration of the theta constants with characteristics near the boundary of $\overline{\mathcal{M}}_3$, which requires an understanding of the degeneration of the period matrices.

4. Degeneration of periods

The variational formula of the period matrices in plumbing coordinates is studied by Yamada [Yam80] and Fay [Fay73] for the case when the stable curve has only 1 node. For the more general cases, Taniguchi [Tan89] computes the logarithmic term in the variational formula of the period matrices. However, their methods do not provide a variational formula for an arbitrary abelian differential, and hence are not enough for further usage such as computing the degeneration of period coordinates.

In the collaboration with C. Norton [HN17], we compute the Taylor expansion of any stable differentials in plumbing coordinates, and give the explicit variational formula for the degeneration of any periods of the differential near an arbitrary stable curve, in particular the period matrices, generalizing the results of Yamada-Fay and Taniguchi.

4.1. **Plumbing coordinates.** Given a stable nodal curve X with n nodes, the standard *plumbing* construction cuts out neighborhoods at the two pre-images q_e and q_{-e} of each node $q_{|e|}$ of X, and identifies their boundaries (called *seams*, denoted by $\gamma_{\pm e}$) via a gluing map I_e sending z_e to $z_{-e} := s_e/z_e$, where $|s_e| \ll 1$ is called the *plumbing parameter* and z_e and z_{-e} are chosen local coordinates near q_e and q_{-e} respectively. The plumbing construction gives a family of curves $\mathcal{X} \to \Delta$ with the central fiber identified with X, where Δ is the small polydisc neighborhood of $0 \in \mathbb{C}^n$ with coordinates given by the plumbing parameters $\underline{s} := (s_1, \ldots, s_n)$. Depending on circumstances $(s_1, \ldots, s_n)$ are also called the *plumbing coordinates*, as they give versal deformation coordinates on $\overline{\mathcal{M}}_g$ to the boundary stratum containing the point X.

4.2. **Degeneration of abelian differentials.** The technique we use to construct the degenerating family $\{X_{\underline{s}}, \Omega_{\underline{s}}\}$ is called (solving) the jump problem, which was developed and used in the real-analytic setting by Grushevsky-Krichever-Norton [GKN17]. Roughly speaking, given a stable differential Ω on X, we have the mis-matches $\{\Omega|_{\gamma_e} - I_e^*(\Omega|_{\gamma_{-e}})\}$ (which we call the *jumps* of Ω) on the seams $\gamma_{\pm e}$ at opposite sides of each node. The solution to the jump problem with initial conditions from Ω is a"correction" differential η that matches the jumps of Ω. By subtracting η from Ω on each irreducible component, one obtains new differentials with zero jumps, which can thus be glued to get a global meromorphic differential $\Omega_{\underline{s}}$ on $X_{\underline{s}}$. In [HN17], we construct explicitly the solution to the jump problem:

Theorem 4.1 ([HN17, Theorem 3.3])**.** *Let $(X, \Omega) \in \partial\Omega\overline{\mathcal{M}}_g$ be a stable differential. Let Ω_v be the restriction of Ω on the connected component X_v. For any $|\underline{s}|$ small enough, $\{\Omega_{v,\underline{s}} := \Omega_v + \eta_v\}$ defines a meromorphic differential $\Omega_{\underline{s}}$ on $X_{\underline{s}}$ satisfying $\Omega_v = \lim_{\underline{s}\to 0} \Omega_{\underline{s}}|_{X_v}$, where $\eta_v = \sum_{k=1}^{\infty} \eta_v^{(k)}$ is the unique solution with vanishing A-periods to the jump problem with the initial conditions from Ω. Furthermore, we have $||\eta_{v,\underline{s}}||_{L^2} = O(\sqrt{|\underline{s}|})$.*

Furthermore, we compute the leading term of the $\underline{s}$-expansion for $\eta_v^{(k)}$, which in particular gives the linear term of the $\underline{s}$-expansion for $\Omega_{\underline{s}}$. Let $l_v^k = (e_1, \ldots, e_k)$ be a path of length k starting from a given vertex $v = v(e_1)$ in the dual graph Γ_X. Let $\omega_v(z, w)$ be the fundamental normalized bidifferential on X_v and $\beta_{e,e'} := \mathrm{hol}(\omega_v)(q_e, q_{e'})$. The $\underline{s}$-expansion of $\eta_v^{(k)}$ is given by:

$$\eta_v^{(k)}(z) = (-1)^k \sum_{l_v^k} \prod_{i=1}^{k} s_{e_i} \cdot \omega_v(z, q_{e_1}) \prod_{j=1}^{k-1} \beta_{-e_j, e_{j+1}} \,\mathrm{hol}(\Omega)(q_{-e_k}) + O(|\underline{s}|^{k+1}), \tag{4.1}$$

where $z \in X_{\underline{s}}$. In the one node case, this expansion is the same as in [Yam80].

4.3. **Degeneration of periods.** Denote $r_e := \mathrm{res}_{q_e} \Omega$. We have $r_e = -r_{-e}$. Let α be any oriented loop in X. Let $\{q_1, \ldots, q_N\}$ be the ordered collection of nodes that α passes through (with possible repetition). Let $\alpha_{\underline{s}}$ be a perturbation of α such that its restrictions on each X_v minus the caps at each node glue correctly to give a loop on $X_{\underline{s}}$.

Theorem 4.2 ([HN17, Theorem 4.1])**.** *The variational formula of the period of $\Omega_{\underline{s}}$ over $\alpha_{\underline{s}}$ is given by:*

$$\int_{\alpha_{\underline{s}}} \Omega_{\underline{s}} = \sum_{i=1}^{N} (r_{e_i} \ln |s_{e_i}| + c_i + l_i) + O(|\underline{s}|^2),$$

here c_i and l_i are the constant and linear terms in $\underline{s}$ respectively, which are explicitly given.

The expression for the constant and linear terms can be found in our paper [HN17, Theorem 4.1]. This theorem in particular shows that $\int_{\alpha_{\underline{s}}}\Omega_{\underline{s}} - \sum_{i=1}^{N} r_{e_i}\ln|s_{e_i}|$ is holomorphic in $\underline{s}$. This observation along with the following corollary gives the main result in [Tan89].

For the degeneration of the period matrices, we choose a suitable symplectic basis $\{A_{k,\underline{s}}, B_{k,\underline{s}}\}_{k=1}^{g}$ of $H_1(X_{\underline{s}},\mathbb{Z})$. We take a normalized basis $\{v_1,\ldots,v_g\}$ of $H^1(X,\mathbb{C})$, such that after applying the jump problem construction, the resulting set of differentials $\{v_{k,\underline{s}}\}_{k=1}^{g}$ is a normalized basis of $H^1(X_{\underline{s}},\mathbb{C})$. We then have:

Corollary 4.3 ([HN17, Corollary 4.5]). *For any fixed h,k, the expansion of $\tau_{h,k}(\underline{s})$ is given by*

$$\tau_{h,k}(\underline{s}) = \sum_{e\in E_X}\frac{N_{|e|,h}\cdot N_{|e|,k}}{2}\cdot\ln|s_e| + c_{h,k} + l_{h,k},$$

where $N_{|e|,k} := \gamma_{|e|}\times B_{k,\underline{s}}$, and E_X is the set of oriented edges of the dual graph of X. Moreover, let $\{q_{|e_i|}\}_{i=0}^{N-1}$ be the set of nodes B_h passes through, then the constant term $c_{h,k}$ and the linear term $l_{h,k}$ are given explicitly as follows:

$$\begin{aligned} c_{h,k} &= \lim_{\underline{s}\to 0}\sum_{i=1}^{N}\left(\int_{z_{-e_{i-1}}^{-1}(\sqrt{|s_{e_{i-1}}|})}^{z_{e_i}^{-1}(\sqrt{|s_{e_i}|})} v_k - N_{|e_i|,h}N_{|e_i|,k}\ln|s_{e_i}|\right)\\ l_{h,k} &= -\sum_{e\in E_X} s_e\left(\mathrm{hol}(v_k)(q_e)\,\mathrm{hol}(v_h)(q_{-e})\right) + O(|\underline{s}|^2), \end{aligned}$$

4.4. **Alternative approach to the incidence variety compactification of strata.** As introduced in section 1.3, in Bainbridge-Chen-Gendron-Grushevsky-Möller [BCGGM16] the authors construct the *incidence variety compactification* (IVC) of strata, and they give the necessary and sufficient conditions for a stable pointed differential to lie in the boundary of the IVC. The obviously harder direction in their proof is the sufficiency of the conditions, which essentially requires a construction of a degenerating family of abelian differentials in the stratum to the given limit differential (X,Ω) with the compatible data (Ξ, l). In [BCGGM16], the authors give two such constructions by using a plumbing argument and a flat geometry argument respectively. Both arguments only give rise to one-parameter degenerating families.

In [HN17, Theorem 6.4] we construct, via the jump problem approach, a degenerating family that is different to the two mentioned above. The number of parameters in our degenerating family is equal to the number of levels in the level graph Γ_X minus 1, which is the maximal number of parameters allowed in such degenerating family. In particular, we reprove the sufficiency of the conditions for a stable differential to lie in the boundary of the IVC.

5. Work in progress

5.1. **Degeneration of period coordinates and algebraicity of affine invariant submanifolds.** Using Theorem 4.2 and [HN17, Theorem 6.4], I expect similar variational formulas can be written down for the period coordinates on the strata. There is a small difficulty: the solution of the jump problem η will break a zero (of the original differential) with higher multiplicity into a number of nearby simple zeroes. In our proof of the sufficiency of the IVC, we applied a local surgery to merge these simple zeroes back to the correct multiplicity. The expansion for the differential after this local surgery remains to be computed. Fortunately, this is a minor problem that is expected to be solved by applying the jump problem technique again.

The understanding of the degeneration of period coordinates will allow me to study the local structures near the boundary of the strata. Therefore I expect to extend the results in Mirzakhani-Wright [MW17] to the boundary of the IVC compactification. Furthermore, since the affine invariant submanifolds are defined locally to be $\mathbb{R}$-linear in period coordinates, I want to study the closure of these objects in the IVC by extending the local defining equations onto the boundary. This will lead to an alternative proof of the algebraicity of the strata (Theorem 2.2). The current proof in [Fil16] uses Hodge theory and dynamics, and no other proof is known.

5.2. **Classification of Teichmüller curves in genus 3.** Theta constants have proven to be useful in finding Teichmüller curves (see Möller-Zagier [MZ16]), since they can be viewed as Hilbert modular forms when we pull them back from $\mathcal{A}_g$ to the Hilbert modular varieties. Therefore both the theta-null modular form and our hyperflex modular form can be seen as Hilbert modular forms. Since the Hilbert modular varieties parametrize abelian varieties with real multiplication, these are the only possible places for Teichmüller curves to exist. Therefore the Teichmüller curves in the hyperelliptic locus in genus three and the stratum $\Omega\mathcal{M}_3^{odd}(4)$ must lie in the locus cut out by the corresponding modular forms.

It is a known fact that Teichmüller curves will always intersect the deepest boundary strata whose points correspond to totally degenerate curves. In [HN17] we compute as an example the explicit variational formula for period matrices of totally degenerate curves, which was previously unknown. One can now study the totally degenerate theta constants and give a description of the possible locus in the boundary where Teichmüller curves are allowed to intersect.

5.3. **More general setting for the jump problem.** The idea of solving the jump problem is to glue the "corrected" differential over the seams near the nodes of the Riemann surface. More generally we can apply this approach to smooth the sections of any locally free sheaf over a nodal Riemann surface. In this way one can construct a degenerating family of pairs $(X_{\underline{s}}, \theta_{\underline{s}})$, where $\theta_{\underline{s}}$ is a global section of some vector bundle $E_{\underline{s}}$ on $X_{\underline{s}}$. Interesting examples include the Higgs bundles. Via the jump problem approach, I hope to write down similar variational formulas for the degeneration of a Higgs field.

I also want to consider the moduli spaces of semi-stable vector bundles of fixed rank and degree over a Riemann surface X, and study the degeneration of such moduli spaces when X degenerates. In [Pa96], Pandharipande constructs such a compactification of the universal moduli spaces of slope-semistable vector bundles using GIT. I hope to study the boundary of this compactification in our analytic jump problem approach.

5.4. **Global geometry of strata.** There is little known about the algebro-gemetric properties of the affine invariant submanifolds. We try to understand the cohomology of the (open) strata by studying the complete subvarieties in them. A recent result by Chen [Ch17] shows that there are no complete curves in strata of strictly meromorphic abelian differentials.

In the well-studied moduli spaces of curves, we have the theorem by Diaz [Di85] saying that $\mathcal{M}_g$ does not contain complete (complex) subvarieties of dimension greater than $g-2$. This result was later reproved by Grushevsky-Krichever [GK09] and again by Krichever [Kr11], whose approaches can hopefully be applied to the strata. Their proofs use the *real-normalized* differentials, i.e., differentials whose absolute periods are real. They work on the *rel foliation* of the real-normalized differential where the absolute periods of the differential are fixed. Since the absolute periods are real, the imaginary parts of the relative periods gives local coordinates on the leaves of the foliation. The relative periods are holomorphic, therefore their imaginary parts are harmonic functions on the leaves. As a result, by restricting any complete subvarieties of dimension larger than $g-2$ to such leaves, one gets that imaginary parts of the relative periods (as local coordinates) become constants on the restriction. This proves Diaz's theorem.

I want to show a Diaz-type theorem for the strata of abelian differentials, namely, I am trying to give a bound on the dimension of the complete subvarieties contained in the strata by imitating the approach used in [GK09]. I start by studying the intersection of the leaves of the rel foliation of a real-normalized differential and a stratum of abelian differentials. Understanding the interplay of these two differentials is most certain to lead to some interesting results.

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