MAT 126 Midterm

Name: ______________________________

Problem	1	2	3	4	5	Total	Bonus
Points	**30**	**30**	**20**	**10**	**10**	**100**	**15**
Scores							

This midterm has five problems, weighted as shown. Please show your work – full credit may not be given if only the answers appear. **No calculators or books will be allowed on this test.** When calculating indefinite integrals, the answers should be in explicit forms, i.e. don't use part1 of Fundamental Theorem of Calculus, unless otherwise stated.

1. Evaluate each of the following definite integrals.

(a) $\int_0^1 (x^3 + 1)\, dx$

antiderivative for x^3+1 is $\frac{1}{4}x^4+x$

so the integral $= \frac{1}{4}x^4+x\Big|_0^1 = \frac{1}{4}\cdot 1^4+1-0$

$= \frac{1}{4}+1 = \boxed{\frac{5}{4}}$

(b) $\int_0^1 (x+1)^3\, dx$

(u-substitution) let $u = x+1 \Rightarrow du = dx$

$\int_0^1 (x+1)^3 dx = \int_0^1 u^3 du = \frac{1}{4}u^4$

$= \frac{1}{4}(x+1)^4\Big|_0^1$

$= \frac{1}{4}\cdot 2^4 - \frac{1}{4}\cdot 1^4$

$= 4 - \frac{1}{4} = \boxed{\frac{15}{4}}$

(c) $\int_1^2 x^2 \ln x \, dx$

(intergration by Parts)

let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x^2 dx \qquad v = \frac{1}{3} x^3$

$$\int_1^2 x^2 \ln x \, dx = \int_1^2 u \, dv = u \cdot v - \int v \, du$$

$$= \ln x \cdot \left(\frac{1}{3} x^3\right)\Big|_1^2 - \int_1^2 \frac{1}{3} x^{\cancel{3}^{2}} \cdot \frac{1}{\cancel{x}} dx$$

$$= \frac{1}{3} \ln x \cdot x^3 \Big|_1^2 - \frac{1}{3} \int_1^2 x^2 dx$$

$$= \frac{1}{3} \ln x \cdot x^3 \Big|_1^2 - \frac{1}{3} \cdot \frac{1}{3} x^3 \Big|_1^2$$

$$= \left(\frac{1}{3} \ln x \cdot x^3 - \frac{1}{9} x^3\right)\Big|_1^2$$

$$= \left(\frac{1}{3} \cdot 8 \cdot \ln 2 - \frac{1}{9} \cdot 8\right) - \left(\frac{1}{3} \underbrace{\ln 1}_{0} \cdot 1^3 - \frac{1}{9}\right)$$

$$= \frac{8}{3} \ln 2 - \frac{8}{9} + \frac{1}{9}$$

$$= \boxed{\frac{8}{3} \ln 2 - \frac{7}{9}.}$$

2. Calculate each of the following indefinite integrals.

(a) $\int x^3 e^{-x^2}\,dx$

let $y = -x^2 \quad dy = -2x\,dx$

$\int x^3 e^{-x^2} dx = \int x^2 e^{-x^2} (x\,dx)$

$= \int (-y) \cdot e^{y} (-\frac{1}{2}) dy$

$= \frac{1}{2} \int y\, e^{y}\, dy$

integration by Parts: let $u = y \quad du = dy$

$dv = e^{y} dy \quad v = e^{y}$

$\frac{1}{2}\int y e^{y} dy = \frac{1}{2}[y \cdot e^{y} - \int e^{y} dy]$

$= \frac{1}{2}[y \cdot e^{y} - e^{y}] \Rightarrow \boxed{\frac{1}{2}[(-x^2)e^{-x^2} - e^{-x^2}] + C}$

(b) $\int \frac{3x+4}{x^2+x-6}\,dx$

let $\frac{3x+4}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$

then $\cancel{\frac{3x+4}{x^2+x-6}}\ \frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x-2)+B(x+3)}{(x+3)\cdot(x-2)}$

So $A(x-2) + B(x+3) = 3x+4$

let $x = 2 \Rightarrow A \cdot 0 + B(2+3) = 3 \cdot 2 + 4$

$\Leftrightarrow 5 \cdot B = 10 \Rightarrow B = 2$

let $x = -3 \Rightarrow A(-3-2) + B \cdot 0 = (-3) \cdot 3 + 4$

$\Leftrightarrow -5 \cdot A = -5 \Rightarrow A = 1$

So $\int \frac{3x+4}{x^2+x-6} dx = \int \frac{1}{x+3} dx + \int \frac{2}{x-2} dx$

$= \boxed{\ln|x+3| + 2 \cdot \ln|x-2| + C}$

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(c) $\int \cos^2 x \sin^2 x \, dx$

Use $\sin(2x) = 2\sin x \cdot \cos x$

then $\cos^2 x \sin^2 x = \left(\frac{1}{2}\sin(2x)\right)^2$

$= \frac{1}{4}\sin^2(2x)$

$\int \cos^2 x \sin^2 x \, dx = \frac{1}{4}\int \sin^2(2x)\, dx$

let $u = 2x$
$du = 2\,dx$ $\Rightarrow \frac{1}{4}\int \sin^2(2x)\, dx$

$= \frac{1}{4}\int \sin^2 u \cdot \frac{1}{2}\, du$

$= \frac{1}{8}\int \sin^2 u \, du$

Use double angle formula:

$\sin^2 u = \frac{1}{2}(1 - \cos(2x))$

then the integral $= \frac{1}{8}\int \frac{1}{2}(1 - \cos 2x)\, dx$

$= \frac{1}{16}\int 1 - \cos 2x \, dx$

$= \frac{1}{16}\left(x - \frac{1}{2}\sin 2x\right)$

$= \boxed{\frac{1}{16}x - \frac{1}{32}\sin 2x}$

3. Calculate each of the following indefinite integrals.

(a) $\int \sin^3 x \, dx$

$$\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx$$
$$= \int (1-\cos^2 x) \cdot \sin x \, dx$$

let $u = \cos x$ $\quad du = -\sin x \, dx$

$$\int (1-\cos^2 x) \sin x \, dx = \int (1-u^2)(-du)$$
$$= -\int (1-u^2) \, du$$
$$= -\int 1 \, du + \int u^2 \, du$$
$$= -u + \frac{1}{3}u^3$$
$$= -\cos x + \frac{1}{3}\cos^3 x + C.$$

(b) $\int \frac{x^3}{\sqrt{1-x^2}} \, dx$

let $x = \sin\theta \Rightarrow \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

and $dx = \cos\theta \, d\theta$

$$\int \frac{x^3}{\sqrt{1-x^2}} dx = \int \frac{\sin^3\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos\theta} \, d\theta$$
$$= \int \sin^3\theta \, d\theta$$
$$= -\cos\theta + \frac{1}{3}\cos^3\theta + C \quad \text{(by the last problem)}$$

4. Express the following limit as a definite integral. Do not evaluate the definite integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(\sin\frac{2i}{n}\right)^3$$

$\frac{2}{n}$ should be $\Delta x = \frac{b-a}{n}$

then $b=2$ $a=0$

so $x_i = 0 + i \cdot \left(\frac{2}{n}\right)$

$= \frac{2i}{n}$

so $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(\sin\frac{2i}{n}\right)^3$

$= \lim_{n\to\infty} \sum_{i=1}^{n} (\sin x_i)^3 \cdot \Delta x$

$= \int_1^2 (\sin x)^3\, dx$

5. Find the derivative of the following function.

$$f(x) = \int_{\ln x}^{e^x} \arctan t \, dt$$

$$f(x) = \int_{\ln x}^{e^x} \arctan(t) dt = \int_{\ln x}^{0} \arctan(t) dt$$

$$+ \int_{0}^{e^x} \arctan(t) dt$$

$$= - \int_{0}^{\ln x} \arctan(t) dt$$

$$+ \int_{0}^{e^x} \arctan(t) dt$$

take derivatives, by FTC and Chain Rule

have to mention both!

$$f'(x) = - \arctan(\ln x) \cdot (\ln x)' + \arctan(e^x)(e^x)'$$

$$= \boxed{- \arctan(\ln x) \cdot \frac{1}{x} + \arctan(e^x) \cdot e^x}$$

6. (Bonus) Evaluate the following indefinite integral.

$$\int e^{\arcsin x}\, dx$$

let $y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow dx = \cos y \cdot dy$

$$\int e^{\arcsin x} dx = \int \underbrace{e^{y}}_{e^{\arcsin x}} \underbrace{\cos y\, dy}_{dx}$$

let $u = e^y$ $\quad du = e^y dy$

$dv = \cos y\, dy$ $\quad v = \sin y$

then int. by Parts:

$$\int u\, dv = u \cdot v - \int v\, du = e^y \cdot \sin y - \int \sin y \cdot e^y dy$$

for $\int \sin y \cdot e^y dy$, int by Parts again:

let $u = e^y$ $\quad du = e^y dy$

$dv = \sin y\, dy$ $\quad v = -\cos y$

$$\int \sin y \cdot e^y dy = e^y \cdot (-\cos y) + \int \cos y \cdot e^y dy$$

then $\int e^y \cdot \cos y\, dy = e^y \sin y - (-e^y \cos y + \int \cos y\, e^y dy)$

$= e^y \sin y + e^y \cdot \cos y - \int \cos y\, e^y dy$

$\Rightarrow \int e^y \cdot \cos y\, dy = \frac{1}{2} e^y (\sin y + \cos y) + C$

$\Rightarrow \int e^{\arcsin x} dx = \frac{1}{2} e^{\arcsin x} (x + \sqrt{1-x^2}) + C$