MAT 126 Practice Final

Name: ______________________________

Problem	1	2	3	4	5	**Total:**
Points	**50**	**10**	**20**	**10**	**10**	**100**
Scores						

No calculators or books will be allowed on this test. When calculate indefinite integrals, the answers should be in explicit forms, i.e. not using part1 of Fundamental Theorem of Calculus, unless otherwise stated.

1. Evaluate each of the following limits, definite/indefinite or improper integrals.

(a) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \cos\frac{3i}{n}$

$\Delta x = \frac{1}{n} = \frac{b-a}{n} \Rightarrow b-a=1$ can set $a=0$, $b=1$

$x_i = a + i\left(\frac{b-a}{n}\right) = 0 + i\cdot\frac{1}{n} = \frac{i}{n}$

$\Rightarrow \cos\frac{3i}{n} = \cos(3x_i)$

So $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \cos\frac{3i}{n} = \lim_{n\to\infty} \Delta x \cdot \sum_{i=1}^{n} \cos 3x_i = \int_0^1 \cos(3x)\,dx$

let $u = 3x$ $du = 3dx \Rightarrow \int_0^1 \cos(3x)\,dx$

$= \int_0^1 \cos u \cdot \frac{1}{3}du$

$= \frac{1}{3}\sin(3x)\Big|_0^1 = \frac{1}{3}\sin 3 - \frac{1}{3}\sin 0$

$= \boxed{\frac{1}{3}\sin 3}$

(b) $\int \arctan(x+2)\,dx$

substitute $y = x+2$
$dy = dx$

$\int \arctan(y)\,dy$ int. by Parts $u = \arctan y$ $du = \frac{1}{1+y^2}dy$
$dv = dy$ $v = y$

$uv - \int v\,du = y\cdot\arctan y - \int \frac{y}{1+y^2}dy$

let $z = 1+y^2$ $dz = 2y\,dy$

$\int \frac{y}{1+y^2}dy = \int \frac{1}{z}\left(\frac{1}{2}dz\right) = \frac{1}{2}\ln|z| = \frac{1}{2}\ln|1+y^2|$

so $\int \arctan(x+2) = \boxed{(x+2)\cdot\arctan(x+2) - \frac{1}{2}\ln\left|1+(x+2)^2\right| + C}$

(c) $\displaystyle\int \frac{x-5}{3x^2-2x-1}\,dx$

$3x^2-2x-1 = (3x+1)(x-1)$

so $\dfrac{x-5}{3x^2-2x-1} = \dfrac{A}{3x+1} + \dfrac{B}{x-1} \Rightarrow A(x-1)+B(3x+1) = x-5$

or let $x=1 \Rightarrow 4B=-4$

$B=-1$

$x=-\frac{1}{3} \Rightarrow A\cdot\left(-\frac{4}{3}\right) = -\frac{16}{3}$

$A=4$

$\Rightarrow (A+3B)x+(B-A) = x-5$

$\Rightarrow A+3B=1$, $B-A=5$ $\Rightarrow$ $A=4$, $B=-1$

so $\displaystyle\int \frac{x-5}{3x^2-2x-1}\,dx = \int \frac{4}{3x+1}\,dx + \int \frac{-1}{x-1}\,dx$

$\displaystyle = 4\cdot \int \frac{1}{3x+1}\,dx - \int \frac{1}{x-1}\,dx$

$= 4\cdot\frac{1}{3}\ln|3x+1| - \ln|x-1| + C.$

(d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

let $x = \sin\theta$ then $\sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\, d\theta$

So $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{\sin\theta}{\cancel{\cos\theta}}\cdot \cancel{\cos\theta}\, d\theta$

$\displaystyle = \int \sin\theta\, d\theta$

$= -\cos\theta + C$

$= -\cos(\arcsin x) + C$

or $= -\sqrt{1-x^2} + C$

(e) $\int_5^6 \frac{x^2}{(x-6)^3}\,dx$

6 is not in the domain of $\frac{x^2}{(x-6)^3}$, so

~~$\int_5^7 \frac{x^2}{(x-6)^3}dx = \int_5^6 \frac{x^2}{(x-6)^3}dx + \int_6^7 \frac{x^2}{(x-6)^3}dx$~~

~~both are improper integrals.~~

$$\int_5^6 \frac{x^2}{(x-6)^3}dx = \lim_{t\to 6}\int_5^t \frac{x^2}{(x-6)^3}dx$$

let $u = x-6$, $du = dx$

and $x = u+6$.

So $\int_5^t \frac{x^2}{(x-6)^3}dx = \int_{-1}^{t-6} \frac{(u+6)^2}{u^3}du$

$$= \int_{-1}^{t-6} \frac{u^2+12u+36}{u^3}du$$

$$= \int_{-1}^{t-6} u^{-1}+12u^{-2}+36u^{-3}\,du$$

$$= \ln|u| - 12u^{-1} - 18u^{-2}\Big|_{-1}^{t-6}$$

$$= \ln|x-6| - 12\frac{1}{x-6} - 18\frac{1}{(x-6)^2}\Big|_5^t$$

and $\lim_{t\to 6} \ln|x-6| - \frac{12}{x-6} - \frac{18}{(x-6)^2} = 0 - \infty$

so the improper integral is <u>divergent</u>.

2. Find the area enclosed by parabolas $x = y^2 - y$ and $x = 3y - y^2$.

$$A = \int_a^b (y^2 - y) - (3y - y^2)\,dy$$

Find a, b. set $y^2 - y = 3y - y^2$

$$\Rightarrow 2y^2 = 4y \Rightarrow 2y(y-2) = 0$$

$$\Rightarrow y = 0 \text{ or } y = 2$$

so $A = \int_0^2 (y^2 - y) - (3y - y^2)\,dy$

$$= \int_0^2 2y^2 - 4y\,dy$$

$$= \cancel{\int_0^2} \frac{2}{3}y^3 - 2y^2 \Big|_0^2$$

$$= \frac{2}{3}\cdot 8 - 2\cdot 4 = -\frac{8}{3}.$$

hence switch the order of $y^2 - y$ and $3y - y^2$,

$$A = \int_0^2 (3y - y^2) - (y^2 - y)\,dy$$

$$= \boxed{\frac{8}{3}}$$

3. Find the following volumes, please indicate which method (washer/cylindrical shell) you are using.

(a) The solid obtained by rotating about x-axis the region enclosed by $x = 0$, $x = 1$, $y = e^{2x}$ and $y = (e^2 - 1)x + 1$.

should use washer method

$$\text{so } V = \int_0^1 \pi r_{out}^2(x) - \pi r_{in}^2(x)\, dx$$

$$= \int_0^1 \pi (e^{2x})^2 - \pi((e^2-1)x+1)^2\, dx$$

$$= \int_0^1 \pi \cdot e^{4x}\, dx - \int_0^1 \pi \cdot ((e^2-1)^2 x^2 + 2(e^2-1)x + 1)\, dx$$

$$= \frac{1}{4} \cdot \pi \cdot e^{4x}\Big|_0^1 - \pi\left[\frac{(e^2-1)^2}{3}x^3 + (e^2-1)x^2 + x\right]_0^1$$

$$\boxed{= \left(\frac{1}{4}\pi \cdot e^4 - \frac{1}{4}\pi\right) - \pi\left[\frac{(e^2-1)^2}{3} + (e^2-1) + 1\right]}$$

Don't need to switch / determine if this is positive.

(also in Final)

(b) The solid obtained by rotating about y-axis the region enclosed by $x = 2$, $x = 3$, $y = e^x + x + 2$ and $y = 0$.

Use shell method

$$V = \int_2^3 2\pi\, r(x) \cdot h(x)\, dx \qquad r(x) = x$$

$$h(x) = e^x + x + 2 - 0.$$

$$= \int_0^3 2\pi \cdot x\,(e^x + x + 2)\, dx$$

$$= \int_0^3 2\pi\,(x \cdot e^x + x^2 + 2x)\, dx$$

$$= 2\pi \cdot \left(\int_0^3 x \cdot e^x\, dx + \int_0^3 x^2\, dx + \int_0^3 2x\, dx \right)$$

For $\int_0^3 x \cdot e^x dx$, use int. by Parts:

let $u = x \qquad du = dx$

$dv = e^x dx \qquad v = e^x$

$$\int_0^3 x \cdot e^x\, dx = x \cdot e^x \Big|_0^3 - \int_0^3 e^x\, dx$$

$$= x \cdot e^x \Big|_0^3 - e^x \Big|_0^3$$

$$= 3 \cdot e^3 - e^3 + e^0 = 2e^3 + 1$$

$$\int_0^3 x^2\, dx = \frac{1}{3} x^3 \Big|_0^3 = 9.$$

$$\int_0^3 2x\, dx = x^2 \Big|_0^3 = 9$$

so $V = 2\pi \cdot (2e^3 + 1 + 9 + 9)$

$$= 2\pi \cdot (2e^3 + 19)$$

4. $y(x) = \int_1^x \sqrt{t^2 + 2t}\, dt$. Find the exact length of the graph of y on the interval $[1, 4]$ (namely $1 \le x \le 4$).

Arc length formula :

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

FTC : $\frac{d}{dx}\left(\int_1^x \sqrt{t^2+2t}\,dt\right) = \sqrt{x^2+2x}$

$$= \int_1^4 \sqrt{1 + \left(\sqrt{x^2+2x}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{x^2+2x+1}\, dx$$

$$= \int_1^4 \sqrt{(x+1)^2}\, dx$$

$$= \int_1^4 (x+1)\, dx$$

$$= \frac{1}{2}x^2 + x \Big|_1^4$$

$$= (8+4) - \left(\frac{1}{2}+1\right)$$

$$= 10\frac{1}{2} = \boxed{\frac{21}{2}}$$

5. Find number c inside the interval $[-1, 3]$ such that $f(c)$ is equal to the average value of $f(x) = x^2 - 2x - 8$ on that interval.

Ave of $f(x)$ on $[-1,3]$:

$$f_{ave} = \frac{1}{3-(-1)} \int_{-1}^{3} x^2 - 2x - 8 \, dx$$

$$= \frac{1}{4} \cdot \left(\frac{1}{3}x^3 - x^2 - 8x\right)\Big|_{-1}^{3}$$

$$= \frac{1}{4} \cdot \left[\left(\frac{1}{3} \cdot 3^3 - 9 - 24\right) - \left(\frac{1}{3}(-1)^3 - 1 + 8\right)\right]$$

$$= \frac{1}{4}\left[(9 - 9 - 24) - \left(-\frac{1}{3} + 7\right)\right]$$

$$= \frac{1}{4}\left(-\frac{92}{3}\right) = -\frac{23}{3}$$

$$f(c) = c^2 - 2c - 8 = -\frac{23}{3}$$

then $3c^2 - 6c - 24 = -23$

$\Rightarrow 3c^2 - 6c - 1 = 0$

$$c = \frac{6 \pm \sqrt{36+12}}{6} = \frac{6 \pm 4\sqrt{3}}{6}$$

$$= \boxed{\frac{3 \pm 2\sqrt{3}}{3}}$$