Meet spin geometry

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A dizzyingly quick intro to "Spin geometry" by Lawson & Michelsohn

Spin structures and spin manifolds

We begin with the notion of a spin structure on a general oriented vector bundle. Recall that, if $E \to B$ is a rank n vector bundle, then it is determined by its transition functions g_{ij} mapping into the general linear group $GL(n, \mathbb{R})$. If we further know that E is equipped with a continuous choice of inner product on its fibers, we may take our transition functions to map into the orthogonal group $O(n)$. This is called a *reduction* of the structure group. We can reduce the group further to $SO(n)$ assuming orientability of E. For our oriented vector bundle, there is an associated principal $SO(n)$-bundle over B, the *bundle of frames* of E, which we denote by $P_{so}(E)$. Each fiber of this associated bundle is the set of orthonormal bases for a fiber of E. Note that as $SO(n)$ acts freely and transitively on frames, we may identify each fiber (although not canonically) with $SO(n)$.

As $\pi_1(SO(n)) = \mathbb{Z}/2$ for $n \geq 3$, there is a simply connected 2-fold covering of $SO(n)$, known as $Spin(n)$. For $n = 2$, we take $Spin(n)$ to be the circle, double covering itself. As coverings of Lie groups are Lie groups, $Spin(n)$ itself is a Lie group. If we can *lift* the structure group to $Spin(n)$, then we say that our oriented bundle E is *spinnable*.

Definition. *For a spinnable vector bundle $E \to B$, a* spin structure *$P_{spin}(E) \to B$ is a principal $Spin(n)$-bundle that nontrivially and equivariantly double covers $P_{so}(E)$ on each fiber, and such that the following diagram commutes:*

$$\begin{tikzcd} P_{spin}(E) \arrow[rr] \arrow[dr] & & P_{so}(E) \arrow[dl] \\ & B & \end{tikzcd}$$

where the horizontal map is the equivariant nontrivial double cover, and the slanted maps are the bundle projections.

There is some subtlety regarding the equivalence of spin structures. It is not enough that they are principal bundle isomorphic, i.e., that there exists a map on the total spaces that commutes with the action of the bundle and covers the identity over B. Such an isomorphism should *also* cover the identity over $P_{so}(E)$ (so that the following diagram commutes):

$$\begin{tikzcd} P_{spin}(E) \arrow[rr, "\simeq"] \arrow[dr] \arrow[ddr] & & P'_{spin}(E) \arrow[dl] \arrow[ddl] \\ & P_{so}(E) \arrow[d] & \\ & B & \end{tikzcd}$$

where the horizontal map is a principal $Spin(n)$-bundle isomorphism (it is $Spin(n)$-equivariant) and the slanted double covering maps are also equivariant.

Example. A word of caution regarding the above subtlety: two spin structures may be equivalent as bundles, but not equivalent as spin structures. Consider spin structures over the circle S^1.

We will see later that spin structures are classified up to equivalence by H^1 with $\mathbb{Z}/2$ coefficients, so there are two distinct spin structures over the circle. However, any principal G-bundle over the circle with G connected is bundle isomorphic to the trivial one, since such bundles are classified up to bundle isomorphism by homotopy classes of maps $[S^1 \to BG] = \pi_1 BG = \pi_0 G = 0$. To be equivalent as spin structures imposes additional requirements on the diagram, as seen above.

A reduction/lift of the structure group for vector bundles can be formulated in terms of lifts of classifying maps to the appropriate classifying spaces, as per classical bundle homotopy theory.

Proposition. *$E \to B$ is spinnable iff there is a lift of the classifying map $B \to BSO(n)$ to a map $B \to BSpin(n)$:*

$$\begin{tikzcd} & BSpin(n) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & BSO(n) \end{tikzcd}$$

Thus, the only obstruction to E being spinnable is the second Stiefel-Whitney class $w_2(E)$.

The above remark follows from the fact that the fiber on the right hand side is $K(\mathbb{Z}/2, 1) = \mathbb{RP}^\infty$ so the only obstruction to extending a section is in $H^2(BSO(n), \pi_1(\mathbb{RP}^\infty)) = H^2(BSO(n), \mathbb{Z}/2) = \mathbb{Z}/2$, which is the universal second Stiefel-Whitney class. Note that the equivalence of spin structures can be formulated in terms of classifying maps as two lifts *covering the same map* being *vertically* homotopic (i.e., for each $b \in B$ we have $H_b(t)$ in the fiber over the image of b for all t). As w_2 is the primary obstruction to trivializing the bundle E over the 2-skeleton of B, we note that the geometric interpretation of a bundle being spinnable is equivalent to it being trivial over the 2-skeleton. As a Cech $\mathbb{Z}/2$-cocycle, w_2 determines whether a consistent choice of lift for the transition maps to $Spin(n)$ can be made globally.

Lemma. *An oriented vector bundle $E \to B$ of rank $n \geq 3$ is spinnable iff for every continuous map $f : \Sigma \to B$ from a surface Σ we have that f^*E is trivial.*

Proof. $w_2(E) = 0$ iff $f^* w_2(E) = w_2(f^*E) = 0$ for all $f : \Sigma \to B$, since we have

$$\langle w_2(f^*E), [\Sigma] \rangle = \langle f^* w_2(E), [\Sigma] \rangle = \langle w_2(E), f_*[\Sigma] \rangle$$

and $H_2(B, \mathbb{Z}/2)$ is generated by all such $f_*[\Sigma]$ elements. Note the above does not use the rank or orientability of our bundle. However, a rank $n \geq 3$ oriented vector bundle over a surface is trivial iff $w_2 = 0$ since, being oriented, w_2 is the primary obstruction to finding a section from the 2-skeleton to the rank $n \geq 3$ oriented frame bundle. $\square$

Thus the geometric picture for a bundle being spinnable is that it is trivial over the 2-skeleton of your base. Recalling some obstruction theory, we know that the obstruction to *uniqueness* up to homotopy of a lift lies in $H^1(B, \mathbb{Z}/2)$. Here we must be careful as we want to distinguish two lifts only if they fail to be *vertically* homotopic. However it suffices to simply consider the entire cohomology group, as it turns out the equivalence classes of spin structures for a given oriented bundle is a $H^1(B, \mathbb{Z}/2)$-torsor:

Proposition. *If $E \to B$ is spinnable, then the group $H^1(B, \mathbb{Z}/2)$ acts freely and transitively on the set of equivalence classes of spin structures of E over a fixed principal $SO(n)$-bundle.*

Proof. We present two different proofs. We can first identify $H^1(B, \mathbb{Z}/2)$ with $\check{H}^1(B, \mathbb{Z}/2)$, where 1-cocycles $\alpha_{ij} : U_i \cap U_j \to \mathbb{Z}/2$ are defined on a cover of local trivializations for a given spin structure ξ with transition functions $g_{ij} : U_i \cap U_j \to Spin(n)$. Given a 1-cocycle α and a spin structure determined by functions g_{ij}, we can define new transition functions h_{ij} on $U_i \cap U_j$ that determine an inequivalent spin structure (but over the same principal $SO(n)$-bundle) by setting

$$h_{ij} = \begin{cases} g_{ij} & \text{if } \alpha_{ij} = 0 \\ -g_{ij} & \text{if } \alpha_{ij} = 1 \end{cases}$$

where -1 denotes pointwise multiplication by the nontrivial element in $Spin(n)$ over the fiber of the identity. Freeness follows immediately, while transitivity follows from the fact that both spin structures are double covers of the same $SO(n)$-bundle, so that any such two differ by multiplication by -1 in the fiber over appropriate trivializations.

Alternatively, we can return to our lifting diagram above. We have that the classifying space functor B takes group objects to group objects; thus $B\mathbb{Z}/2$ is an *H-space*. Then we have that $B\mathbb{Z}/2$ acts on the fibration on the right freely and transitively fiberwise, up to homotopy. Thus, any two lifts to $BSpin(n)$ of the same map to $BSO(n)$ differ by an element in $B\mathbb{Z}/2$ pointwise; this defines a map $B \to B\mathbb{Z}/2$. But such maps are in correspondence with $H^1(B, \mathbb{Z}/2)$. $\square$

Thus a spin structure can be thought of as a *choice* of homotopy class of trivialization on the 1-skeleton. We say that an orientable riemannian manifold *is spinnable* if its tangent bundle is spinnable. A *spin manifold* is a spinnable manifold *equipped* with a spin structure on its tangent bundle. As we've seen, an orientable manifold is spinnable iff its second Stiefel-Whitney class vanishes. This gives us a large plethora of examples of manifolds that are spinnable and not spinnable.

Example. Every parallelizable manifold is clearly spinnable (e.g., Lie groups, compact orientable 3-manifolds, Euclidean space, etc.) In fact, every stably parallelizable manifold is spinnable (e.g., spheres, orientable surfaces, preimages of regular values of smooth maps from $\mathbb{R}^{n+k} \to \mathbb{R}^n$). Any 2-connected manifold is spinnable (e.g., frame manifolds).

Example. Recall that the tangent bundle of $\mathbb{RP}^n$ is isomorphic to $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)$ where γ_n^1 denotes the tautological line bundle and $\gamma^\perp$ its dual line bundle. From this isomorphism, we obtain that that the total Stiefel-Whitney class of $\mathbb{RP}^n$ is $w(\mathbb{RP}^n) = (1+a)^{n+1}$ where a is the degree 1 generator for $H^*(\mathbb{RP}^n, \mathbb{Z}/2) \simeq \mathbb{Z}/2[a]/(a^{n+1})$. Since we require both w_1 and w_2 to vanish, this forces n to be odd and $\frac{n(n+1)}{2}$ to be even. Thus $\mathbb{RP}^n$ is spinnable iff $n \equiv 3 \bmod 4$.
Moreover, we have that the total Chern class of $\mathbb{CP}^n$ is $c(\mathbb{CP}^n) = (1+a)^{n+1}$ where a is now the degree 2 generator of the cohomology ring, so that $\mathbb{CP}^n$ is spinnable iff n odd, since $w_2 \equiv c_1 \bmod 2$. Finally, $\mathbb{HP}^n$ is spinnable for all n as it is built from a 0-cell, a 4-cell, an 8-cell, etc., so that it has no first or second cohomology.

Example. Consider the hypersurface X in $\mathbb{CP}^3$ cut out by a degree 4 homogeneous polynomial i.e., the K3 surface. By splitting principle, we have that $c_1(E) = c_1(\wedge^{top} E)$ so $c_1(TX) = -c_1(K_X)$ where K_X denotes the canonical bundle, which is trivial for K3 surfaces. Thus X has $c_1 = 0$ and is spinnable. More generally, for a degree d hypersurface X in $\mathbb{CP}^n$ we have the adjunction formula $K_X \otimes \nu X^* = K_{\mathbb{CP}^n}|_X$ so that $c_1(X) = (n+1-d)a$ for a the generator, so X is spinnable iff $n-d$ is odd.

Note that as the Stiefel-Whitney classes are homotopy invariants by Wu's formula, being spinnable is also a homotopy invariant. Assuming that our manifold M has $H_2(M, \mathbb{Z})$ generated by maps of compact *orientable* surfaces (such as when M is simply connected), we have the analogous geometric statement for a manifold being spinnable as follows.

Theorem. *Let M be an oriented n-manifold whose second homology is generated by maps of compact orientable surfaces, with $n \geq 5$. Then, M is spinnable iff every compact orientable surface embedded in M has trivial normal bundle. If $n = 4$, then M is spinnable iff every compact orientable surface embedded in M has even Euler class on its normal bundle.*

Proof. For $n \geq 5$, by Whitney embedding, we can homotope a given map of a compact orientable surface so that the surface is *embedded* in M. For $n = 4$, we can homotope the map to a self-transversal immersion. At each point of intersection, we can remove a small disk and attach a handle (as in surgery); the resulting manifold will be embedded and homologous to the original immersed one. Thus all second homology classes can be represented by *embeddings* of compact orientable surfaces. Then, we have that $w_2(TM) = 0$ iff $f^*w_2(TM) = 0$ for all embeddings $f : \Sigma \to M$. But for an embedding, we have that $f^*TM = TM|_\Sigma = T\Sigma \oplus \nu\Sigma$ such that $w_2(f^*TM) = w_2(T\Sigma \oplus \nu\Sigma) = w_2(T\Sigma) + w_1(T\Sigma)w_1(\nu\Sigma) + w_2(\nu\Sigma)$. All terms but $w_2(\nu\Sigma)$ vanish since Σ is stably parallelizable; the remaining term vanishes iff $\nu\Sigma$ is trivial, again since rank $\nu\Sigma \geq 3$. The same equation shows that if $n = 4$, then the last term vanishes iff $e(\nu\Sigma)$ is even, since the Euler class mod 2 is the top Stiefel-Whitney class. $\square$

Corollary. *If M simply connected of dimension ≥ 5, then M is spinnable iff every embedded 2-sphere has trivial normal bundle.*

Proof. Apply Hurewicz to obtain that $H_2(M, \mathbb{Z})$ is represented by embeddings of 2-spheres.

Corollary. *If M compact, simply-connected 4-manifold, then M is spinnable iff $\langle y \cup y, [M] \rangle \equiv 0$ mod 2 for every $y \in H^2(M, \mathbb{Z})$.*

Proof. Every $y \in H^2(M, \mathbb{Z})$ uniquely determines a map (up to homotopy) $f : M \to K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$. We can cellularly deform the map so that we have a map $f : M \to \mathbb{CP}^3$. We can again homotope so that the map becomes transverse to $\mathbb{CP}^2 \subset \mathbb{CP}^3$ and take its preimage of $\mathbb{CP}^2$ to obtain a surface $\Sigma \subset M$. As $y = f^*a$ for a the degree 2 generator in $\mathbb{CP}^3$, we have that its Poincare dual is the unique 2-cycle such that its intersection count with other 2-cycles is how y acts on other 2-cycles. But y acts on 2-cycles σ by counting intersections of $\mathbb{CP}^2$ with $f_*\sigma$, by definition of pullback and as a Poincare dual to $\mathbb{CP}^2$. Pulling back the intersection sets, we see Σ is Poincare dual to y. Moreover $\langle y \cup y, [M] \rangle$ is the self-intersection number of Σ with itself, which is also equal to the Euler class of its normal bundle. $\square$

A recurring theme for topological results in spin geometry is that the spin condition on a manifold allows for more subtle and refined invariants, while also imposing a certain degree of rigidity on underlying structure; indeed, many of these invariants turn out to be either integers, or must satisfy some divisibility condition or another. For example, we have:

Theorem. *(Rokhlin) The signature of a smooth compact spin 4-manifold is divisible by 16.*

The above follows from a lengthy discussion on the $\hat{A}$-genus and the fact that the signature of a 4-manifold is always 8 times the $\hat{A}$-genus, which, for a compact spin manifold of dimension 4 mod 8, is an even integer. The condition of spinnability, a homotopy condition, also surprisingly imposes restrictions on when a manifold can admit certain types of metrics (such as of positive scalar curvature).

We turn our attention to the *spin bordism* groups. It is not so simple as simply cobordism with spin manifolds. First we need the right notion of a *spin structure-preserving* diffeomorphism. Given an orientation-preserving diffeomorphism $f : M \to M$ between two spin manifolds, there is an induced diffeomorphism $df : P_{gl+}(M) \to P_{gl+}(M)$ on the bundle of oriented tangent frames, given by exactly the differential. This map is fiber-preserving and thus induces a permutation of the possible spin structures on your spin manifold (since spin structures are exactly double coverings of the bundle of oriented tangent frames). If the given spin structure remains fixed, then f is a *spin structure-preserving diffeomorphism.*

Now note that if we have vector bundles $E \oplus E' = E''$, then by Whitney sum, any two being spinnable implies the third is spinnable (just as any two being orientable implies the third is orientable). The spin structure on the third is also canonically determined by the other two. Thus, any submanifold of a spin manifold with a spin structure on its normal bundle is then canonically a spin manifold itself. This implies if a spin manifold M has boundary ∂M, its boundary is automatically spin. Moreover, we can concretely realize the spin structure on the boundary as follows: there is an embedding $P_{so}(\partial M) \subset P_{so}(M)$ by taking a tangent frame on ∂M and completing it to a tangent frame on M by adding an inward-pointing normal vector. The spin structure on M, as a double covering of $P_{so}(M)$, can then be restricted to be a double covering of $P_{so}(\partial M)$, i.e., a spin structure on ∂M.

We say that two compact spin n-manifolds M, M' are *spin bordant* if there exists an $(n+1)$-manifold W who is spin, such that ∂W is diffeomorphic to $M \coprod M'$ with respect to a spin structure-preserving map, where the orientation and spin structure on ∂W is the one induced from W as above. A compact spin manifold is *spin bordant to zero* if it is spin bordant to a boundary. Let Ω_n^{spin} denote the free abelian group generated by the set of equivalence classes of compact connected spin n-manifolds, modulo the subgroup generated by $[X_1] + ... + [X_k]$ where $X_1 \coprod ... \coprod X_k$ is spin bordant to zero and where equivalence means up to spin structure-preserving diffeomorphism. This is the *n-dimensional spin bordism group.* Again from the Whitney sum, we know that the product of two spin manifolds is again spin. Moreover, the product of spin manifolds has a uniquely determined spin structure, as is true for bundles. This makes $\Omega_*^{spin} = \bigoplus_{n=0}^{\infty} \Omega_n^{spin}$ into a graded ring, called the *spin bordism ring.* The equivalence class (and therefore the cobordism class) of a spin manifold here is independent of the choice of metric.

Note that surgery may affect the property of being spin. However, there are certain surgeries that allow one to preserve the property of being spin, so long as one is careful. For example, given two spin n-manifolds, we can take their connect sum (an $S^0 \times D^n$ type surgery) and obtain a *connected* spin manifold, so that every spin bordism class is representable by a connected manifold. For $n \geq 3$ (as we need to be less than half the dimension and, for $n = 3$, as every compact oriented 3-manifold is surgically equivalent to S^3), we can similarly apply surgery to kill π_1 and find *simply connected* representatives. For $n = 5$, we can kill π_1 and π_2 (as there are no obstructions to surgery in this dimension) so that by Poincare duality, every spin 5-manifold is spin bordant to the disk; that is, $\Omega_5^{spin} = 0$.

Example. We will compute Ω_1^{spin}. The only principal $SO(1)$-bundle over the circle S^1 is itself, identified as a product with a point. However, we have two distinct spin structures: one is a connected double cover of the circle, and the other is the trivial double cover. If we realize S^1 as the boundary of the 2-disk, which has a unique spin structure, we see that the induced spin structure is the *connected* double cover: since the bundle of frames on the disk is $D^2 \times SO(2)$, we can imagine a solid torus where each point on the parameterizing circle assigns a 2-frame for all points in the orthogonal disk slice. Then, the bundle of frames for S^1 as the boundary of the disk is realized as a path along the boundary of this solid torus: pairing a point on the boundary of the disk with the unique frame corresponding to a unique (respecting orientation) tangent vector and the inward unit normal, we obtain a point on the boundary of the solid torus. Varying continuously, we obtain a single path along the longitudinal boundary of this solid torus. This path is then the bundle of frames $S^1 \times SO(1) \simeq S^1$. As this path is along the longitude, the restriction of the principal spin bundle will be the connected double cover of the circle. Thus, S^1 equipped with the connected double cover for its spin structure is spin bordant to zero. If we equip the disconnected double cover, then we see that it is not spin bordant to zero, as there is no spin structure-preserving diffeomorphism between this circle and the circle that bounds the disk (the double coverings are not diffeomorphic, as one is connected and the other is not). Moreover, by the same picture above, it fails to be induced as a spin boundary (using that surfaces with boundary have a unique principal $SO(2)$-bundle). However, two copies of S^1 with the disconnected spin structure *is* spin bordant to zero, since we can connect sum it to obtain a spin structure on the circle. But this is either the disconnected or connected one, and as the disconnected one is not the connected one, two copies of it must be (i.e., $x + x = x$ implies $x = 0$). This gives us that $\Omega_1^{spin} = \mathbb{Z}/2$.

Example. We have that the square of the bad spin structure over S^1 is not a boundary (by a heavy computation, according to Milnor [3]), but twice of it is spin bordant to one, again by algebra. Moreover, any other surface of genus g can be surgered into a finite union of tori, so that again it is either spin bordant to a boundary, or the nontrivial spin structure on the torus. Thus, $\Omega_2^{spin} = \mathbb{Z}/2$. By surgery, every compact oriented 3-manifold is representable by a simply connected manifold (so we avoid the thrice-power of the bad spin structure on S^1). Moreover, every closed oriented 3-manifold is obtained by surgery along a framed link to S^3. Thus, we have $\Omega_3^{spin} = 0$. We have that $\Omega_4^{spin} \simeq \mathbb{Z}$ generated by the K3 surface, and that $\Omega_5^{spin} = 0$ by a previous discussion using surgery (as every embedded 2-sphere in a spin manifold has trivial normal bundle) and Poincare duality.

Clifford algebras and spinor bundles

In this section, we will explicitly define the Spin group using Clifford algebras, and see how the Clifford algebra formalism is imported to the bundle setting. This will lead us naturally to the notion of a Dirac operator and eventually its relation with certain topological invariants.

Definition. *Define the* Clifford algebra Cl_n *as* $\mathcal{T}(\mathbb{R}^n)/\mathcal{I}$ *where* $\mathcal{T}(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} (\mathbb{R}^n)^{\otimes k}$ *is the free tensor algebra generated by* $\mathbb{R}^n$ *and* $\mathcal{I}$ *is the ideal generated by elements of the form* $v \cdot v + ||v||^2 \cdot 1$ *for* $|| \cdot ||$ *the standard Euclidean norm on* $\mathbb{R}^n$*.*

The imposed relation forces vectors in $\mathbb{R}^n$ to satisfy $v \cdot w + w \cdot v = -2(v, w)$ where $(\cdot, \cdot)$ is the standard inner product. Before proving anything about the Clifford algebra, we first establish its universal property. Most of the resulting lemmas and propositions are then a result of abusing the universal property with respect to an appropriate map. We have a natural embedding $\mathbb{R}^n \hookrightarrow Cl_n$ via including $\mathbb{R}^n$ as the 0-th graded part of the tensor algebra and then taking quotients.

Proposition. *Given a linear map $f : \mathbb{R}^n \to \mathcal{A}$ to an associative algebra with unit, such that the map satisfies $f(v) \cdot f(v) = -||v||^2$ for all v, we have that the map factors uniquely:*

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[d, hook] & \mathcal{A} \\ Cl_n \arrow[ur, dashed, "\tilde{f}"] & \end{tikzcd}$$

so that the diagram commutes. Furthermore, Cl_n is the unique associative algebra with this property.

Proof. The linear map induces an algebra map on the free tensor algebra. The resulting algebra map then descends to the Clifford algebra as the map satisfies $f(v) \cdot f(v) = -||v||^2$. $\square$

The motivation for the Clifford algebra construction historically comes from physics. It should be regarded as a refinement of the standard exterior algebra on a vector space, taking into account the additional data of the quadratic form. This can be seen as follows: the tensor algebra has a natural filtration $\tilde{\mathcal{F}}^0 \subset \tilde{\mathcal{F}}^1 \subset ... \subset \mathcal{T}$ where $\tilde{\mathcal{F}}^m = \bigoplus_{k=0}^m (\mathbb{R}^n)^{\otimes k}$. This filtration has the property that $\tilde{\mathcal{F}}^l \otimes \tilde{\mathcal{F}}^m \subset \tilde{\mathcal{F}}^{l+m}$. Projecting down by the quotient map π and setting $\mathcal{F}^m = \pi(\tilde{\mathcal{F}}^m)$ we obtain a filtration of the Clifford algebra $\mathcal{F}^0 \subset \mathcal{F}^1 \subset ... \subset Cl_n$. Here, we have from the projection that $\mathcal{F}^l \cdot \mathcal{F}^m \subset \mathcal{F}^{l+m}$, i.e., graded multiplication is also respected with Clifford multiplication. Thus Cl_n is a *filtered algebra*. Setting $\mathcal{G}^m = \mathcal{F}^m / \mathcal{F}^{m-1}$ we retain the graded multiplication and therefore obtain the *associated graded algebra* of Cl_n, defined as $\mathcal{G}^* = \bigoplus_{m \geq 0} \mathcal{G}^m$.

Proposition. *The associated graded algebra $\mathcal{G}^*$ of Cl_n is naturally isomorphic to $\wedge^* \mathbb{R}^n$.*

Proof. We invoke the universal property of the exterior algebra and define an alternating map $f : \bigotimes^m \mathbb{R}^n \to \mathcal{F}^m \to \mathcal{F}^m / \mathcal{F}^{m-1}$ by sending $v_{i_1} \otimes ... \otimes v_{i_m}$ to $[v_{i_1} \cdots v_{i_m}]$. Using the relation $v \cdot w + w \cdot v = -2(v, w)$ we obtain that the map is alternating, therefore descending to a map $\tilde{f}$ from $\wedge^m \mathbb{R}^n$ (since multiplication by the scalar lowers the degree, which is quotiented out). Surjectivity follows immediately, since the first map in the composition is the projection onto the m-th degree of the filter from inclusion into the tensor algebra (and as all lower degree elements vanish). The kernel of f is exactly comprised of m-homogeneous parts of elements φ in the ideal $\mathcal{I}$ of degree $\leq m$ with respect to the filtration. By definition of being in the ideal, such elements can be written as a finite sum $\varphi = \sum a_i \otimes (v_i \otimes v_i + ||v_i||^2) \otimes b_i$ where a_i, b_i are pure tensors of degree $\leq m - 2$ and the v_i are in $\mathbb{R}^n$. Then the m-homogeneous parts are exactly $\varphi_m = \sum a_i \otimes v_i \otimes v_i \otimes b_i$ where the a_i, b_i have degree exactly $m - 2$. But such elements vanish in the exterior algebra. So $\tilde{f}$ is injective. Therefore $\tilde{f}$ is a graded algebra isomorphism between $\wedge^* \mathbb{R}^n$ and $\mathcal{G}^*$. $\square$

Proposition. *There is a canonical* vector space *isomorphism $\wedge^* \mathbb{R}^n \simeq Cl_n$ respecting the filtrations.*

Proof. Define a linear map from the m-fold direct product $\mathbb{R}^n \times ... \times \mathbb{R}^n \to Cl_n$ by

$$f(v_1, ..., v_m) = \tfrac{1}{m!} \textstyle\sum_\sigma sgn(\sigma) v_{\sigma(1)} \cdots v_{\sigma(m)} \text{ for } \sigma \in S_m.$$

This map is alternating as $sgn(\sigma)$ changes sign when there is a transposition, so f descends to a linear map $\tilde{f}$ from $\wedge^m \mathbb{R}^n$ with image in $\mathcal{F}^m$ (as a sum of Clifford products of m elements). Composing this map with $\mathcal{F}^m \to \mathcal{F}^m / \mathcal{F}^{m-1}$ gives us the map from the exterior algebra in the above proposition (by reducing the sum to one term given distinct ordered points), so that $\tilde{f}$ is injective. Taking direct sums over the maps gives us the canonical vector space isomorphism. $\square$

This canonical vector space isomorphism will come into play later in the bundle setting; indeed it will give us that the Euler characteristic of a manifold is the index of a Dirac operator. The graded algebra structure is completely destroyed in this isomorphism, as one can tell from seeing what Clifford multiplication corresponds to: for $v \in \mathbb{R}^n$ and $\varphi \in Cl_n$, we see that $v \cdot \varphi \simeq v \wedge \varphi - v \llcorner \varphi$ where $\llcorner$ denotes contraction in the exterior algebra. As wedging increases degree and contraction decreases it, we see that the algebra structure is not at all preserved, which is consistent with $\wedge^* \mathbb{R}^n$ being independent of the quadratic form. This change in grading is exactly what is needed for the Euler characteristic to come into play later, using harmonic forms and the Hodge decomposition.

Let $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ be the "antipodal" map, i.e., the map that sends $v \mapsto -v$. This linear map extends to the tensor algebra as an algebra map and descends to the Clifford algebra, resulting in an eigenspace decomposition $Cl_n = Cl_n^0 \oplus Cl_n^1$ where Cl_n^0 denotes the *even* part, i.e., the φ where $\alpha(\varphi) = \varphi$, and Cl_n^1 denotes the *odd* part, i.e., the φ where $\alpha(\varphi) = -\varphi$. Note that even (odd) φ are a product of an even (odd) number of vector space elements.

Definition. *Let $Pin(n)$ denote* the Pin group*, the multiplicative subgroup of Cl_n generated by vector space elements of unit norm. Let $Spin(n)$ denote* the Spin group*, the subgroup of $Pin(n)$ of even elements; i.e., $Spin(n) = Pin(n) \cap Cl_n^0$.*

The Pin and Spin groups are the multiplicative subgroups generated by the unit sphere. For each φ in $Pin(n)$, consider the twisted conjugation map $\psi \mapsto \alpha(\varphi)\psi\varphi^{-1}$. This gives us a map called the *twisted adjoint representation* $\tilde{\mathrm{Ad}}$ from $Pin(n)$ to $O(n)$. Note that restricting to the even part equates it with the standard conjugation map Ad so that we have Ad: $Spin(n) \to SO(n)$. One way to see why we map to the orthogonal group is that the conjugation map on a *vector* is equal to a *reflection* across the orthogonal hyperplane of that vector. As all elements in Cl_n are products of such vectors and α is an algebra map, we can then realize the adjoint map as mapping to compositions of various reflections. By a classical theorem of Cartan-Dieudonné, $O(n)$ is actually *generated* by reflections. Similarly, the even elements are sent to an even number of reflections, which are orientation-preserving (reflections alone are never orientation-preserving) so $Spin(n)$ lands in $SO(n)$. Since an element and its antipode share the same reflecting hyperplanes, the kernel of both adjoint maps is exactly $\mathbb{Z}/2$. Thus these maps are precisely the double covering maps that define $Spin(n)$ and $Pin(n)$ as the universal covers of $SO(n)$ and $O(n)$. To summarize, we have the following exact sequences:

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow Spin(n) \xrightarrow{\mathrm{Ad}} SO(n) \longrightarrow 1$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow Pin(n) \xrightarrow{\tilde{\mathrm{Ad}}} O(n) \longrightarrow 1$$

Let us say a brief word on importing this algebraic data to the bundle setting. Any orthogonal transformation on $\mathbb{R}^n$ induces an automorphism on Cl_n by extension to the tensor algebra and descending. Then, for any oriented vector bundle $E \to B$, we can define the *Clifford bundle* as $Cl(E) \equiv P_{so}(E) \times_\rho Cl_n$ where $\rho : SO(n) \to Aut(Cl_n)$ is the standard $SO(n)$-representation of Cl_n (i.e., induced from standard representation $\rho : SO(n) \to SO(\mathbb{R}^n)$) and the product is the Borel construction (take $P_{so}(E) \times Cl_n$ and quotient by the relation $(p, \varphi) \sim (pg^{-1}, \rho(g)\varphi)$ for $g \in SO(n)$). Alternatively, the Clifford bundle can be realized by the Clifford construction at each fiber $Cl(E)_x \equiv \mathcal{T}(E_x)/\mathcal{I}(E_x)$. The antipodal map above then extends to a bundle isomorphism so that we can decompose our Clifford bundle into ± 1 eigenbundles, so that $Cl(E) = Cl(E)^0 \oplus Cl(E)^1$. In the vector space setting, as α is an algebra map, we have that $Cl_n^i \cdot Cl_n^j \subset Cl_n^{(i+j) \bmod 2}$ so that Cl_n becomes a $\mathbb{Z}/2$-*graded algebra*. Similarly, the Clifford bundle $Cl(E)$ becomes a *bundle of algebras.*

We should note that the α-eigenbundle decomposition is not the only type of splitting we have on $Cl(E)$. Consider the *oriented volume element* $\omega = e_1 \cdots e_n$ for $(e_1, ..., e_n)$ a choice of orthonormal basis. Any other choice of orthonormal basis will scale the element by the determinant of the corresponding change of basis matrix, which is 1, so that ω is well defined. We now have to consider different cases depending on the dimension. Keep in mind that by orthonormality, $e_ie_i = -1$ and $e_ie_j = -e_je_i$ in the Clifford algebra.

Lemma. *We have that the oriented volume element ω satisfies the following identities:*

$$\omega v = (-1)^{n-1} v\omega \text{ for } v \in \mathbb{R}^n$$
$$\omega^2 = (-1)^{\frac{n(n+1)}{2}}$$

In particular, if n is odd, then ω is central in Cl_n, i.e., commutes with all elements. If n is even, then $\varphi\omega = \omega\alpha(\varphi)$.

Proof. The above identities follow from purely formal computation, using the relations of the Clifford algebra. $\square$

The second identity above tells us that for $n \equiv 0,\ 3 \bmod 4$ we have $\omega^2 = 1$. In the bundle setting, we can realize ω as a global section on the Clifford bundle, with multiplication by ω as a bundle automorphism. If $n \equiv 0, 3 \bmod 4$, then multiplication by ω has eigenvalues $\pm$ 1 so that the Clifford bundle decomposes yet again into eigenbundles $Cl^+(E) \oplus Cl^-(E)$. We can actually require more: these eigenbundles are isomorphic *bundles of subalgebras* for $n \equiv 3 \bmod 4$, when ω is central. Indeed, we can define $\pi^+ = \frac{1}{2}(1 + \omega)$ and $\pi^- = \frac{1}{2}(1 - \omega)$ which are the projection maps onto each eigenspace (so they satisfy $\pi^+ + \pi^- = 1$, $(\pi^\pm)^2 = \pi^\pm$, and $\pi^+\pi^- = \pi^-\pi^+ = 0$). For $n \equiv 3 \bmod 4$, we have ω is central over each fiber so these projections are central (so $Cl_n^\pm = \pi^\pm \cdot Cl_n = Cl_n \cdot \pi^\pm$). Thus these eigenspaces are ideals, and therefore extend to bundles of subalgebras in the bundle setting. Moreover, in this dimension ω is an odd element, so α switches these two subbundles/subalgebras: i.e., $\alpha(Cl^\pm(E)) = Cl^\mp(E)$. We can also decompose modules over the Clifford algebra (and therefore bundles of modules over bundles of Clifford algebras).

Proposition. *Let $n \equiv 0 \bmod 4$. If M is a Cl_n-module, i.e., M is a real vector space with an algebra homomorphism $Cl_n \to \mathrm{Hom}(M, M)$, then M decomposes*

$$M = M^+ \oplus M^-$$

into the $+1$ and -1 eigenspaces for multiplication by ω. In fact we have $M^\pm = \pi^\pm \cdot M$ and that module multiplication by any nonzero $v \in \mathbb{R}^n$ gives an isomorphic swap of the eigenspaces $M^\pm \to M^\mp$.

Note that ω in this dimension is *not* central. The above proposition follows from the relations between the projection maps above and the fact that $\omega \cdot \pi^\pm = \pm\pi^\pm$. The isomorphisms for nonzero vectors follow from $v\pi^+ = \pi^- v$ and $v\pi^- = \pi^+ v$ with $v \cdot v = -||v||^2 \cdot 1$. An example of such a module above that decomposes is the Clifford algebra as a module over itself. Thus, we have another way of inducing a $\mathbb{Z}/2$ grading on the Clifford algebra.

Before we proceed, let us discuss the complex analogue of the above discussion. Define the *complex volume element* as $\omega_\mathbb{C} = i^{\lfloor \frac{n+1}{2} \rfloor}\omega$. Note for $n = 2m$ we have $\omega_\mathbb{C} = i^m\omega$ and that $\omega_\mathbb{C} = \omega$ only in dimensions 0, 7 mod 8. If n is odd, then both ω and $\omega_\mathbb{C}$ are central. Whereas $\omega^2 = 1$ only if $n \equiv 0$, 3 mod 4, we have that $\omega_\mathbb{C}^2 = 1$ for all n. Denote $\mathbb{C}l_n$ as the *complexified Clifford algebra* $Cl_n \otimes \mathbb{C}$. Just as Cl_n decomposes into two isomorphic subalgebras for $n \equiv 3 \bmod 4$, we have that $\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$ for n odd (i.e., when $\omega_{\mathbb{C}}$ is central).

Proposition. *Let n even. We again have $M = M^+ \oplus M^-$ for M a $\mathbb{C}l_n$-module, and $M^\pm$ the ± 1 eigenbundles from Clifford multiplication by $w_{\mathbb{C}}$. The $M^\pm$ are in fact Cl_n^0-modules.*

We can now discuss similar decompositions for bundles of modules over bundles of Clifford algebras. This is where we require our oriented vector bundle $E \to B$ to be spin. Define a *spinor bundle* as $S(E) \equiv P_{spin}(E) \times_\mu M$ where M is a module over Cl_n and $\mu : Spin(n) \to SO(M)$ is Clifford multiplication on M (we have that multiplication by Clifford elements is orthogonal with respect to some inner product, by averaging a given one over the Clifford group). Our spinor bundle is a bundle of modules over a bundle of algebras, with the fiberwise action of Cl_n on M defined smoothly over the whole base. We also have that the sections of our spinor bundle form a module over the algebra of sections of our Clifford bundle, again by pointwise multiplication. We can also define a complex analog $S_{\mathbb{C}}(E) \equiv P_{spin}(E) \times_\mu M_{\mathbb{C}}$ where $M_{\mathbb{C}}$ is a complex module over $\mathbb{C}l_n$.

Proposition. *For a spinor bundle $S(E)$ for $n \equiv 0 \bmod 4$, we have the following decomposition:*

$$S(E) = S^+(E) \oplus S^-(E)$$

into the $+1$ and -1 eigenbundles for multiplication by ω, a global section of $Cl(E)$. For n even,

$$S_{\mathbb{C}}(E) = S_{\mathbb{C}}^+(E) \oplus S_{\mathbb{C}}^-(E)$$

into the eigenbundles for multiplication by $\omega_{\mathbb{C}}$, a global section of $\mathbb{C}l(E)$.

Assuming our bundle is spin, we can also rewrite its Clifford bundle as $Cl(E) = P_{spin}(E) \times_{\mathrm{Ad}} Cl_n$ since the adjoint representation exactly descends to the one induced from the standard representation.

As the above definitions of spinor bundles depend on modules over Clifford algebras, we will mention a bit on their classification – in particular, the classification of representations of the Spin group. First we observe that the *even* part Cl_n^0 of the Clifford algebra in its antipodal decomposition is itself a Clifford algebra.

Proposition. *Cl_n^0 is isomorphic to Cl_{n-1} as an algebra.*

Proof. (Sketch) Choose an orthonormal basis of $\mathbb{R}^n$ and define $f : \mathbb{R}^{n-1} \to Cl_n$ by sending $e_i \mapsto e_n \cdot e_i$. Extend the map linearly and apply the universal property.

This realization of the even part of the Clifford algebra of $\mathbb{R}^n$ as the Clifford algebra of $\mathbb{R}^{n-1}$ will come into play when considering irreducible representations. After a minor discussion on $\mathbb{Z}/2$-graded tensor products, one can see that the dimension of Cl_n is 2^n. We can actually see explicitly what the Clifford algebras are in lower dimensions:

$$\begin{gathered} Cl_1 \simeq \mathbb{C} \\ Cl_1^* \simeq \mathbb{R} \oplus \mathbb{R} \\ Cl_2 \simeq \mathbb{H} \\ Cl_2^* \simeq \mathbb{R}(2) \end{gathered}$$

where Cl_n^* denotes the dual of Cl_n (i.e., take the quadratic form $-||\cdot||^2$ instead of the standard form $||\cdot||^2$) and $\mathbb{R}(2)$ denotes 2×2 real matrices. We can now classify all Clifford algebras.

Proposition. *We have the following isomorphisms:*

$$Cl_{n+2} \simeq Cl_n^* \otimes Cl_2$$
$$Cl_{n+2}^* \simeq Cl_n \otimes Cl_2^*$$

Proof. (Sketch) Again one chooses an orthonormal basis to define a map from $\mathbb{R}^{n+2} \to Cl_n \otimes Cl_2^*$ to use the universal property.

Theorem. *(Bott periodicity) We have the following isomorphisms:*

$$Cl_{n+8} \simeq Cl_n \otimes Cl_8$$
$$Cl_{n+8}^* \simeq Cl_n^* \otimes Cl_8^*$$
$$\mathbb{C}l_{n+2} \simeq \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

Proof. (Sketch) The above isomorphisms use the above proposition and the fact that

$$\mathbb{R}(n) \otimes \mathbb{R}(m) \simeq \mathbb{R}(nm)$$
$$\mathbb{R}(n) \otimes_{\mathbb{R}} K \simeq K(n) \text{ for } K = \mathbb{C} \text{ or } \mathbb{H}$$
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$$
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathbb{C}(2)$$
$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathbb{R}(4)$$

From this classification scheme, we see that all Clifford algebras are familiar matrix algebras with coefficients in $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. This makes their representation theory relatively simple. Recall that *a representation* of the Clifford algebra is an algebra homomorphism $\rho : Cl_n \to \text{Hom}(W, W)$ where W is a finite dimensional real vector space called a *module over* Cl_n, and the action of the representation is called *Clifford multiplication.* Two representations $\rho_j : Cl_n \to \text{Hom}(W_j, W_j)$ are *equivalent* if there exists a $\mathbb{R}$-linear isomorphism $T : W_1 \to W_2$ such that $T \circ \rho_1(\varphi) = \rho_2(\varphi) \circ T$ for all $\varphi \in Cl_n$, i.e., the map is equivariant with respect to the action of Clifford multiplication.

Theorem. *Let $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and consider the algebra of K-matrices over $\mathbb{R}$. The natural representation $\rho : K(n) \to Mat(n, K)$ is, up to equivalence, the only irreducible real representation of $K(n)$. The algebra $K(n) \oplus K(n)$ has exactly two equivalence classes of irreducible real representations, given by $\rho_1(\varphi_1, \varphi_2) = \rho(\varphi_1)$ and $\rho_2(\varphi_1, \varphi_2) = \rho(\varphi_2)$ acting on K^n.*

The irreducible representations of Clifford algebras turn out to be related to the K-theory of a point, via the Atiyah-Bott-Shapiro construction which we will outline later. Their importance in the context of these notes is relegated to the moral that there are not too many spin representations or Clifford algebras, and that they are all determined by the Clifford algebras of $\mathbb{R}^n$ for $n = 1, ..., 8$ by Bott periodicity. We have the following two propositions regarding irreducible representations, mimicking our earlier propositions about the Clifford and spinor decompositions:

Proposition. *For $\rho : Cl_n \to \text{Hom}_{\mathbb{R}}(W, W)$ where $n \equiv 3 \mod 4$, we have that*

$$\rho(\omega) = \text{Id} \textit{ or } \rho(\omega) = -\text{Id} \textit{ for } \omega \textit{ the volume element.}$$

Both possibilities can occur and the corresponding representations are equivalent. Similarly, for the complex case, we have for n odd

$$\rho(\omega_{\mathbb{C}}) = \text{Id} \textit{ or } \rho(\omega_{\mathbb{C}}) = -\text{Id} \textit{ for } \omega_{\mathbb{C}} \textit{ the complex volume element.}$$

Proposition. *For $\rho : Cl_n \to \text{Hom}_{\mathbb{R}}(W, W)$ where $n \equiv 0 \mod 4$ so that $W = W^+ \oplus W^-$ for $W^{\pm} = (1 \pm \rho(\omega)) \cdot W$, we have that each of the subspaces W^+ and W^- are invariant under the even subalgebra Cl_n^0. Under the isomorphism $Cl_n^0 \simeq Cl_{n-1}$, these spaces correspond to the two distinct irreducible real representations of Cl_{n-1}. The analogous statements are true for $\mathbb{C}l_n$ for n even, with respect to $\omega_{\mathbb{C}}$.*

The importance of these statements is that in dimensions $4k$, we have a unique complex spinor bundle, given by the irreducible complex Clifford module, that decomposes via the complex volume element into a sum of two complex irreducible (with respect to representations one dimension lower) Clifford modules. This gives us the index theorem for the $\hat{A}$-genus for spin $4k$-manifolds. One can do a similar analysis for general $4k$-manifolds instead, by splitting the Clifford bundle itself with the complex volume element, and this gives us the index theorem for the signature.

For completion, we mention that a *real spin representation* $\Delta_n : Spin(n) \to GL(S)$ is defined by restricting an irreducible real representation $Cl_n \to \mathrm{Hom}_{\mathbb{R}}(S,S)$ to $Spin(n)$. Similarly a *complex spin representation* $\Delta_n^{\mathbb{C}} : Spin(n) \to GL_{\mathbb{C}}(S)$ is given by restricting an irreducible complex representation $\mathbb{C}l_n \to \mathrm{Hom}_{\mathbb{C}}(S,S)$ to $Spin(n)$. Based on the dimension, these spin representations are irreducible, a direct sum of two equivalent irreducible representations, or a direct sum of two inequivalent irreducible representations. These really only come into play to set up spinor bundles in the first place, and again the moral is that there are not too many of them up to isomorphism, due to Bott periodicity and the classification of Clifford algebras. Moreover, they do not necessarily descend to representations of $SO(n)$, unlike the analogous situation for $\widetilde{GL(n,\mathbb{R})}$ and $GL(n,\mathbb{R})$.

The upshot is that irreducible representations, both real and complex, of Clifford algebras are completely known and determined by the first eight, where in most cases there is only one up to isomorphism. For the real case, there are two inequivalent ones in $n \equiv 3, 7 \bmod 4$, and for the complex case there are two for n odd. These pairs allow for an irreducible module in $n \equiv 0 \bmod 4$ to decompose into two irreducible (again with respect to a subrepresentation on the even part) modules, which in the bundle setting gives us self-adjoint elliptic operators. This in turn allows us to realize many topological invariants as the indices of such operators restricted to parts of the decomposition.

Dirac bundles and Dirac operators

Let us first recall the notion of a connection on a principal G-bundle. A *connection* τ is a smooth collection of G-invariant horizontal tangent subspaces of the tangent bundle of P_G, where horizontal means complementary to the tangent space of the fiber. We say a vector bundle has a connection if it has one on its principal $SO(n)$-bundle of frames. The picture is that the connection allows one to "connect" different fibers together so that we may take derivatives of sections, much like the picture for the Lie derivative of vector fields. Identifying $T^*M \otimes E$ with $\mathrm{Hom}(TM, E)$ we see that we are assigning to each section of E, a linear map from TM to E; in other words, for each vector field, we are assigning a derivative in the direction of that vector field, exactly like a directional derivative. Indeed, for a fixed vector field V, we have that ∇_V is a map $\Gamma(E) \to \Gamma(E)$. We define our notion of derivative for sections as follows:

Definition. *Given a connection on a vector bundle over a smooth manifold $E \to M$, a* covariant derivative *is a linear assignment $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$ for a smooth function f, and a section σ.*

At each point $p \in P_G$, we have that τ_p determines a linear projection $T_pP_G \to T_p(G \cdot p)$ where $G \cdot p$ is the orbit of p under the action of G (i.e., we have a projection to the tangent space of the fiber). Although the fiber is not canonically isomorphic to G, we have that there is a canonical isomorphism for a *fixed* p from $T_p(G \cdot p) \simeq \mathfrak{g}$ the tangent space to the identity of G. We then have a linear map $TP_G \to \mathfrak{g}$ which is a Lie algebra-valued 1-form, called *the connection 1-form* ω. We will be mainly interested in the case of $G = SO(n)$ with $\mathfrak{g} = \mathfrak{so}(n)$ the Lie algebra of real, skew symmetric $n \times n$-matrices (so that $\omega_{ij} = -\omega_{ji}$). This connection 1-form uniquely determines the connection (since $\tau_p = \ker(\omega_p)$) and vice versa. Thus it also determines the covariant derivative, which is why the covariant derivative crucially depends on a connection.

Proposition. *Let ω be a connection 1-form on $P_{so}(E)$. Then ω determines a unique covariant derivative on E by the rule*

$$\nabla e_i = \textstyle\sum_{j=1}^{n} \tilde{\omega}_{ji} \otimes e_j$$

where $\mathcal{E} = (e_1, ..., e_n)$ is a local family of pointwise orthonormal sections of E, i.e., a local section of $P_{so}(E)$, and where $\tilde{\omega} = \mathcal{E}^*\omega$. This covariant derivative satisfies the rule

$$V\langle e, e'\rangle = \langle \nabla_V e, e'\rangle + \langle e, \nabla_V e'\rangle$$

for all $V \in TM$ and $e, e' \in \Gamma(E)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on E. Conversely, any covariant derivative satisfying this relation determines a unique connection 1-form by the local expression above.

In terms of a basis for the skew-symmetric matrices, we have that $\omega = -\sum_{i<j} \omega_{ij} e_i \wedge e_j$. Given a connection on a spin vector bundle $E \to M$, we can associate a connection to an associated spinor bundle as follows. Let $\xi : P_{spin}(E) \to P_{so}(E)$ be the given spin structure and $S(E) = P_{spin}(E) \times_\mu M$ be an associated spinor bundle, with M some module over Cl_n. The connection τ on $P_{so}(E)$ lifts via the double covering to a connection τ' on P_{spin} (one can pullback the connection 1-form). From there, we can extend τ' trivially to $\tau' \times 0$ to a connection on $P_{spin}(E) \times M$. As τ' is already $Spin(n)$-invariant, we can take a quotient by the action and induce a connection $\tilde{\tau}$ on $S(E)$. This in turn induces a covariant derivative on our spinor bundle. Similarly we have an induced connection on the Clifford bundle, by the same construction. It turns out one can relate the old connection 1-form on a principal G-bundle with the induced one on the associated riemannian vector bundle exactly by the induced Lie algebra homomorphism from the G-representation. Looking at the $SO(n)$-representation acting on Cl_n, we have that the corresponding Lie algebra homomorphism maps into *derivations* of the Clifford algebra.

Proposition. *The covariant derivative ∇ on $Cl(E)$ acts as a derivation on the algebra of sections, i.e.,*

$$\nabla(\varphi \cdot \psi) = (\nabla\varphi) \cdot \psi + \varphi \cdot (\nabla\psi)$$

for any two sections φ and ψ of $Cl(E)$. Furthermore, under the canonical identification $Cl(E) \simeq \wedge^*(E)$, the covariant derivative ∇ preserves the subbundles $\wedge^p(E)$ and agrees there with the covariant derivative induced by the representation $\wedge^p\rho_n$ (i.e., the usual covariant derivative).

Corollary. *The subbundles $Cl^0(E)$ and $Cl^1(E)$ are preserved by ∇. Furthermore, the oriented volume form ω is globally parallel, i.e.,*

$$\nabla\omega = 0.$$

Therefore when $n \equiv 0, 3 \bmod 4$ the eigenbundles of ω are also preserved by ∇.

Proposition. *The covariant derivative ∇ on $S(E)$ acts as a derivation with respect to the module structure over $Cl(E)$, i.e.,*

$$\nabla(\varphi \cdot \sigma) = (\nabla\varphi) \cdot \sigma + \varphi \cdot (\nabla\sigma)$$

for any section φ of $Cl(E)$ and any section σ of $S(E)$.

Recall that for a Cl_n-module W, we can always choose an inner product on W such that Clifford multiplication by unit vectors in $\mathbb{R}^n$ is orthogonal, by averaging a given inner product over the Clifford group, i.e., the finite group $F_n \subset Cl_n^\times$ generated by an orthonormal basis of $\mathbb{R}^n$ (finite due to the relations between the basis elements from the Clifford algebra). This extends to spinor bundles in that we can always choose a bundle metric such that fiberwise Clifford multiplication by unit vectors in TM is orthogonal.

Definition. *A* Dirac bundle *over a riemannian manifold M is a bundle S of modules over $Cl(M)$ together with a riemannian metric such that Clifford multiplication by unit vectors in TM is orthogonal, and a connection such that its covariant derivative is a derivation with respect to the module multiplication of $Cl(M)$.*

Example. For $S = Cl(M)$ with its canonical connection, viewed as a bundle of modules over itself, is a Dirac bundle. The fact that the covariant derivative is a derivation follows from the above propositions.

Example. For M a spin manifold, any spinor bundle S associated to TM is a Dirac bundle, again from the above propositions.

Example. Let S be a Dirac bundle, and E be any vector bundle over M. Then $S \otimes E$ is a bundle of modules over $Cl(M)$, where module multiplication is given on simple tensors by $\varphi\cdot(\sigma\otimes e) \equiv (\varphi\cdot\sigma)\otimes e$ for $\varphi \in Cl(M)$, $\sigma \in S$, and $e \in E$. The tensor product metric is again orthogonal with respect to Clifford multiplication by unit tangent vectors. Finally, $S \otimes E$ has the tensor product connection $\nabla \equiv \nabla^S \otimes \nabla^E$ which is defined on sections of the form $\sigma \otimes e$ by $\nabla(\sigma \otimes e) = (\nabla^S\sigma) \otimes e + \sigma \otimes (\nabla^E e)$. This connection is again a derivation. Thus, $S \otimes E$ is a Dirac bundle *for any vector bundle E*. This construction gives us an enormous amount of Dirac bundles.

Definition. *Let M be any riemannian manifold with Clifford bundle $Cl(M)$ and let S be any bundle of modules over $Cl(M)$ with a riemannian connection and metric. The first-order differential operator $D : \Gamma(S) \to \Gamma(S)$ defined by*

$$D\sigma = \textstyle\sum_{j=1}^n e_j \cdot \nabla_{e_j}\sigma$$

is called the Dirac operator*, where the above local expression is at $x \in M$ with $e_1, ..., e_n$ an orthonormal basis of T_xM, ∇ the covariant derivative on S, and "$\cdot$" Clifford multiplication.*

The operator D^2 is called the *Dirac laplacian*. Recall that the *principal symbol* of a differential operator $D : \Gamma(E) \to \Gamma(E)$ is a linear map that associates to each point $x \in M$ and each cotangent vector $\xi \in T_x^*(M)$ a linear map $\sigma_\xi(D) : E_x \to E_x$. If in local coordinates we have

$$D = \textstyle\sum_{|\alpha|\le m} A_\alpha(x)\frac{\partial^{|\alpha|}}{\partial x^\alpha} \text{ and } \xi = \sum_k \xi_k dx_k$$

where m is the order of D, then $\sigma_\xi(D) \equiv i^m \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$ where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for α a multi-index. The operator is *elliptic* if the principal symbol is an isomorphism for every $\xi \neq 0$.

Lemma. *Let D be the Dirac operator of the bundle of modules S over a Clifford bundle $Cl(M)$. Then for any $\xi \in T^*M \simeq TM$ we have*

$$\sigma_\xi(D) = i\xi$$
$$\sigma_\xi(D^2) = ||\xi||^2$$

where the symbol on the right denotes Clifford multiplication by the vector ξ and scalar $||\xi||^2$ respectively. In particular, both D and D^2 are elliptic operators.

Proof. The covariant derivative is a first-order operator, with $\nabla_{e_j} = \partial/\partial x_j$ + zero-order terms. So D is a first-order operator, with $D = \sum e_j(\partial/\partial x_j)$ + zero-order terms. Thus for any cotangent vector $\xi = \sum \xi_j dx_j$, we have by definition that $\sigma_\xi(D) = i\sum e_j\xi_j = i\xi$ i.e., the sum of the vector components with respect to the basis is the vector. The second follows from $\sigma_\xi(D^2) = \sigma_\xi(D)\circ\sigma_\xi(D) = -\xi\cdot\xi = ||\xi||^2$. $\square$

Thus any Dirac bundle has a canonically associated Dirac operator.

Theorem. *Let D be the Dirac operator of any Dirac bundle over a compact riemannian manifold M. Then* $\ker D = \ker D^2$ *and this space has finite dimension.*

Note: we made an implicit assumption that all spinor bundles were bundles of *left* modules over itself with respect to Clifford multiplication. However, for the Clifford bundle, it is also true that it is a *right* module over itself and still satisfies that its canonical connection is a derivation. This allows one to define a "right-handed" Dirac operator $\hat{D}$ by setting $\hat{D}\varphi = \sum_{j=1}^n (\nabla_{e_j}\varphi)\cdot e_j$ with its principal symbol being right multiplication by $i\xi$. Recall that the exterior algebra has two canonical first order operators: the exterior derivative $d : \wedge^*(M) \to \wedge^*(M)$ and its adjoint $d^* : \wedge^*(M) \to \wedge^*(M)$. The adjoint is given by

$$d^* = (-1)^{np+n+1} * d *$$

on $\wedge^p(M)$ where $* : \wedge^p(M) \to \wedge^{n-p}(M)$ is the Hodge star operator, i.e., the linear map defined by the condition that $\varphi \wedge *\psi = \langle\varphi,\psi\rangle * 1$ where $*1$ is the volume form.

Theorem. *Under the canonical isomorphism $Cl(M) \simeq \wedge^*(M)$, the Dirac operators of $Cl(M)$ satisfy the following.*

$$D \simeq d + d^*$$
$$\hat{D} \simeq (-1)^p(d - d^*)$$

Consequently since $d^2 = (d^*)^2 = 0$ we also have

$$D^2 = \hat{D}^2 = dd^* + d^*d = \Delta$$
$$D\hat{D} = \hat{D}D$$

where Δ is the Hodge laplacian.

Recall that the *index* of an operator D is $\mathrm{ind}(D) \equiv \dim(\ker D)$ - $\dim(\mathrm{coker}\, D)$, which for an elliptic operator over a compact manifold is always defined, as the kernel and cokernel are always finite dimensional (compactness is crucial here). We will now discuss some examples of fundamental Dirac operators for different spinor bundles decomposed in different ways.

Let S be a Dirac bundle with D a Dirac operator over a riemannian manifold M, and suppose S is $\mathbb{Z}/2$-graded, i.e., there is a parallel decomposition $S = S^0 \oplus S^1$ so that $Cl(M)^i \cdot S^j \subset S^{i+j}$ for $i,j \in \mathbb{Z}/2$. Then D decomposes as

$$D = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix}$$

where $D^0 : \Gamma(S^0) \to \Gamma(S^1)$ and $D^1 : \Gamma(S^1) \to \Gamma(S^0)$ are adjoint to each other, as D is self-adjoint. Recall that we have two important methods on endowing a spinor bundle with a $\mathbb{Z}/2$ grading: we can either decompose it using the antipodal α eigenbundles, or using the complex volume element $\omega_{\mathbb{C}}$ in dimensions $4k$ using the representation theory of Clifford algebras. Both methods relate well known topological invariants with the indices of these Dirac operators.

Example. Let $S = Cl(M) = Cl^0(M) \oplus Cl^1(M)$ be the Clifford bundle considered as a Dirac bundle, split into the eigenbundles of α the antipodal map. Under the canonical isomorphism $Cl(M) \simeq \wedge^*(M)$ we see that $D^0 : \Gamma(Cl^0(M)) \to \Gamma(Cl^1(M))$ corresponds to the operator

$$d + d^* : \wedge^{even}(M) \to \wedge^{odd}(M)$$

so that ind $D^0 = \dim \mathbf{H}^{even} - \dim \mathbf{H}^{odd} = \chi(M)$ where $\mathbf{H} \equiv \bigoplus_{p=0}^{n} \mathbf{H}^p = \ker(\Delta)$ is the space of *harmonic p-forms*. Thus, the Euler characteristic is the index of the even part of the Dirac operator defined on the Clifford bundle.

Example. Let M be a $4k$-manifold and now let $S = Cl(M) = Cl^+(M) \oplus Cl^-(M)$, this time decomposed into the eigenbundles of $\omega_{\mathbb{C}} = (-1)^k\omega$ the complex volume element; although $\omega_{\mathbb{C}}$ decomposes *complex* Clifford modules into eigenspaces in every *complex* dimension, it is in dimensions $4k$ (i.e., *even* complex dimensions) that the decomposition is exactly the complexification of the splitting in the real setting (since it equals, up to sign, exactly the real volume element). Again take D to be the Dirac operator corresponding to $d + d^*$. There is a corresponding decomposition ker D = ker $D^+ \oplus$ ker D^-. Since $\omega_{\mathbb{C}}$ is parallel (as $0 = d\langle \omega, \omega \rangle = \langle \nabla\omega, \omega \rangle$ as ω has unit norm globally), it preserves ker D and the subspaces ker $D^{\pm}$ are exactly the ± 1 eigenspaces under multiplication by $\omega_{\mathbb{C}}$ on ker D (again by computation showing anticommutativity of D and ω). Thus, we have

$$\ker D^{\pm} = (1 \pm \omega_{\mathbb{C}})\ker D$$

Under the canonical isomorphism $Cl(M) \simeq \wedge^*(M)$ we again have that ker $D \simeq \mathbf{H} = \mathbf{H}^0 \oplus ... \oplus \mathbf{H}^{4k}$ the space of harmonic forms. Under this isomorphism, *left multiplication by* $\omega_{\mathbb{C}}$ *corresponds to the Hodge* $*$*-operator*. So for each $p = 0, ..., 2k$ we have an isomorphism $\omega_{\mathbb{C}} : \mathbf{H}^p \to \mathbf{H}^{4k-p}$. This in turn implies that the space $\mathbf{H}(p) \equiv \mathbf{H}^p \oplus \mathbf{H}^{4k-p}$ for $p < 2k$ has a decomposition $\mathbf{H}(p) = \mathbf{H}^+(p) \oplus \mathbf{H}^-(p)$ where the subspaces $\mathbf{H}^{\pm}(p) \equiv (1 \pm \omega_{\mathbb{C}})\mathbf{H}(p)$ are of the same dimension (via an explicit isomorphism, mapping $\alpha \oplus \beta$ to $-\alpha \oplus \beta$). Since ker $D^{\pm} = \mathbf{H}^{\pm} = \mathbf{H}^{\pm}(0) \oplus ... \oplus \mathbf{H}^{\pm}(2k-1) \oplus (\mathbf{H}^{2k})^{\pm}$ where $(\mathbf{H}^{2k})^{\pm} = (1 \pm \omega_{\mathbb{C}})\mathbf{H}^{2k}$ we can restrict D to the "positive" part of this grading (and therefore maps to the "negative" part by anticommutativity with $\omega_{\mathbb{C}}$ in even dimensions) and conclude

$$\text{ind } D^+ = \dim(\mathbf{H}^{2k})^+ - \dim(\mathbf{H}^{2k})^- = \text{sig}(M)$$

where sig(M) denotes the *signature* of M, under the Hodge/de Rham isomorphism. Indeed, the signature is just the difference of the dimensions of the $+1$ and -1 eigenspaces of $*$ on $\mathbf{H}^{2k}$, since $* \simeq \omega_{\mathbb{C}}$ in dimension $2k$.

Example. Let M be a $4k$-spin manifold and let $\not{S}_{\mathbb{C}}$ be the unique complex spinor bundle with Dirac operator $\not{D}$, called the *Atiyah-Singer operator*. Again we have a decomposition $\not{S}_{\mathbb{C}} = \not{S}_{\mathbb{C}}^+ \oplus \not{S}_{\mathbb{C}}^-$ into eigenbundles of the complex volume element $\omega_{\mathbb{C}}$. It is a corollary of the Atiyah-Singer index theorem that ind$(\not{D}) = \hat{A}(M)$ where $\hat{A}(M)$ is the $\hat{A}$*-genus*.

In general the $\hat{A}$-genus is *not* an integer. For example, for a compact 4-manifold X, we have that $-\hat{A}(X) = \frac{1}{8}\text{sig}(X) = \frac{1}{24}p_1(X)$, where p_1 is the first Pontryagin number of X. In particular, for $\mathbb{CP}^2$, as $H^2(\mathbb{CP}^2,\mathbb{Z}) \simeq \mathbb{Z}$, its signature is 1, so its $\hat{A}$-genus is $-\frac{1}{8}$. Thus, the above result tells us that *the $\hat{A}$-genus of a compact spin manifold is always an integer.*

To even begin to define the $\hat{A}$-genus, we first define a multiplicative sequence.

Definition. *Let $\mathbb{Q}[[x]]^\frown$ denote the set of formal power series in x with rational coefficients and constant term 1. Fix an element $f(x) \in \mathbb{Q}[[x]]^\frown$ and for each $n \in \mathbb{N}$ consider the formal power series in n indeterminates given by $f(x_1)\cdots f(x_n)$. This is symmetric in the x_j and so has an expansion of the form*

$$f(x_1)\cdots f(x_n) = 1 + F_1(\sigma_1) + F_2(\sigma_1,\sigma_2) + F_3(\sigma_1,\sigma_2,\sigma_3) + ...$$

where

$$\sigma_k(x_1,...,x_n) \equiv \sum_{i_1<...<i_k} x_{i_1}\cdots x_{i_k} \text{ for } 1 \le k \le n$$

denotes the kth elementary symmetric function in $x_1,...,x_n$ and where F_k is weighted homogeneous of degree k. This sequence of polynomials $\{F_k(\sigma_1,...,\sigma_k)\}_{k=1}^\infty$ is called the multiplicative sequence *determined by the formal power series $f(x)$.*

Example. Associated to the formal power series

$$l(x) \equiv \frac{\sqrt{x}}{\tanh\sqrt{x}} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + ...$$

is a multiplicative sequence $\{L_m\}$ called the *Hirzebruch L-sequence.* Replace the symmetric functions σ_k with p_k the kth pontryagin class. The first few terms of this sequence are

$$L_1(p_1) = \frac{1}{3}p_1$$
$$L_2(p_1,p_2) = \frac{1}{45}(7p_2 - p_1^2)$$
$$L_3(p_1,p_2,p_3) = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3)$$

Given a real bundle $E \to X$, the total L-class of E is the sum

$$\mathbf{L}(E) \equiv 1 + L_1(p_1E) + L_2(p_1E, p_2E) + ...$$

The L-genus $L(X)$ is then the number $\langle\langle\mathbf{L}(TX),[X]\rangle$ which is equal to 0 if dim X is not $4k$ and is equal to $\langle L_k(p_1(TX),...,p_k(TX)),[X]\rangle$ if dim X is $4k$. By the Hirzebruch signature theorem, we have that $L(X) = \text{sig}(X)$.

Example. Associated to the formal power series

$$\hat{a}(x) \equiv \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{1}{24}x + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^2 + ...$$

is a multiplicative sequence $\{\hat{A}_m\}$ called the $\hat{A}$-sequence. Again, using pontryagin classes, we have that the first few terms in this $\hat{A}$-sequence are

$$\hat{A}_1 = -\frac{1}{24}p_1$$
$$\hat{A}_2 = \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2)$$
$$\hat{A}_3 = -\frac{1}{2^{10} \cdot 3^3 \cdot 5 \cdot 7}(16p_3 - 44p_2p_1 + 31p_1^3)$$

so that we can define the total $\hat{A}$-class of a vector bundle $E \to X$ as the sum

$$\hat{\mathbf{A}}(E) = 1 + \hat{A}_1(p_1E) + \hat{A}_2(p_1E, p_2E) + ...$$

Thus, we can now define the $\hat{A}$-genus $\hat{A}(X)$ as the number $\langle \hat{\mathbf{A}}(TX), [X] \rangle$ which is 0 if dim X is not $4k$ and is $\langle \hat{A}_k(p_1(TX), ..., p_k(TX)), [X] \rangle$ if dim X is $4k$.

Note that $L(X)$ is always an integer. This is not the case with $\hat{A}(X)$ from the example of $\mathbb{CP}^2$; we have in fact that $\hat{A}(X) = -\frac{1}{8}$ sig(X) when X is a 4-manifold. From the Atiyah-Singer index theorem however, it follows that, for spin manifolds, the $\hat{A}$-genus is always an integer.

Example. Let us compute $\hat{A}(\mathbb{CP}^2)$. We have

$$\begin{aligned}
\hat{A}(\mathbb{CP}^2) &= -\frac{1}{24} p_1(\mathbb{CP}^2) \\
&= \frac{1}{24} c_2(T\mathbb{CP}^2 \otimes \mathbb{C}) \\
&= \frac{1}{24} c_2(T\mathbb{CP}^2 \oplus \overline{T\mathbb{CP}^2}) \\
&= \frac{1}{24} (c_2(\mathbb{CP}^2) + c_1(\mathbb{CP}^2) c_1(\overline{\mathbb{CP}^2}) + c_2(\overline{\mathbb{CP}^2}) \\
&= \frac{1}{24} (2c_2(\mathbb{CP}^2) - c_1(\mathbb{CP}^2)^2) \\
&= \frac{1}{24} (6 - 9) \\
&= -\frac{1}{8}
\end{aligned}$$

Example. Let us compute $\hat{A}(K3)$ for $K3$ the degree 4 hypersurface in $\mathbb{CP}^3$.

$$\begin{aligned}
\hat{A}(K3) &= -\frac{1}{24} p_1(K3) \\
&= \frac{1}{24} (2c_2(K3) - (c_1(K3))^2) \\
&= \frac{48}{24} = 2
\end{aligned}$$

since $c_1(K3) = 0$ and as the top Chern class is the Euler class, which evaluates the Euler characteristic $\chi(K3) = 24$. As signature is additive, this also gives us $\hat{A}$ for connect sums of K3s.

Representations and K-theory

This section discusses a little more on the irreducible representations of Clifford algebras and their relation to K-theory. In particular, we aim to define the Atiyah-Bott-Shapiro map.

Definition. *Let X be a compact manifold, $V(X)$ the set of all isomorphism classes of complex vector bundles over X, and $F(X)$ the free abelian group generated by the elements of $V(X)$, which form an abelian semigroup with respect to Whitney sum. Let $E(X)$ be the subgroup of $F(X)$ generated by elements of the form $[V] + [W] - ([V] \oplus [W])$, where $+$ denotes addition in $F(X)$ and $\oplus$ denotes addition in $V(X)$. The* K-group *of X is defined to be the quotient*

$$K(X) \equiv F(X)/E(X)$$

We can define the analogous *real* K-group for X, denoted $KO(X)$ by taking isomorphism classes of *real* vector bundles; all following properties hold through. A fact that we will not prove here (but is not too difficult using some algebra) is that every element in $K(X)$ can be written in the form $[V]-[W]$ for $[V],[W]\in V(X)$. Notice that for every vector bundle V over X, there is a "complementary" bundle $V^\perp$ such that $V\oplus V^\perp$ is trivial. One sees this first by realizing that for every vector bundle V, there is some index N and a continuous map from $V\to\mathbb{C}^N$ which is injective and linear on the fibers, by using the local trivializations to embed the bundle fibers locally into some large enough Euclidean space and then using a partition of unity (as in the realization of the Grassmannian as a classifying space). Then one takes this embedding and takes a quotient to obtain the complementary bundle. However, this complementary bundle is not unique – one can add trivial bundles. The upshot is that every element of $K(X)$ can be written in the form $[U]-[\tau^n]$ for some n: since we have that every element is of the form $[V]-[W]$, we take a trivializing complement to $[W]$ to obtain $[V]-[W]=[V]+[W^\perp]-[W^\perp]-[W]=[V]+[W]^\perp-[\tau^n]=[U]-[\tau]^n$.

Define the *reduced* K-group to be $\tilde{K}(X)\equiv\ker\,(K(X)\to K(\mathrm{pt})\simeq\mathbb{Z})$ where pt is a distinguished point, so that the exact sequence splits:

$$0\to\tilde{K}(X)\to K(X)\to K(\mathrm{pt})\to 0$$

One can think of the reduced K-group as the group of isomorphism classes of complex vector bundles up to *stable* equivalence, i.e., two elements in the reduced K-group are equivalent if the bundles representing them are isomorphic upon addition of trivial bundles. The ordinary K-group records the additional data of the dimension at which these bundles trivialize, as noted above. Note that we can define a product structure on the K-group by taking tensor powers and then pulling back by the diagonal. This makes $K(X)$ into a ring.

Definition. *Let Y be a nonempty closed subset of X. Then the* relative K-groups *are*

$$K(X,Y)\equiv\tilde{K}(X/Y)$$

where X/Y is taken to have Y as its basepoint. If Y is empty, then define X/Y as the space $X^+\equiv X\cup\{\tilde{\mathrm{pt}}\}$ where $\tilde{\mathrm{pt}}$ is a disjoint point which plays the role of basepoint.

If X is not basepointed, we have that $K(X)\simeq\tilde{K}(X^+)=K(X,\varnothing)$.

Definition. *For X a compact basepointed space, or when (X,Y) is a compact pair, we define*

$$\tilde{K}^{-i}(X)\equiv\tilde{K}(\Sigma^i(X))$$
$$K^{-i}(X,Y)\equiv\tilde{K}^{-i}(X/Y)\equiv\tilde{K}(\Sigma^i(X/Y))$$

where Σ denotes the *reduced suspension* of X, and Σ^i is the *i-fold suspension.* For spaces that are not necessarily basepointed, we define

$$K^{-i}(X)\equiv K^{-i}(X,\varnothing)\equiv\tilde{K}(\Sigma^i(X^+))$$

We have that $\tilde{K}^{-i}(X\times Y)\simeq\tilde{K}^{-i}(X\wedge Y)\oplus\tilde{K}^{-i}(X)\oplus\tilde{K}^{-i}(Y)$ for all i and for X,Y basepointed. Moreover we have a pairing given by tensor product $\tilde{K}^{-i}(X)\otimes\tilde{K}^{-j}(Y)\to\tilde{K}^{-i-j}(X\wedge Y)$ for any X,Y and for any $i,j\geq 0$. Combining these two and replacing X with X^+ and Y with Y^+, we have a pairing $K^{-i}(X)\otimes K^{-j}(Y)\to K^{-i-j}(X\times Y)$ so that $K^{-*}(\mathrm{pt})$ becomes a graded ring. Indeed, $K^{-*}(\mathrm{pt})$ is well known:

$$K^{-i}(\mathrm{pt})=\begin{cases}\mathbb{Z} & \text{if } i \text{ even}\\ 0 & \text{if } i \text{ odd}\end{cases}$$

Indeed $K^{-i}(\mathrm{pt}) = \tilde{K}(S^i)$ are the complex vector bundles up to stable equivalence on the i-sphere, which are known by Bott periodicity. Moreover, for any basepointed space (X, pt), by the above pairing, we have that $K^{-*}(X)$ is a graded module over $K^{-*}(\mathrm{pt})$. We want to define the K-groups in another way, however, so as to define the Atiyah-Bott-Shapiro map. For a pair (X, Y) consider the set of elements $\mathbf{V} = (V_0, V_1; \sigma_1)$ where V_0, V_1 are vector bundles on X and where σ_1 is an isomorphism between these two bundles restricted to Y. Say two such tuples are *isomorphic* when there are bundle isomorphisms $\varphi_i : V_i \to V_i'$ over X so that the following diagram commutes:

$$\begin{tikzcd} V_0|_Y \arrow[r, "\sigma_1"] \arrow[d, "\varphi_0"] & V_1|_Y \arrow[d, "\varphi_1"] \\ V_0'|_Y \arrow[r, "\sigma_1'"] & V_1'|_Y \end{tikzcd}$$

An element $\mathbf{V} = (V_0, V_1; \sigma_1)$ is *elementary* if $V_0 = V_1$ and $\sigma_1 = \mathrm{Id}$. One can then say that two elements $\mathbf{V}, \mathbf{V'}$ are *equivalent* if there exist elementary elements $\mathbf{E}_1$, ..., $\mathbf{E}_k, \mathbf{F}_1, ..., \mathbf{F}_l$ and an isomorphism

$$\mathbf{V} \oplus \mathbf{E}_1 \oplus ... \oplus \mathbf{E}_k \simeq \mathbf{V'} \oplus \mathbf{F}_1 ... \oplus \mathbf{F}_l$$

The equivalence class of such an element will be denoted by $[V_0, V_1; \sigma_1]$. It turns out that this group $L(X, Y)$ will be isomorphic to $K(X, Y)$. The proof of the equivalence is the "difference bundle construction" which is a generalization of the clutching map picture for bundles over the sphere.

Given an element $\mathbf{V} = [V_0, V_1; \sigma]$, we associate to it an element $\chi(V) \in K(X, Y)$ via the following difference bundle construction. Set $X_k = X \times k$ for $k = 0, 1$ and consider the space $Z = X_0 \cup X_1$ obtained from the disjoint union $X_0 \coprod X_1$ by identifying $y \times \{0\}$ with $y \times \{1\}$ for all y (much like how the sphere is obtained from the disjoint union of two disks glued along their boundaries). The natural sequence

$$0 \longrightarrow K(Z, X_1) \xrightarrow{j^*} K(Z) \xrightarrow{i^*} K(X_1) \longrightarrow 0$$

is split exact since there is a retraction $\rho : Z \to X_1$. We also have an isomorphism $\varphi : K(Z, X_1) \to K(X, Y)$ induced by the map of pairs $(X, Y) \to (Z, X_1)$ which identifies X with X_0. Now we apply the same picture for the clutching map over the sphere: define a new vector bundle W over Z by setting $W|_{X_k} \equiv V_k$ and identifying over Y via the isomorphism σ. Setting $W_1 \equiv \rho^*(V_1)$ we have $[W] - j^*\varphi^{-1}\chi(\mathbf{V}) = [W] - [W_1]$ where $\chi(\mathbf{V}) = \sum_{k=0}^{n} (-1)^k [V_k]$. This defines the unique equivalence of functors $\chi : L(X, Y) \to K(X, Y)$.

Let us finally come to the Atiyah-Bott-Shapiro isomorphism $\varphi_* : (\hat{\mathfrak{M}}_*^{\mathbb{C}} / i^* \hat{\mathfrak{M}}_{*+1}^{\mathbb{C}}) \to K^{-*}(\mathrm{pt})$ and similarly the real case $\varphi_* : (\hat{\mathfrak{M}}_* / i^* \hat{\mathfrak{M}}_{*+1}) \to KO^{-*}(\mathrm{pt})$ where $\hat{\mathfrak{M}}_*^{\mathbb{C}}$ and $\hat{\mathfrak{M}}_*$ denote the Grothendieck group of irreducible complex and real $\mathbb{Z}/2$-graded representations of the complex and real Clifford algebra, respectively. We will only define the map here, and not prove that it is an isomorphism (for details, see [1]). Let $W = W^0 \oplus W^1$ be a $\mathbb{Z}/2$-graded module over Cl_n. Define a vector bundle over the sphere by the clutching construction: let $E_k = D^n \times W^k$ for $k = 0, 1$ be the trivial bundle over the n-disk D^n, and glue the two copies together by the isomorphism μ defined on S^{n-1} where

$$\mu(x, w) = (x, x \cdot w)$$

i.e., μ is Clifford multiplication by the elements of the sphere, thought of as being in $\mathbb{R}^n \subset Cl_n$. This defines a vector bundle $\varphi_*(W) = [E_0, E_1; \mu] \in K(D^n, S^{n-1}) = K(S^n)$. Note however the natural inclusion map $i : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ pulls back Clifford representations via restriction. If the $\mathbb{Z}/2$-graded module W is actually the pullback of a representation from a higher dimensional module, then the clutching map (in this case given by multiplication by elements of the equatorial sphere on the module elements) extends over the disk as an isomorphism by setting

$$\mu(x, w) = (x, (x + \sqrt{1 - ||x||^2} e^{n+1}) \cdot w)$$

where e^{n+1} is the extra vector in $\mathbb{R}^{n+1}$. Thus, if the module arises as a pullback from a higher dimensional module, then the resulting bundle is trivial, as the clutching map extends over the disk (and is therefore nullhomotopic). Therefore we have the following theorem:

Theorem. *(Atiyah-Bott-Shapiro isomorphism) The maps φ_* induce graded ring isomorphisms*

$$\varphi_* : (\hat{\mathfrak{M}}_*^{\mathbb{C}} / i^* \hat{\mathfrak{M}}_{*+1}^{\mathbb{C}}) \to K^{-*}(\mathrm{pt})$$
$$\varphi_* : (\hat{\mathfrak{M}}_* / i^* \hat{\mathfrak{M}}_{*+1}) \to KO^{-*}(\mathrm{pt})$$

The existence of these isomorphisms is a profound and important fact towards explaining the fundamental role played by Clifford algebras in the index theory for elliptic operators.

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