The Atiyah-Singer Index Theorem

April 6, 2015

$$\dim \mathrm{Ker} \not{D}_E - \dim \mathrm{Coker} \not{D}_E = \int_M \widehat{A}(M) \cdot ch(E)$$

The Atiyah-Singer Index Theorem (1963) forms a bridge between topology and analysis. It is a topological formula that computes the index of operators associated to elliptic partial differential equations and other related equations. This index is the "formal" dimension of the space of solutions to the equation in the sense that it is the dimension of the space of solutions minus the dimension of the obstruction space associated with the equation[1]. It good cases, the latter vanishes and the topological formula gives the actual dimension of the space of solutions. The topological nature of the formula means that in practice in many situations the index, or formal dimension, can be computed. The fact that the theorem applies to many equations and operators appearing naturally in geometry and physics means that it has important applications in both these fields.

One of its first uses in theoretical physics (1978) was to compute the dimension of the moduli space of anti-self-dual connections, including the special case of instantons on $\mathbb{R}^4$. This was one of the first direct connections between topology and modern theoretical high-energy physics. It was a significant early step in the resurgence of interactions between these fields that continues to play an important role in each of these fields.

[1] The case shown M is a spin manifold and the formula is for the index of $\not{D}_E$, the Dirac operator from positive spinors on M with values in an auxiliary bundle E to negative spinors on M with values in the same bundle, $ch(E)$ is the Chern character of E, and $\widehat{A}(M)$ is a topological class associated to the spin manifold M.